Extremal Processes of Gaussian Processes Indexed by Trees

Anton Bovier
Louis-Pierre Arguin, Lisa Hartung, Nicola Kistler

Institute for Applied Mathematics Bonn

Markov Processes, Mixing Times and Cutoff, Durham, 27.07.2017

hausdorff center for mathematics

universitätbonn iam

SFB 1060

PROBABILISTIC STRUCTURES IN EVOLUTION
DFG SPP 1590

Motivation

Motivation

- **Spin glasses:** What is the structure of ground states for (mean field) spin glasses?

Motivation

- **Spin glasses:** What is the structure of ground states for (mean field) spin glasses?
- **Extreme value theory:** What are the extreme values and the extremal process of dependent random processes?

Motivation

- **Spin glasses:** What is the structure of ground states for (mean field) spin glasses?
- **Extreme value theory:** What are the extreme values and the extremal process of dependent random processes?
- **Spatial branching processes:** Describe the cloud of spatial branching processes, in particular near their propagation front!

Motivation

- **Spin glasses:** What is the structure of ground states for (mean field) spin glasses?
- **Extreme value theory:** What are the extreme values and the extremal process of dependent random processes?
- **Spatial branching processes:** Describe the cloud of spatial branching processes, in particular near their propagation front!
- **Reaction diffusion equations:** Characterise convergence to travelling wave solutions in certain non-linear pdes!

Motivation

- **Spin glasses:** What is the structure of ground states for (mean field) spin glasses?
- **Extreme value theory:** What are the extreme values and the extremal process of dependent random processes?
- **Spatial branching processes:** Describe the cloud of spatial branching processes, in particular near their propagation front!
- **Reaction diffusion equations:** Characterise convergence to travelling wave solutions in certain non-linear pdes!

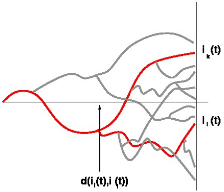
Gaussian processes labelled by trees

Gaussian processes labelled by trees

- A time-homogeneous tree. Label individuals at time t as $\mathbf{i}_1(t), \ldots, \mathbf{i}_{n(t)}(t)$.

Gaussian processes labelled by trees

- A time-homogeneous tree. Label individuals at time t as $\mathbf{i}_1(t), \ldots, \mathbf{i}_{n(t)}(t)$.
- Canonical tree-distance: $d(\mathbf{i}_\ell(t), \mathbf{i}_k(t)) \equiv$ time of most recent common ancestor of $\mathbf{i}_\ell(t)$ and $\mathbf{i}_k(t)$

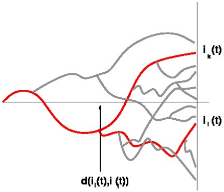


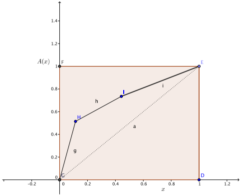
Gaussian processes labelled by trees

- A time-homogeneous tree. Label individuals at time t as $\mathbf{i}_1(t), \ldots, \mathbf{i}_{n(t)}(t)$.
- Canonical tree-distance: $d(\mathbf{i}_\ell(t), \mathbf{i}_k(t)) \equiv$ time of most recent common ancestor of $\mathbf{i}_\ell(t)$ and $\mathbf{i}_k(t)$
- For fixed time horizon t, define Gaussian process, $(x_k^t(s), k \leq n(t), s \leq t)$, with covariance

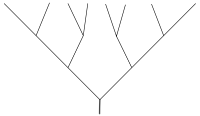
$$\mathbb{E} x_k^t(r) x_\ell^t(s) = tA(t^{-1} d(\mathbf{i}_k(r), \mathbf{i}_\ell(s)))$$

for $A : [0,1] \to [0,1]$, increasing.





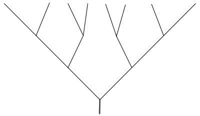
Examples



Examples

Binary tree, branching at integer times

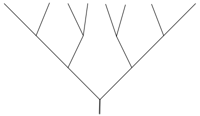
Examples



Binary tree, branching at integer times

- $A(x) = x$: Branching random walk [Harris '63]

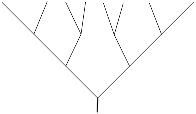
Examples



Binary tree, branching at integer times

- $A(x) = x$: Branching random walk [Harris '63]
- A step function: Generalised Random Energy models (GREM) [Gardner-Derrida '82]

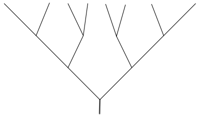
Examples



Binary tree, branching at integer times

- $A(x) = x$: Branching random walk [Harris '63]
- A step function: Generalised Random Energy models (GREM) [Gardner-Derrida '82]
- Special case $A(x) = 0, x < 1$, $A(1) = 1$: Random energy model (REM), i.e. $n(t)$ iid $\mathcal{N}(0, t)$ r.v.s

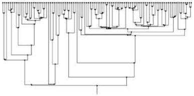
Examples



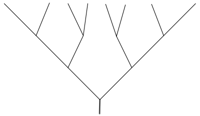
Binary tree, branching at integer times

- $A(x) = x$: Branching random walk [Harris '63]
- A step function: Generalised Random Energy models (GREM) [Gardner-Derrida '82]
- Special case $A(x) = 0, x < 1$, $A(1) = 1$: Random energy model (REM), i.e. $n(t)$ iid $\mathcal{N}(0, t)$ r.v.s

Supercritical Galton-Watson tree



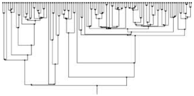
Examples



Binary tree, branching at integer times

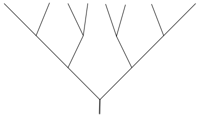
- $A(x) = x$: Branching random walk [Harris '63]
- A step function: Generalised Random Energy models (GREM) [Gardner-Derrida '82]
- Special case $A(x) = 0, x < 1$, $A(1) = 1$: Random energy model (REM), i.e. $n(t)$ iid $\mathcal{N}(0,t)$ r.v.s

Supercritical Galton-Watson tree



- $A(x) = x$: Branching Brownian motion (BBM) [Moyal '62]

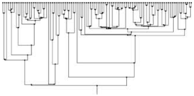
Examples



Binary tree, branching at integer times

- $A(x) = x$: Branching random walk [Harris '63]
- A step function: Generalised Random Energy models (GREM) [Gardner-Derrida '82]
- Special case $A(x) = 0, x < 1$, $A(1) = 1$: Random energy model (REM), i.e. $n(t)$ iid $\mathcal{N}(0, t)$ r.v.s

Supercritical Galton-Watson tree



- $A(x) = x$: Branching Brownian motion (BBM) [Moyal '62]
- General A: variable speed BBM [Derrida-Spohn '88, Fang-Zeitouni '12]

Extreme value theory

Extreme value theory

In the class of models we have described, we are interested in three main questions:

Extreme value theory

In the class of models we have described, we are interested in three main questions:

- How big is $M(t)/t \equiv \max_{k \leq n(t)} x_k(t)/t$, as $t \uparrow \infty$?

Extreme value theory

In the class of models we have described, we are interested in three main questions:

- How big is $M(t)/t \equiv \max_{k \leq n(t)} x_k(t)/t$, as $t \uparrow \infty$?
- Is there a rescaling $u_t(x)$, such that

$$\mathbb{P}\left(M(t) \leq u_t(x)\right) \to F(x)?$$

Extreme value theory

In the class of models we have described, we are interested in three main questions:

- How big is $M(t)/t \equiv \max_{k\leq n(t)} x_k(t)/t$, as $t \uparrow \infty$?
- Is there a rescaling $u_t(x)$, such that

$$\mathbb{P}\left(M(t) \leq u_t(x)\right) \to F(x)?$$

- Is there a limiting extremal process, $\mathcal{P}$, such that

$$\sum_{k\leq n(t)} \delta_{u_t^{-1}(x_k(t))} \to \mathcal{P}?$$

Reference: The REMs

Reference: The REMs

If $x_k(t)$ are just $n(t)$ iid Gaussian rv's with variance t:

Reference: The REMs

If $x_k(t)$ are just $n(t)$ iid Gaussian rv's with variance t:

- $M(t)/t \to \sqrt{2 \lim_{t\uparrow\infty} t^{-1} \ln n(t)} \equiv \sqrt{2r}$

Reference: The REMs

If $x_k(t)$ are just $n(t)$ iid Gaussian rv's with variance t:

- $M(t)/t \to \sqrt{2\lim_{t\uparrow\infty} t^{-1}\ln n(t)} \equiv \sqrt{2r}$

With $u_t(x) = t\sqrt{2r} - \frac{\ln(rt)}{2\sqrt{2r}} + \frac{x}{\sqrt{r}} + \frac{\ln(n(t)/\mathbb{E}n(t))}{\sqrt{2r}}$, where $n(t)/\mathbb{E}n(t) \to RV$, a.s.

- $$\mathbb{P}\left(M(t) \leq u_t(x)\right) \to \exp\left(-\frac{1}{4\pi}e^{-\sqrt{2}x}\right)$$

Reference: The REMs

If $x_k(t)$ are just $n(t)$ iid Gaussian rv's with variance t:

- $M(t)/t \to \sqrt{2\lim_{t\uparrow\infty} t^{-1}\ln n(t)} \equiv \sqrt{2r}$

With $u_t(x) = t\sqrt{2r} - \frac{\ln(rt)}{2\sqrt{2r}} + \frac{x}{\sqrt{r}} + \frac{\ln(n(t)/\mathbb{E}n(t))}{\sqrt{2r}}$, where $n(t)/\mathbb{E}n(t) \to RV$, a.s.

- $$\mathbb{P}\left(M(t) \le u_t(x)\right) \to \exp\left(-\frac{1}{4\pi} e^{-\sqrt{2}x}\right)$$

- $$\sum_{k\le n(t)} \delta_{u_t^{-1}(x_k(t))} \to \mathrm{PPP}\left(\frac{1}{4\pi} e^{-\sqrt{2}x} dx\right)$$

where PPP(μ) : Poisson Point Process with intensity μ.

Universality 1: the order of the maximum

Universality 1: the order of the maximum

Order of the maximum is function of the growth rate of $n(t)$ and concave hull $\bar{A}$ of the function A:

Universality 1: the order of the maximum

Order of the maximum is function of the growth rate of $n(t)$ and concave hull $\bar{A}$ of the function A:

$$\lim_{t\to\infty} t^{-1}M(t) = \sqrt{2\lim_{t\to\infty} t^{-1}\ln n(t)}\int_0^1 \sqrt{\frac{d}{ds}\bar{A}(s)}ds$$

[B-Kurkova 01, for binary tree, Fang-Zeitouni 11, GW tree]

Universality 1: the order of the maximum

Order of the maximum is function of the growth rate of $n(t)$ and concave hull $\bar{A}$ of the function A:

$$\lim_{t\to\infty} t^{-1}M(t) = \sqrt{2\lim_{t\to\infty} t^{-1}\ln n(t)}\int_0^1 \sqrt{\frac{d}{ds}\bar{A}(s)}ds$$

[B-Kurkova 01, for binary tree, Fang-Zeitouni 11, GW tree]

In particular, as long as $A(s) \leq s$ for all $s \leq 1$, then $\bar{A}(s) = s$, and the order of the maximum is the same as in the REM.

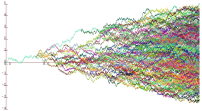
hausdorff center for mathematics universität bonn iam SFB 1060 PSE PROBABILISTIC STRUCTURES IN EVOLUTION DFG SPP 1590

Branching Brownian motion

(BBM) is a classical object in probability, combining the standard models of random motion and random genealogies into one: Each particle of the Galton-Watson process performs Brownian motion independently of any other. This produces an immersion of the Galton-Watson process in space.

Branching Brownian motion

(BBM) is a classical object in probability, combining the standard models of random motion and random genealogies into one: Each particle of the Galton-Watson process performs Brownian motion independently of any other. This produces an immersion of the Galton-Watson process in space.



Picture by **Matt Roberts**, Bath

BBM is the canonical model of a spatial branching process.

The F-KPP equation

The F-KPP equation

One of the simplest reaction-diffusion equations is the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation:

$$\partial_t v(x,t) = \frac{1}{2}\partial_x^2 v(x,t) + v - v^2$$

The F-KPP equation

One of the simplest reaction-diffusion equations is the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation:

$$\partial_t v(x,t) = \frac{1}{2}\partial_x^2 v(x,t) + v - v^2$$

Fischer used this equation to model the evolution of biological populations. It accounts for:

- birth: v,
- death: $-v^2$,
- diffusive migration: $\partial_x^2 v$.

F-KPP equation and BBM

F-KPP equation and BBM

Lemma (McKeane '75, Ikeda, Nagasawa, Watanabe '69)

Let $f : \mathbb{R} \to [0,1]$ *and* $\{x_k(t) : k \leq n(t)\}$ *BBM.*

$$u(t,x) = \mathbb{E}\left[\prod_{k=1}^{n(t)} f(x - x_k(t))\right]$$

Then $v \equiv 1 - u$ *is the solution of the F-KPP equation with initial condition* $v(0,x) = 1 - f(x)$.

Travelling waves

Travelling waves

Theorem (KPP '37,......, Bramson '78)

The equation

$$\frac{1}{2}\omega'' + \sqrt{2}\omega' - \omega^2 + \omega = 0.$$

has a unique solution satisfying $0 < \omega(x) < 1$, $\omega(x) \to 0$, *as* $x \to +\infty$, *and* $\omega(x) \to 1$, *as* $x \to -\infty$, *up to translation.*

Travelling waves

Theorem (KPP '37,......, Bramson '78)

The equation

$$\frac{1}{2}\omega'' + \sqrt{2}\omega' - \omega^2 + \omega = 0.$$

has a unique solution satisfying $0 < \omega(x) < 1$, $\omega(x) \to 0$, *as* $x \to +\infty$, *and* $\omega(x) \to 1$, *as* $x \to -\infty$, *up to translation.*
For suitable initial conditions,

$$u(t, x + m(t)) \to \omega(x),$$

where $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t$, *where* ω *is one of the stationary solutions.*

Examples

Examples

Choosing suitable initial conditions, this theorem applies to

Examples

Choosing suitable initial conditions, this theorem applies to

- $u(t,x) = \mathbb{P}(\max_{k \leq n(t)} x_k(t) \leq x)$.

Examples

Choosing suitable initial conditions, this theorem applies to

- $u(t,x) = \mathbb{P}(\max_{k \leq n(t)} x_k(t) \leq x)$.

  This gives Bramson's celebrated result

$$\lim_{t\to\infty} \mathbb{P}(\max_{k\leq n(t)} x_k(t) - m(t) \leq x) = \omega(x)$$

Examples

Choosing suitable initial conditions, this theorem applies to

- $u(t,x) = \mathbb{P}(\max_{k \le n(t)} x_k(t) \le x)$.

  This gives Bramson's celebrated result

$$\lim_{t\to\infty} \mathbb{P}(\max_{k\le n(t)} x_k(t) - m(t) \le x) = \omega(x)$$

  and

- the Laplace functional $u(t,x) = \mathbb{E}\exp(-\sum_{k\le n(t)} \phi(x_k(t)))$
  Allows to characterise the extremal process...

The derivative martingale

The derivative martingale

Lalley-Sellke, 1987: $\omega(x)$ is random shift of Gumbel-distribution

$$\omega(x) = \mathbb{E}\left[e^{-CZe^{-\sqrt{2}x}}\right]$$

The derivative martingale

Lalley-Sellke, 1987: $\omega(x)$ is random shift of Gumbel-distribution

$$\omega(x) = \mathbb{E}\left[e^{-C\mathbf{Z}e^{-\sqrt{2}x}}\right]$$

$Z \stackrel{(d)}{=} \lim_{t\to\infty} Z(t)$, where $Z(t)$ is the derivative martingale,

$$Z(t) = \sum_{k\leq n(t)} \{\sqrt{2}t - x_k(t)\} e^{-\sqrt{2}\{\sqrt{2}t - x_k(t)\}}$$

The derivative martingale

Lalley-Sellke, 1987: $\omega(x)$ is random shift of Gumbel-distribution

$$\omega(x) = \mathbb{E}\left[e^{-C\boldsymbol{Z}e^{-\sqrt{2}x}}\right]$$

$Z \stackrel{(d)}{=} \lim_{t\to\infty} Z(t)$, where $Z(t)$ is the derivative martingale,

$$Z(t) = \sum_{k\leq n(t)} \{\sqrt{2}t - x_k(t)\} e^{-\sqrt{2}\{\sqrt{2}t - x_k(t)\}}$$

Description of the extremal process of BBM

Description of the extremal process of BBM

Poisson Point Process: $\mathcal{P}_Z = \sum_{i\in\mathbb{N}} \delta_{p_i} \equiv \mathrm{PPP}\left(CZe^{-\sqrt{2}x}dx\right)$

Description of the extremal process of BBM

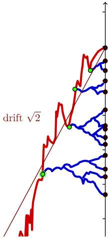
Poisson Point Process: $\mathcal{P}_Z = \sum_{i\in\mathbb{N}} \delta_{p_i} \equiv \mathrm{PPP}\left(CZe^{-\sqrt{2}x}dx\right)$

Cluster process:

$$\Delta(t) \equiv \sum_k \delta_{x_k(t)-\max_{j\leq n(t)} x_j(t)}.$$

conditioned on the event $\left\{\max_{j\leq n(t)} x_j(t) > \sqrt{2}t\right\}$ converges in law to point process, Δ.

[Chauvin, Rouault '90]

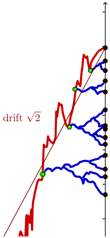


Description of the extremal process of BBM

Poisson Point Process: $\mathcal{P}_Z = \sum_{i\in\mathbb{N}} \delta_{p_i} \equiv \text{PPP}\left(CZe^{-\sqrt{2}x}dx\right)$

Cluster process:

$$\Delta(t) \equiv \sum_k \delta_{x_k(t)-\max_{j\leq n(t)} x_j(t)}.$$



conditioned on the event $\left\{\max_{j\leq n(t)} x_j(t) > \sqrt{2}t\right\}$ converges in law to point process, Δ.

[Chauvin, Rouault '90]

$$\mathcal{E} \equiv \sum_{i,j\in\mathbb{N}} \delta_{p_i+\Delta_j^{(i)}}, \qquad \Delta^{(i)} \text{ iid copies of } \Delta$$

The extremal process

The extremal process

Theorem (Arguin-B-Kistler [PTRF'13, Aidékon-Brunet-Berestycki-Shi [PTRF'13)

The point process $\mathcal{E}_t \equiv \sum_{i=1}^{n(t)} \delta_{x_i(t)-m(t)} \to \mathcal{E}$.

The extremal process

Theorem (Arguin-B-Kistler [PTRF'13, Aidékon-Brunet-Berestycki-Shi [PTRF'13)

The point process $\mathcal{E}_t \equiv \sum_{i=1}^{n(t)} \delta_{x_i(t)-m(t)} \to \mathcal{E}$.

Extended convergence

Can we find a way to encode simultaneously information on the correlation structure of $\mathcal{E}$?

Answer [B-Hartung '14]:
Step 1: Turn the process into a two-dimensional one using its underlying tree structure!

Extended convergence

Can we find a way to encode simultaneously information on the correlation structure of $\mathcal{E}$?

Answer [B-Hartung '14]:
Step 1: Turn the process into a two-dimensional one using its underlying tree structure!

Extended convergence

Can we find a way to encode simultaneously information on the correlation structure of $\mathcal{E}$?

Answer [B-Hartung '14]:
Step 1: Turn the process into a two-dimensional one using its underlying tree structure!

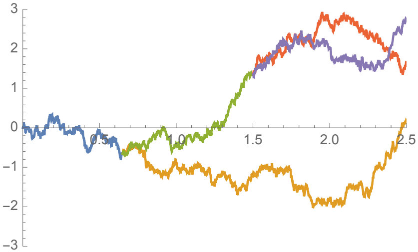


Figure: Particles in space

hausdorff center for mathematics

Extended convergence

Can we find a way to encode simultaneously information on the correlation structure of $\mathcal{E}$?

Answer [Bovier-Hartung '14]:
Step 1: Turn the process into a two-dimensional one using its underlying tree structure!

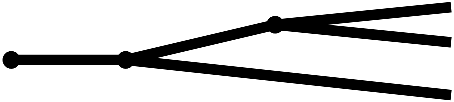
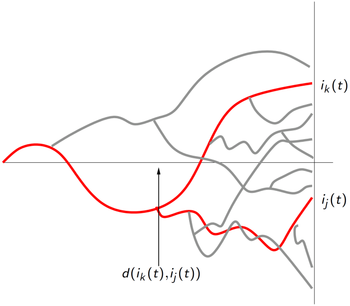


Figure: Underlying branching structure

Embedding

Construct embedding $\gamma : \{1, \ldots, n(t)\} \to \mathbb{R}_+$ of the tree

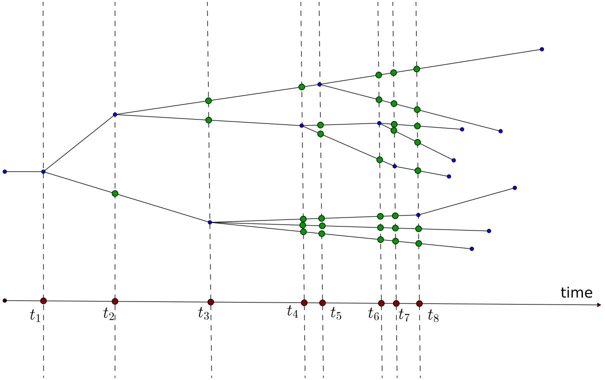
$$|\gamma(i_k(t)) - \gamma(i_j(t))| \sim e^{-d(i_k(t), i_j(t))}$$



hausdorff center for mathematics

Embedding

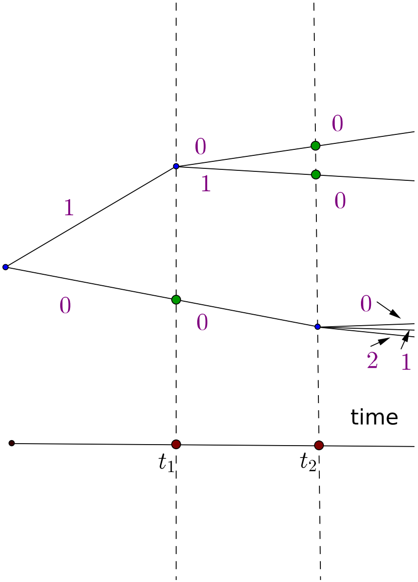
Let $W(t)$ = number of branchings in $[0, t]$.
$(t_1, \ldots, t_{W(t)})$ corresponding time points.
Add extra vertices to the underlying Galton Watson tree:



Embedding of the GW tree

Label at each time t_i the edges starting from 0.
Leaf i_k in GW-tree T_t $\hat{=}$multilabel $u^k(t)$

$$\gamma(u^k(t)) = \sum_{j=1}^{W(t)} u_j^k(t) e^{-t_j}$$



hausdorff center for mathematics

Extended convergence of the extremal process

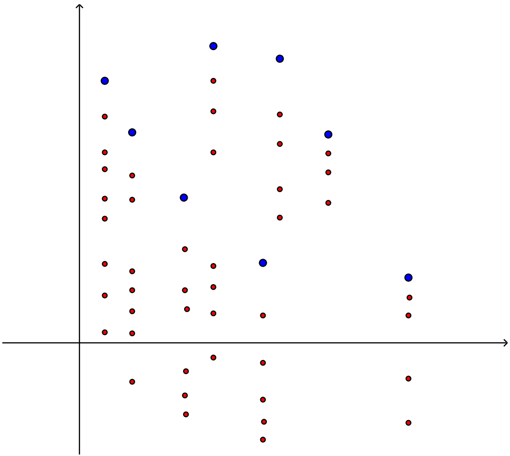
Theorem (B-Hartung '14, AAP)

The point process $\mathcal{E}_t \equiv \sum_{k=1}^{n(t)} \delta_{(\gamma(i_k(t)),x_k(t)-m(t))} \to \widetilde{\mathcal{E}}$ *on* $\mathbb{R}_+ \times \mathbb{R}$*, where*

$$\widetilde{\mathcal{E}} \equiv \sum_{i,j} \delta_{(q_i,p_i)+(0,\Delta_j^{(i)})},$$

with (q_i, p_i) *atoms of a Cox process on* $\mathbb{R}_+ \times \mathbb{R}$ *with intensity measure* $Z(du) \times Ce^{-\sqrt{2}x}dx$*, where* $Z(du)$ *is some random measure and* $\Delta_j^{(i)}$ *as before.*

Extended convergence of the extremal process



Blue: PPP   Red: cluster

The measure

Define for $u \in \mathbb{R}_+$,

$$Z(r,t,u) \equiv \sum_{k:\gamma(i_k(r))\leq u} \{\sqrt{2}t - x_k(t)\} e^{-\sqrt{2}\{\sqrt{2}t - x_k(t)\}}$$

The measure

Define for $u \in \mathbb{R}_+$,

$$Z(r,t,u) \equiv \sum_{k:\gamma(i_k(r))\leq u} \{\sqrt{2}t - x_k(t)\} e^{-\sqrt{2}\{\sqrt{2}t - x_k(t)\}}$$

Lemma (B-Hartung '14)

$$\lim_{r\uparrow\infty}\lim_{t\uparrow\infty} Z(r,t,u) \equiv Z(u)$$

exists almost surely. Moreover, $0 \leq Z(u) \leq Z$ *and* $Z(u)$ *is almost surely non-atomic.*

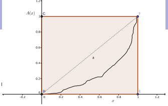
Related Results

- The 2-dim DGFF (Biskup and Louidor '16)

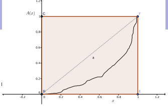
$$\sum_{1\leq i,j\leq n} \delta_{(i/n,j/n),X_{(i,j)}-m_n} \Rightarrow \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \delta_{x_i,p_i+\Delta_j^{(i)}},$$

where (x_i, p_i) are the atoms PPP on $(0,1]^2 \times \mathbb{R}$ with random intensity measure $Z(dx) \times e^{-\sqrt{2}u}du$, where $Z(dx)$ is some random measure on $(0,1]^2$. $\Delta_j^{(i)}$ are the atoms of some cluster process.

Variable speed BBM.....below the straight line...



Variable speed BBM.....below the straight line...

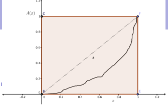


Theorem (B-Hartung [EJP'14, ALEA'15])

Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$.

hausdorff center for mathematics

Variable speed BBM.....below the straight line...



Theorem (B-Hartung [EJP'14, ALEA'15])

Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$.
Then $\exists$ $C(b)$ *and a r.v.* Y_a *such that*

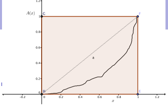
hausdorff center for mathematics

universitätbonn iam

SFB 1060

PSE PROBABILISTIC STRUCTURES IN EVOLUTION DFG SPP 1590

Variable speed BBM.....below the straight line...

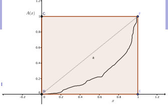


Theorem (B-Hartung [EJP'14, ALEA'15])

Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$. *Then* $\exists$ $C(b)$ *and a r.v.* Y_a *such that*

- $\mathbb{P}\left(M(t) - \tilde{m}(t) \leq x\right) \to \mathbb{E}e^{-C(b)Y_a e^{-\sqrt{2}x}}$

Variable speed BBM.....below the straight line...

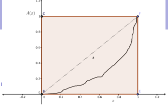


Theorem (B-Hartung [EJP'14, ALEA'15])

Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$.
Then $\exists$ $C(b)$ *and a r.v.* Y_a *such that*

- $\mathbb{P}\left(M(t) - \tilde{m}(t) \leq x\right) \to \mathbb{E}e^{-C(b)Y_a e^{-\sqrt{2}x}}$
- $\sum_{k \leq n(t)} \delta_{x_k(t) - \tilde{m}(t)} \to \mathcal{E}_{a,b} = \sum_{i,j} \delta_{p_i + b\Delta_j^{(i)}}$

Variable speed BBM.....below the straight line...



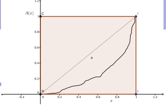
Theorem (B-Hartung [EJP'14, ALEA'15])

Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$. *Then* $\exists$ $C(b)$ *and a r.v.* Y_a *such that*

- $\mathbb{P}\left(M(t) - \tilde{m}(t) \leq x\right) \to \mathbb{E}e^{-C(b)Y_a e^{-\sqrt{2}x}}$
- $\sum_{k \leq n(t)} \delta_{x_k(t) - \tilde{m}(t)} \to \mathcal{E}_{a,b} = \sum_{i,j} \delta_{p_i + b\Delta_j^{(i)}}$

- $\tilde{m}(t) \equiv \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t$.
- p_i: atoms of a $\mathrm{PPP}(C(b)Y_a e^{-\sqrt{2}x}dx)$,
- $Y_a = \lim \sum_{i=1}^{n(s)} e^{-s(1+a^2)+\sqrt{2}x_i(s)}$

Variable speed BBM.....below the straight line..

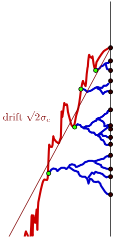


Theorem (B-Hartung [EJP'14, ALEA'15])

Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$. Then $\exists$ $C(b)$ and a r.v. Y_a such that

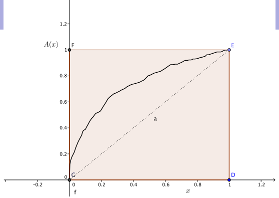
- $\mathbb{P}\left(M(t) - \tilde{m}(t) \leq x\right) \to \mathbb{E}e^{-C(b)Y_a e^{-\sqrt{2}x}}$
- $\sum_{k \leq n(t)} \delta_{x_k(t) - \tilde{m}(t)} \to \mathcal{E}_{a,b} = \sum_{i,j} \delta_{p_i + b\Delta_j^{(i)}}$

- $\tilde{m}(t) \equiv \sqrt{2}t - \frac{1}{2\sqrt{2}} \ln t$.
- p_i: atoms of a $\mathrm{PPP}(C(b)Y_a e^{-\sqrt{2}x}dx)$,
- $Y_a = \lim \sum_{i=1}^{n(s)} e^{-s(1+a^2)+\sqrt{2}x_i(s)}$
- Δ: BBM conditioned on $\{\max_k x_k(t) \geq \sqrt{2}bt\}$.

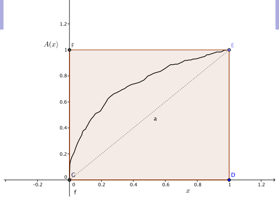


hausdorff center for mathematics

Above the straight line

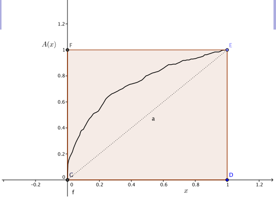


Above the straight line



When the concave hull of A is above the straight line, everything changes.

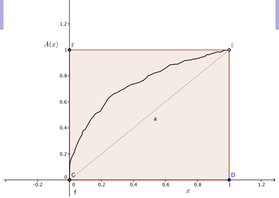
Above the straight line



When the concave hull of A is above the straight line, everything changes.

- If A is piecewise linear, it is quite easy to get the full picture: Cascade of BBM processes.

Above the straight line



When the concave hull of A is above the straight line, everything changes.

- If A is piecewise linear, it is quite easy to get the full picture: Cascade of BBM processes.

- If A is strictly concave, Fang and Zeitouni '12 and Maillard and Zeitouni '13 have shown that the correct rescaling is

$$m(t) = C_\sigma t - D_\sigma t^{1/3} - \sigma^2(1) \ln t + f_t$$

(with explicit constants C_σ and D_σ), and $|f_t|$ bounded and

$$\mathbb{P}[M_T - m(t) \leq x) \to \omega(x/\sigma(0)).$$

Universality

Universality

BBM prototype for extremal processes in many other models:

Universality

BBM prototype for extremal processes in many other models:

- Branching random walk [Bramson '78, Addario-Berry, Aídékon '13 (law of max), Madaule '13 (full extremal process),...]
- Gaussian free field in $d = 2$ [Bolthausen, Deuschel, Giacomin '01, Bramson-Ding-Zeitouni '13, Biskup-Louidor '13, '16 (full extremal process)]

Universality

BBM prototype for extremal processes in many other models:

- Branching random walk [Bramson '78, Addario-Berry, Aídékon '13 (law of max), Madaule '13 (full extremal process),...]
- Gaussian free field in $d = 2$ [Bolthausen, Deuschel, Giacomin '01, Bramson-Ding-Zeitouni '13, Biskup-Louidor '13, '16 (full extremal process)]
- Cover times of random walks [Lawler '9,3 Dembo-Peres-Rosen-Zeitouni '06, Belius-Kistler '14 ....]

Universality

BBM prototype for extremal processes in many other models:

- Branching random walk [Bramson '78, Addario-Berry, Aídékon '13 (law of max), Madaule '13 (full extremal process),...]
- Gaussian free field in $d = 2$ [Bolthausen, Deuschel, Giacomin '01, Bramson-Ding-Zeitouni '13, Biskup-Louidor '13, '16 (full extremal process]
- Cover times of random walks [Lawler '9,3 Dembo-Peres-Rosen-Zeitouni '06, Belius-Kistler '14 ....]
- Spin glasses with log-correlated potentials [Fyodorov, Bouchaud '08,Arguin, Zindy '12..]

Universality

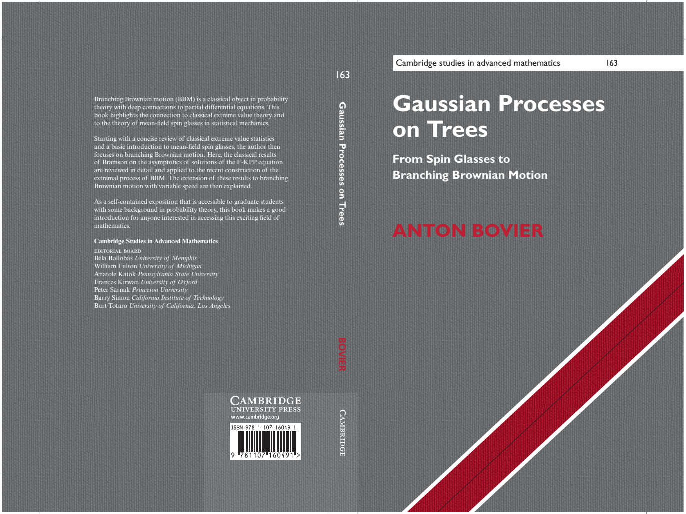
BBM prototype for extremal processes in many other models:

- Branching random walk [Bramson '78, Addario-Berry, Aídékon '13 (law of max), Madaule '13 (full extremal process),...]
- Gaussian free field in $d = 2$ [Bolthausen, Deuschel, Giacomin '01, Bramson-Ding-Zeitouni '13, Biskup-Louidor '13, '16 (full extremal process]
- Cover times of random walks [Lawler '9,3 Dembo-Peres-Rosen-Zeitouni '06, Belius-Kistler '14 ....]
- Spin glasses with log-correlated potentials [Fyodorov, Bouchaud '08,Arguin, Zindy '12..]
- Statistics of zeros of Riemann zeta-function [Fyodorov, Keating '12, Arguin, Belius, Fyodorov '15, Arguin, Belius , Harper, '16]

Thank you for your attention!

hausdorff center for mathematics

Advertisement:



hausdorff center for mathematics

references

- L.-P. Arguin, A. Bovier, and N. Kistler, *The genealogy of extremal particles of branching Brownian motion*, CPAM **64**, 1647–1676 (2011).
- L.-P. Arguin, A. Bovier, and N. Kistler, *Poissonian statistics in the extremal process of branching Brownian motion*, AAP. **22**, 1693–1711 (2012).
- L.-P. Arguin, A. Bovier, and N. Kistler, *The extremal process of branching Brownian motion*, PTRF **157**, 535-574 (2013).
- E. Aïdékon, J. Berestyzki, É. Brunet, and Z. Shi, *Branching Brownian motion seen from its tip*, PTRF **157**, 405-451 (2013).
- A. Bovier and L. Hartung, *The extremal process of two-speed branching Brownian motion*, EJP 19, No 18 (2014)
- A. Bovier and L. Hartung, *Variable speed branching Brownian motion 1. Extremal processes in the weak correlation regime*, ALEA 12, 261–291 (2015).
- A. Bovier and L. Hartung, *Extended convergence of the extremal process of BBM*, AAP 27, 1756-1777 (2017).