Stability of overshoots of zero mean random walks

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Introduction

Random walk

Let $S_n = S_0 + X_1 + \ldots + X_n$ be a zero mean non-degenerate random walk in $\mathbb{R}$ with i.i.d. increments $X_1, X_2, \ldots$ and the starting point S_0 that is a r.v. independent of the increments.

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The Markov chain of overshoots

Define the *crossing times* T_n of the zero level: $T_0 := 0$ and

$$T_{n+1} := \min\{k > T_n : S_{k-1} < 0, S_k \geq 0 \text{ or } S_{k-1} \geq 0, S_k < 0\}.$$

Now, define the corresponding overshoots:

$$O_n := S_{T_n}, \qquad n \geq 0.$$

The sequence $(O_n)_{n\geq 0}$ is a Markov chain starting at $O_0 = S_0$.

The problem

- Does O have a stationary distribution? Is it unique?
- Do O_n stabilise to this distribution in the sense that the laws $\mathbb{P}(O_n \in \cdot | S_0 = x)$ converge to this distribution $\forall x$?
- What is the rate of this convergence?

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Overshoots at up-crossings

Since O has a periodic structure, it suffices to consider the Markov chain $O_n^{\uparrow} := O_{2n}$ of the overshoots at up-crossing times $T_n^{\uparrow} = T_{2n}$, starting at $S_0 \geq 0$.

Stationary distribution

Arithmetic vs non-arithmetic

The random walk S_n is called non-arithmetic if $\mathbb{P}(X_1 \in d\mathbb{Z}) < 1$ for any d. All other walks are called arithmetic. An arithmetic RW is d-arithmetic iff $d = \max\{d' \geq 0 : \mathbb{P}(X_1 \in d'\mathbb{Z}) = 1\}$.

State space

Define the state space $\mathcal{X}_+$ of the walk as $[0, \infty)$ in the non-arithmetic case and as $\{0, d, 2d, \ldots\}$ in the d-arithmetic case.

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Theorem 1

Let λ_+ be either Lebesgue or $d \cdot \#$ (counting) measure on $\mathcal{X}_+$, respectively. Then

$$\pi_+(dx) := \frac{2}{\mathbb{E}|X_1|}\mathbb{P}(X_1 > x)\lambda_+(dx), \qquad x \in \mathcal{X}_+$$

is a stationary distribution for the chain $O_n^{\uparrow}$.

Heuristics

Assume $\mathbb{E}X_1^2 = 1$ and that S_n is aperiodic integer-valued.
Let $L_n^{\uparrow} := \max\{i \geq 0 : T_i^{\uparrow} \leq n\}$ be the *number of up-crossings* of the zero level. Then for any $k \in \{0, 1, 2, \ldots\}$,

$$\sum_{i=0}^{n-1} \mathbb{1}(S_i < 0, S_{i+1} = k) = \sum_{i=1}^{L_n^{\uparrow}} \mathbb{1}(O_i^{\uparrow} = k).$$

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By L-CLT: $\mathbb{P}_x(S_i = -\ell) = \exp(-(\ell + x)^2/2i)/\sqrt{2\pi i} + o(1/\sqrt{i})$

$$\mathbb{E}_x\left[\frac{L_n^{\uparrow}}{\sqrt{n}} \cdot \frac{1}{L_n^{\uparrow}} \sum_{i=1}^{L_n^{\uparrow}} \mathbb{1}(O_i^{\uparrow} = k)\right] = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \mathbb{P}_x(S_i < 0, S_{i+1} = k)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \sum_{\ell=1}^{\infty} \mathbb{P}_x(S_i = -\ell)\mathbb{P}(X_1 = k + \ell)$$

$$\sim \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \frac{1}{\sqrt{2\pi i}} \sum_{\ell=1}^{o(\sqrt{n})} \mathbb{P}(X_1 = k + \ell)$$

$$\mathbb{E}_x\Big[\frac{L_n^{\uparrow}}{\sqrt{n}}\cdot\frac{1}{L_n^{\uparrow}}\sum_{i=1}^{L_n^{\uparrow}}\mathbb{1}(O_i^{\uparrow}=k)\Big]=\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\mathbb{P}_x(S_i<0,S_{i+1}=k)$$

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If we believe in the ergodicity of $O_n^{\uparrow}$, then

$$\pi_+(k)\mathbb{E}_x\Big[\frac{L_n^{\uparrow}}{\sqrt{n}}\Big]\sim c\mathbb{P}(X_1>k).$$

A similar argument gives

$$\mathbb{E}_x\Big[\frac{L_n^{\uparrow}}{\sqrt{n}}\Big]\sim c\sum_{k=0}^{\infty}\mathbb{P}(X_1>k)=c\mathbb{E}|X_1|/2.$$

Proof of Theorem 1 (idea)

For simplicity, consider the non-arithmetic case.
We represent $\mathbb{P}_\mu(O_1^\uparrow \in \cdot) = \mu P Q$, where Q and P are transition probabilities of two new Markov chains defined by

$$\begin{aligned} P(x,dy) &:= \mathbb{P}_x(S_{T_1^\uparrow - 1} \in -dy), \quad x,y \in \mathcal{X}_+ \\ Q(x,dy) &:= \mathbb{P}(X_1 \in dy + x | X_1 > x), \quad x,y \in \mathcal{X}_+. \end{aligned}$$

P corresponds to the undershoot at the up-crossing and Q governs the increment performing the level-crossing.

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Proposition

Assuming $\mathbb{E}X_1 = 0$, the kernels P and Q are reversible with respect to π_+.

Corollary

π_+ is a stationary distribution for P and Q and, consequently, for $O^\uparrow$.

Uniqueness

Theorem 2

Assuming $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 < \infty$, π_+ is a unique stationary distribution of $O_n^{\uparrow}$.

Corollary

The chain $O_n^{\uparrow}$ is ergodic.

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Proof of Theorem 2 (idea)

Combine ε-coupling with the Stone local limit theorem to show that for any bounded Lipschitz $f : \mathcal{X}_+ \to \mathbb{R}$,

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^{n} f(O_i^\uparrow(x)) - \frac{1}{n} \sum_{i=1}^{n} f(O_i^\uparrow(y)) \right| \overset{\mathbb{P}}{=} 0, \quad x, y \in \mathcal{X}_+.$$

Convergence

Smoothness assumption

The distribution of X_1 is called spread out if the distribution of S_k is non-singular for some $k \geq 1$.

Theorem 3

Assume $\mathbb{E}X_1 = 0$ and that the distribution of X_1 is either arithmetic or spread out. Then

$$\lim_{n\to\infty} \|\mathbb{P}_x(O_n^{\uparrow} \in \cdot) - \pi_+(\cdot)\|_{\mathrm{TV}} = 0, \quad x \in \mathcal{X}_+.$$

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Proof

This follows from a general statement for ψ-irreducible aperiodic chains with a stationary distribution (however, it only gives the convergence for π_+-a.e. x). Such setting, where a stationary distribution is known to exist, is typical for MCMC.

Convergence

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Assume $\mathbb{E}X_1 = 0$ and that the distribution of X_1 is either arithmetic or spread out. Then for all $x \in \mathcal{X}_+$ we have

$$\lim_{n\to\infty} \|\mathbb{P}_x(O_n^{\uparrow} \in \cdot) - \pi_+(\cdot)\|_{\mathsf{TV}} = 0. \quad (1)$$

Remark

Eq (1) fails $\forall x \in \mathcal{X}_+$ if X_1 is neither spread out nor arithmetic but with countable support, e.g. $\mathrm{supp}(X_1) = \{-1, \sqrt{2}\}$.

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The Dominated Convergence Theorem implies:

Corollary

$\lim_{n\to\infty} \|\mathbb{P}_\mu(O_n^{\uparrow} \in \cdot) - \pi_+(\cdot)\|_{\mathsf{TV}} = 0$ for any prob. measure μ on $\mathcal{X}_+$. Hence π_+ is the unique stationary measure for $O^{\uparrow}$.

Rate of convergence

Theorem 4

Assume $\mathbb{E}X_1 = 0$ and that the distribution of X_1 is either arithmetic or spread out. In addition, assume that either $\mathbb{E}X_1^2 < \infty$ or $X_1 \in \mathcal{D}(\alpha, \beta)$ for some $\alpha \in (1, 2), |\beta| < 1$.
Then for any $\gamma \in \{0, 1\}$ in the first case and any $\gamma > 0$ small enough in the second case, there exist constants $r \in (0, 1)$ and $c_1 > 0$ such that

$$\|\mathbb{P}_x(O_n^{\uparrow} \in \cdot) - \pi_+(\cdot)\|_{V_\gamma} \leq c_1(1 + x^\gamma)r^n, \qquad x \in \mathcal{X}_+.$$

Idea of proof

Use the so-called Meyn and Tweedie approach. We already have ψ-irreducibility. The Lyapunov function is $V_\gamma(x) := x^\gamma + 1$.

Motivation

Local times of random walks

Let $L_n := \max\{k \geq 0 : T_k \leq n\}$ be the *number of zero-level crossings*, and let ℓ_0 be the local time at 0 at time 1 of a standard Brownian motion.

Perkins('82): $\mathbb{E}X_1^2 < \infty$, then for any x,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{L_n} |O_k| \xrightarrow{\mathcal{D}} \sqrt{\mathrm{Var}(X_1)}\ell_0 \quad \text{under } \mathbb{P}_x.$$

The ergodicity of O_n now yields the limit theorem for L_n, generalising Borodin ('80s): if S is either integer-valued or has density then $L_n/\sqrt{n} \xrightarrow{\mathcal{D}} \frac{\mathbb{E}|X_1|}{\sqrt{\mathrm{Var}(X_1)}}\ell_0$.

What ought to be true?

Recall $S_n = S_0 + X_1 + \ldots + X_n$ and $O_n^{\uparrow} = S_{T_n^{\uparrow}}$, where $T_0^{\uparrow} = 0$,

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Conjecture

If $\mathbb{E}|X_1| \in (0, \infty)$ and $\mathbb{E}X_1 = 0$, the following weak limit $\mathbb{P}_x(O_n^{\uparrow} \in \cdot) \xrightarrow{\mathcal{D}} \pi_+$, as $n \to \infty$, holds for any $x \in \mathcal{X}_+$.

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Recall $S_n = S_0 + X_1 + \ldots + X_n$ and $O_n^\uparrow = S_{T_n^\uparrow}$, where $T_0^\uparrow = 0$,

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Evidence

- Conjecture holds if X_1 is arithmetic or spread out, since $\|\mathbb{P}_x(O_n^\uparrow \in \cdot) - \pi_+(\cdot)\|_{\mathrm{TV}} \to 0$ as $n \to \infty$ for any $x \in \mathcal{X}_+$.
- Conjecture implies uniqueness of the stationary law π_+, which holds if $\mathbb{E}X_1^2 < \infty$ (or if X_1 is either spread out or arithmetic).