Convergence analysis of MCMC algorithms for Bayesian robust multivariate regression

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Outline: I. A robust Bayesian multivariate regression model

II. A data augmentation algorithm

III. The main result

IV. Three slides on drift & minorization calculations

I. A robust Bayesian multivariate regression model

$$Y = X\beta + E$$

- Y is an $n \times d$ matrix of observables
- X is an $n \times p$ matrix of known covariates
- β is an unknown $p \times d$ matrix of regression parameters
- $E = (\varepsilon_1 \cdots \varepsilon_n)^T$ where $\{\varepsilon_i\}_{i=1}^n$ are iid error vectors of dim d

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Alternative model: Use an error density of the form

$$f(\varepsilon) = \int_0^\infty \frac{z^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{z}{2}\, \varepsilon^T \Sigma^{-1} \varepsilon\right\} h(z)\, dz$$

$$\varepsilon | Z \sim \mathrm{N}_d\Big(0, \frac{\Sigma}{Z}\Big) \quad \text{and} \quad Z \sim h(\cdot)$$

$$Y = X\beta + E \quad \text{and} \quad f(\varepsilon) = \int_0^\infty \frac{z^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{z}{2}\, \varepsilon^T \Sigma^{-1} \varepsilon}\, h(z)\, dz$$

Joint density of the observable data (i.e. likelihood) is

$$f(y|\beta,\Sigma) = \prod_{i=1}^{n} \left[\int_0^\infty \frac{z^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{z}{2}\,(y_i - \beta^T x_i)^T \Sigma^{-1} (y_i - \beta^T x_i)}\, h(z)\, dz \right]$$

Default prior for (β, Σ): $\pi(\beta, \Sigma) \propto |\Sigma|^{-\frac{d+1}{2}} I_{\mathcal{S}_d}(\Sigma)$

The (intractable) posterior $\pi : \mathbb{R}^{p \times d} \times \mathcal{S}_d \to (0, \infty)$ is given by:

$$\pi(\beta, \Sigma | y) \propto f(y|\beta,\Sigma)\, \pi(\beta, \Sigma)$$

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The Bayesian wants posterior expectations of the form:

$$E\big[g(\beta,\Sigma)\big|y\big] = \frac{\int_{\mathbb{R}^{p\times d}} \int_{\mathcal{S}_d} g(\beta,\Sigma)\, f(y|\beta,\Sigma)\,\pi(\beta,\Sigma)\, d\Sigma\, d\beta}{\int_{\mathbb{R}^{p\times d}} \int_{\mathcal{S}_d} f(y|\beta,\Sigma)\,\pi(\beta,\Sigma)\, d\Sigma\, d\beta}$$

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MCMC solution: Simulate a Markov chain $\{(\beta_i, \Sigma_i)\}_{i=0}^{\infty}$ with invariant density π and estimate $E\big[g(\beta, \Sigma) \big| y\big]$ with:

$$\bar{g}_m = \frac{1}{m} \sum_{i=0}^{m-1} g(\beta_i, \Sigma_i)$$

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Q: How do we choose m?

Answer: As in classical Monte Carlo, we use the CLT

$$\sqrt{m}\left(\bar{g}_m - E\left[g(\beta,\Sigma)\middle|y\right]\right) \xrightarrow{d} \mathrm{N}(0,\gamma^2)$$

II. A Data augmentation algorithm (C. Liu, 1996, *JASA*)

Latent data model: Let $\{(Y_i, U_i)\}_{i=1}^n$ be independent pairs st

$$Y_i | U_i \sim \mathrm{N}_d\left(\beta^T x_i, \frac{\Sigma}{U_i}\right) \quad \text{and} \quad U_i \sim h(\cdot)$$

Since $f(y|\beta, \Sigma) = \int_{\mathbb{R}_+^n} f(y, u|\beta, \Sigma)\, du$, we have:

$$\pi(\beta, \Sigma | y) = \int_{\mathbb{R}_+^n} \frac{\pi(\beta, \Sigma)\, f(y, u|\beta, \Sigma)}{m(y)}\, du = \int_{\mathbb{R}_+^n} \pi(\beta, \Sigma, u | y)\, du$$

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Mtd for DA: $k(\beta', \Sigma'|\beta, \Sigma) = \int_{\mathbb{R}_+^n} \pi(\beta', \Sigma'|u, y)\, \pi(u|\beta, \Sigma, y)\, du$

Simulating this Markov chain is easy!

Mtd for DA: $k(\beta', \Sigma' | \beta, \Sigma) = \int_{\mathbb{R}_+^n} \pi(\beta', \Sigma' | u, y) \, \pi(u | \beta, \Sigma, y) \, du$

$$\pi(u | \beta, \Sigma, y) = \prod_{i=1}^{n} b(r_i) \, u_i^{\frac{d}{2}} \, e^{-\frac{r_i u_i}{2}} \, h(u_i)$$

$$r_i = r_i(\beta, \Sigma) = (y_i - \beta^T x_i)^T \Sigma^{-1} (y_i - \beta^T x_i)$$

$$\pi(\beta', \Sigma' | u, y) = \pi(\Sigma' | u, y) \, \pi(\beta', \Sigma' | u, y) = \text{IW} \times \text{matrix normal}$$

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Q: Do there exist $C : \mathbb{R}^{p \times d} \times \mathcal{S}_d \to [0, \infty)$ and $\rho \in [0, 1)$ st

$$\int_{\mathcal{S}_d} \int_{\mathbb{R}^{p \times d}} \left| k^m(\beta', \Sigma' | \beta, \Sigma) - \pi(\beta', \Sigma' | y) \right| d\beta\, d\Sigma \leq C(\beta, \Sigma)\, \rho^m$$

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Drift and minorization conditions yield formulas for C and ρ

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Drift and minorization conditions yield formulas for C and ρ

Drift: $\int_{\mathbb{R}^{p \times d}} \int_{\mathcal{S}_d} V(\beta', \Sigma')\, k(\beta', \Sigma' | \beta, \Sigma)\, d\Sigma'\, d\beta' \le \lambda V(\beta, \Sigma) + L$

Minorization: $k(\beta', \Sigma' | \beta, \Sigma) \ge \epsilon\, f^*(\beta', \Sigma')$

III. The main result

$$Y_i | U_i \sim \mathrm{N}_d\left(\beta^T x_i, \frac{\Sigma}{U_i}\right) \quad \text{and} \quad U_i \sim h(\cdot)$$

Mtd for DA: $k(\beta', \Sigma' | \beta, \Sigma) = \int_{\mathbb{R}_+^n} \pi(\beta', \Sigma' | u, y)\, \pi(u | \beta, \Sigma, y)\, du$

Def: h is PNO(c) if $\lim_{u \to 0} \frac{h(u)}{u^c} \in (0, \infty)$

Examples: Gamma, F, Weibull

Def: h is FPNO if for each $c > 0$, $\exists\, \eta_c > 0$ st $\frac{h(u)}{u^c} \uparrow$ for $u \in (0, \eta_c)$

Examples: IG, Log-normal, GIG, Fréchet

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Examples: IG, Log-normal, GIG, Fréchet

Proposition: If h is FPNO or PNO(c) with $c > (n-p)/2$, then the DA Markov chain is geometrically ergodic

IV. Three slides on drift & minorization calculations

Mtd for DA: $k(\beta', \Sigma' | \beta, \Sigma) = \int_{\mathbb{R}_+^n} \pi(\beta', \Sigma' | u, y)\, \pi(u | \beta, \Sigma, y)\, du$

NTS: $\int_{\mathbb{R}^{p \times d}} \int_{\mathcal{S}_d} V(\beta', \Sigma')\, k(\beta', \Sigma' | \beta, \Sigma)\, d\Sigma'\, d\beta' \leq \lambda V(\beta, \Sigma) + L$

Drift fcn: $V(\beta, \Sigma) = \sum_{i=1}^n (y_i - \beta^T x_i)^T \Sigma^{-1} (y_i - \beta^T x_i) = \sum_{i=1}^n r_i$

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Roy & H (2010, *JMVA*) showed that

$$\int_{\mathbb{R}^{p \times d}} \int_{\mathcal{S}_d} V(\beta', \Sigma') \, \pi(\beta', \Sigma' | u, y) \, d\Sigma' \, d\beta' \leq (n - p + d) \sum_{i=1}^n \frac{1}{u_i}$$

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and that when h is gamma

$$(n - p + d) \int_{\mathbb{R}_+^n} \left[\sum_{i=1}^n \frac{1}{u_i} \right] \pi(u | \beta, \Sigma, y)\, du \leq \lambda V(\beta, \Sigma) + L$$

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$$\int_{\mathbb{R}^{p\times d}} \int_{\mathcal{S}_d} V(\beta', \Sigma')\, \pi(\beta', \Sigma' | u, y)\, d\Sigma'\, d\beta' \leq (n - p + d) \sum_{i=1}^{n} \frac{1}{u_i}$$

Recall: $\pi(u | \beta, \Sigma, y) = \prod_{i=1}^{n} b(r_i)\, u_i^{\frac{d}{2}}\, e^{-\frac{r_i u_i}{2}}\, h(u_i)$

Thus, in the general case, we have

$$\int_{\mathbb{R}_+^n} \left[\sum_{i=1}^{n} \frac{1}{u_i} \right] \pi(u | \beta, \Sigma, y)\, du = \sum_{i=1}^{n} \frac{\int_{\mathbb{R}_+} z^{\frac{d-2}{2}}\, e^{-\frac{r_i z}{2}}\, h(z)\, dz}{\int_{\mathbb{R}_+} z^{\frac{d}{2}}\, e^{-\frac{r_i z}{2}}\, h(z)\, dz}$$

Mtd for DA: $k(\beta', \Sigma' | \beta, \Sigma) = \int_{\mathbb{R}_+^n} \pi(\beta', \Sigma' | u, y)\, \pi(u | \beta, \Sigma, y)\, du$

NTS: $\int_{\mathbb{R}^{p \times d}} \int_{\mathcal{S}_d} V(\beta', \Sigma')\, k(\beta', \Sigma' | \beta, \Sigma)\, d\Sigma'\, d\beta' \leq \lambda V(\beta, \Sigma) + L$

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It suffices to show that, for all $s \geq 0$,

$$\frac{\int_{\mathbb{R}_+} z^{\frac{d-2}{2}}\, e^{-\frac{sz}{2}}\, h(z)\, dz}{\int_{\mathbb{R}_+} z^{\frac{d}{2}}\, e^{-\frac{sz}{2}}\, h(z)\, dz} \leq \lambda s + L$$

Mtd for DA: $k(\beta', \Sigma' | \beta, \Sigma) = \int_{\mathbb{R}_+^n} \pi(\beta', \Sigma' | u, y) \, \pi(u | \beta, \Sigma, y) \, du$

Suppose we find $\tilde{f} : \mathbb{R}_+^n \to [0, \infty)$ and $\epsilon \in (0, 1)$ st

$$\pi(u | \beta, \Sigma, y) \geq \epsilon \, \tilde{f}(u) \text{ whenever } V(\beta, \Sigma) \leq t$$

Then we have minorization: For all (β, Σ) st $V(\beta, \Sigma) \leq t$,

$$k(\beta', \Sigma' | \beta, \Sigma) \geq \epsilon \int_{\mathbb{R}_+^n} \pi(\beta', \Sigma' | u, y) \, \tilde{f}(u) \, du = \epsilon \, f^*(\beta', \Sigma')$$

Recall: $\pi(u | \beta, \Sigma, y) = \prod_{i=1}^n b(r_i) \, u_i^{\frac{d}{2}} \, e^{-\frac{r_i u_i}{2}} \, h(u_i)$

But $b(r_i) \geq \left[\int_0^\infty z^{\frac{d}{2}} \, h(z) \, dz \right]^{-1}$ and $e^{-\frac{r_i u_i}{2}} \geq e^{-\frac{t u_i}{2}}$, so

$$\pi(u | \beta, \Sigma, y) \geq \left[\frac{\int_0^\infty z^{\frac{d}{2}} \, e^{-\frac{tz}{2}} \, h(z) \, dz}{\int_0^\infty z^{\frac{d}{2}} \, h(z) \, dz} \right]^n \prod_{i=1}^n \frac{u_i^{\frac{d}{2}} \, e^{-\frac{t u_i}{2}} \, h(u_i)}{\int_0^\infty z^{\frac{d}{2}} \, e^{-\frac{tz}{2}} \, h(z) \, dz}$$

$$:= \epsilon \, \tilde{f}(u)$$