Extended algebras and geometries

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CHALMERS

Based on 1804.04377, 1711.07694 (with Martin Cederwall), 1802.05767 (with MC and Lisa Carbone) and 1507.08828

There is a way of extending geometry for any choice $(\mathfrak{g}, \lambda)$ of

- a Kac-Moody algebra $\mathfrak{g}$ of rank r (this talk: simply laced)
- and an integral dominant highest weight λ of $\mathfrak{g}$, with a corresponding highest weight representation $R(\lambda)$,

that gives ordinary, double and exceptional geometry in the cases $\mathfrak{g} = A_r, D_r, E_r$, respectively, and $\lambda = \Lambda_1$.

Fields depend on coordinates x^M, transforming in the representation $R_1 = R(\lambda)$, subject to the section condition

$$\partial_{\langle M} \otimes \partial_{N \rangle} = 0,$$

with the derivatives projected on the dual of $R_2 \oplus \widetilde{R}_2$, where

$$\begin{aligned} R_2 &= R_1 \vee R_1 \ominus R(2\lambda), \\ \widetilde{R}_2 &= R_1 \wedge R_1 \ominus \bigoplus_{\lambda_i = 1} R(2\lambda - \alpha_i). \end{aligned}$$

Under generalised diffeomorphisms, vector fields transform with the generalised Lie derivative:

$$\begin{aligned}\mathscr{L}_U V^M &= U^N \partial_N V^M - V^N \partial_N U^M + Y^{MN}{}_{PQ} \partial_N U^P V^Q \\ &= U^N \partial_N V^M + Z^{MN}{}_{PQ} \partial_N U^P V^Q\end{aligned}$$

where $Y^{MN}{}_{PQ} = Z^{MN}{}_{PQ} + \delta^M{}_P \delta^N{}_Q$ is a $\mathfrak{g}$ invariant tensor with the upper pair of R_1 indices in $R_2 \oplus \widetilde{R}_2$:

$$Z^{MN}{}_{PQ} = -(T^\alpha)^M{}_Q (T_\alpha)^N{}_P + \big((\lambda, \lambda) - 1\big) \delta^M{}_Q \delta^N{}_P$$

Closure of the generalised diffeomorphisms (up to ancillary $\mathfrak{g}$ transformations) relies on the section condition

$$Y^{MN}{}_{PQ}(\partial_M \otimes \partial_N) = 0$$

and the fundamental identity

$$\begin{aligned} &Z^{NT}{}_{SM} Z^{QS}{}_{RP} - Z^{QT}{}_{SP} Z^{NS}{}_{RM} \\ &- Z^{NS}{}_{PM} Z^{QT}{}_{RS} + Z^{ST}{}_{RP} Z^{NQ}{}_{SM} = 0. \end{aligned}$$

Add two nodes -1 and 0 to the Dynkin diagram of $\mathfrak{g}$, corresponding to simple roots α_0 and α_{-1}, and extend the Cartan matrix $A_{ij} = (\alpha_i, \alpha_j)$ so that

$$
\begin{aligned}
(\alpha_{-1}, \alpha_{-1}) &= 0, & & \\
(\alpha_0, \alpha_{-1}) &= -1, & (\alpha_0, \alpha_0) &= 2, \\
(\alpha_i, \alpha_{-1}) &= 0, & (\alpha_i, \alpha_0) &= -\lambda_i.
\end{aligned}
$$

Associate three generators e_I, f_I, h_I to the each node I ($I = -1, 0, 1, \ldots, r$), where e_{-1}, f_{-1} are odd, the others even.

Let $\mathscr{B}$ be the Lie superalgebra generated by all e_I, f_I, h_I modulo the Chevalley-Serre relations

$$[h_I, e_J] = A_{IJ} e_J \,, \quad [h_I, f_J] = -A_{IJ} f_J \,, \quad [e_I, f_J] = \delta_{IJ} h_J \,,$$

$$(\text{ad } e_I)^{1-A_{IJ}}(e_J) = (\text{ad } f_I)^{1-A_{IJ}}(f_J) = 0 \,.$$

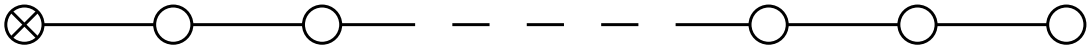
This is a Borcherds(-Kac-Moody) superalgebra.

The Borcherds superalgebra $\mathscr{B}$ decomposes into a $(\mathbb{Z} \times \mathbb{Z})$-grading of $\mathfrak{g}$-modules spanned by root vectors e_α, where $\alpha = n\,\alpha_{-1} + p\,\alpha_0 + \sum_{i=1}^{r} a_i\,\alpha_i$, and h_I for $n = p = 0$.

$\cdots$	$p=-1$	$p=0$	$p=1$	$p=2$	$p=3$	$\cdots$
$\cdots$						$n=0$
$q=3$					$\tilde{\tilde{R}}_3$	$n=1$
$q=2$				$\tilde{R}_2$	$\tilde{R}_3 \oplus \tilde{\tilde{R}}_3$	$n=2$
$q=1$		$\mathbf{1}$	R_1	$R_2 \oplus \tilde{R}_2$	$R_3 \oplus \tilde{R}_3$	$n=3$
$q=0$	$\overline{R}_1$	$\mathbf{1} \oplus \mathbf{adj} \oplus \mathbf{1}$	R_1	R_2	R_3	$\cdots$
$\cdots$	$\overline{R}_1$	$\mathbf{1}$				

Ordinary geometry, $\mathfrak{g} = \mathfrak{sl}(r+1)$, $\mathscr{B} = \mathfrak{sl}(r+2\,|\,1)$:

	$p=-1$	$p=0$	$p=1$
$q=1$		$\mathbf{1}$	$\mathbf{v}$
$q=0$	$\overline{\mathbf{v}}$	$\mathbf{1} \oplus \mathbf{adj} \oplus \mathbf{1}$	$\mathbf{v}$
$q=-1$	$\overline{\mathbf{v}}$	$\mathbf{1}$	

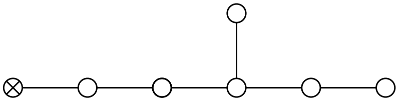


Double geometry, $\mathfrak{g} = \mathfrak{so}(r,r)$, $\mathscr{B} = \mathfrak{osp}(r+1, r+1\,|\,2)$:

	$p=-2$	$p=-1$	$p=0$	$p=1$	$p=2$
$q=1$			$\mathbf{1}$	$\mathbf{v}$	$\mathbf{1}$
$q=0$	$\mathbf{1}$	$\mathbf{v}$	$\mathbf{1}\oplus\mathbf{adj}\oplus\mathbf{1}$	$\mathbf{v}$	$\mathbf{1}$
$q=-1$	$\mathbf{1}$	$\mathbf{v}$	$\mathbf{1}$		

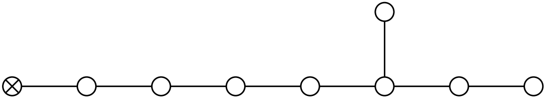
Exceptional geometry, $\mathfrak{g} = \mathfrak{so}(5,5)$:

	$p=-1$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$q=2$						$\mathbf{1}$	$\mathbf{16}$
$q=1$		$\mathbf{1}$	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45} \oplus \mathbf{1}$	$\overline{\mathbf{144}} \oplus \mathbf{16}$
$q=0$	$\overline{\mathbf{16}}$	$\mathbf{1} \oplus \mathbf{45} \oplus \mathbf{1}$	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$	$\overline{\mathbf{144}}$
$q=-1$	$\overline{\mathbf{16}}$	$\mathbf{1}$					



Exceptional geometry, $\mathfrak{g} = E_7$:

	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$
$q = 3$					$\mathbf{1}$
$q = 2$			$\mathbf{1}$	$\mathbf{56}$	$\mathbf{1539} \oplus \mathbf{133} \oplus \mathbf{1} \oplus \mathbf{1}$
$q = 1$	$\mathbf{1}$	$\mathbf{56}$	$\mathbf{133} \oplus \mathbf{1}$	$\mathbf{912} \oplus \mathbf{56}$	$\mathbf{8645} \oplus \mathbf{133} \oplus \mathbf{1539} \oplus \mathbf{133} \oplus \mathbf{1}$
$q = 0$	$\mathbf{1} \oplus \mathbf{133} \oplus \mathbf{1}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{8645} \oplus \mathbf{133}$
$q = -1$	$\mathbf{1}$				



Back to the general case:

$\cdots$	$p=-1$	$p=0$	$p=1$	$p=2$	$p=3$	$\cdots$
$q=4$						$\cdots$
$q=3$					$\tilde{\tilde{R}}_3$	$\cdots$
$q=2$				$\tilde{R}_2$	$\tilde{R}_3 \oplus \tilde{\tilde{R}}_3$	$\cdots$
$q=1$		$\mathbf{1}$	R_1	$R_2 \oplus \tilde{R}_2$	$R_3 \oplus \tilde{R}_3$	$\cdots$
$q=0$	$\overline{R}_1$	$\mathbf{1} \oplus \mathbf{adj} \oplus \mathbf{1}$	R_1	R_2	R_3	$\cdots$
$q=-1$	$\overline{R}_1$	$\mathbf{1}$				

Basis elements:

$\cdots$	$p=-1$	$p=0$	$p=1$	$p=2$	$p=3$	$\cdots$
$q=4$						$\cdots$
$q=3$					$\cdots$	$\cdots$
$q=2$				$[\widetilde{E}_M, \widetilde{E}_N]$	$\cdots$	$\cdots$
$q=1$		f_{-1}	$\widetilde{E}_M$	$[E_M, \widetilde{E}_N]$	$\cdots$	$\cdots$
$q=0$	F^M	$\widetilde{k}\,, T^\alpha,\, k$	E_M	$[E_M, E_N]$	$\cdots$	$\cdots$
$q=-1$	$\widetilde{F}^M$	e_{-1}				

We identify the internal tangent space with the odd subspace spanned by the E_M and write a vector field V as $V = V^M E_M$. It can be mapped to the even element $V^\sharp = [f_{-1}, V] = V^M \widetilde{E}_M$.

The generalised Lie derivative is now given by

$$\mathscr{L}_U V = [[U, \widetilde{F}^N], \partial_N V^\sharp] - [[\partial_N U^\sharp, \widetilde{F}^N], V] \ .$$

The section condition can be written

$$[F^M, F^N]\partial_M \otimes \partial_N = [\widetilde{F}^M, \widetilde{F}^N]\partial_M \otimes \partial_N = 0 \ .$$

It follows from relations in the Lie superalgebra $\mathscr{B}$ whether the transformations close or not.

[Palmkvist: 1507.08828]

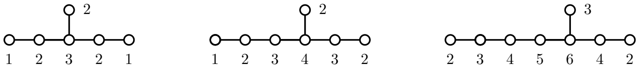
If $\mathfrak{g}$ is finite-dimensional and λ is a fundamental weight Λ_i such that the corresponding Coxeter number c_i is equal to 1, then

$$\mathscr{L}_U \mathscr{L}_V - \mathscr{L}_V \mathscr{L}_U = \mathscr{L}_{[\![U,V]\!]},$$

where

$$[\![U,V]\!] = \frac{1}{2}(\mathscr{L}_U V - \mathscr{L}_V U).$$

This is the 2-bracket of an L_∞ algebra.



[Cederwall, Palmkvist: 1711.07694, 1804.04377]

In addition to the vector fields in R_1 at $(p,q)=(1,0)$, the L_∞ algebra also contains ghosts C_p in R_p at higher levels p and $q=0$, as well as ancillary ghosts K_p in R_p at $p \geqslant p_0$ and $q=1$, where p_0 is the lowest level p such that $\widetilde{R}_{p+1}$ is nonzero.

The 1-bracket is given by $[\![C]\!]=dC$ and $[\![K]\!]=dK+K^\flat$, where $d \sim (\mathrm{ad}\, F^M)\,\partial_M$ and $\flat \sim \mathrm{ad}\; e_{-1}$.

The ancillary ghosts appear when d fails to be covariant.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & K_{p_0} & \xleftarrow[d]{} & K_{p_0+1} & \xleftarrow[d]{} & K_{p_0+2} & \xleftarrow[d]{} & \cdots \\
 & & & & & & & & \downarrow{\scriptstyle\flat} & & \downarrow{\scriptstyle\flat} & & \downarrow{\scriptstyle\flat} & & \\
0 & \xleftarrow[d]{} & C_1 & \xleftarrow[d]{} & \cdots & \xleftarrow[d]{} & C_{p_0-1} & \xleftarrow[d]{} & C_{p_0} & \xleftarrow[d]{} & C_{p_0+1} & \xleftarrow[d]{} & C_{p_0+2} & \xleftarrow[d]{} & \cdots
\end{array}
$$

[Berman, Cederwall, Kleinschmidt, Thompson: 1208.5884]

[Cederwall, Edlund, Karlsson: 1302.6736]

The L_∞ degrees are given by $\ell = p + q$ (with the convention that all brackets have degree -1). Explicit expressions for all brackets can be derived from the Lie superbracket in $\mathscr{B}$.

$\cdots$	$p=-1$	$p=0$	$p=1$	$p=2$	$p=3$	$\cdots$
$\cdots$						
$q=3$					$\tilde{\tilde{R}}_3$	
$q=2$				$\tilde{R}_2$	$\tilde{R}_3 \oplus \tilde{\tilde{R}}_3$	
$q=1$		$\mathbf{1}$	R_1	$R_2 \oplus \tilde{R}_2$	$R_3 \oplus \tilde{R}_3$	
$q=0$	$\overline{R}_1$	$\mathbf{1} \oplus \mathbf{adj} \oplus \mathbf{1}$	R_1	R_2	R_3	$\cdots$
$\cdots$	$\overline{R}_1$	$\mathbf{1}$		$\ell = 1$	$\ell = 2$	$\ell = 3$

[Cederwall, Palmkvist: 1804.04377]

If $\mathfrak{g}$ is infinite-dimensional, or if $\mathfrak{g}$ is finite-dimensional and $(\lambda, \theta) \geqslant 2$, where θ is the highest root, then the generalised diffeomorphisms only close up to ancillary $\mathfrak{g}$ transformations. In order to describe these cases we need to replace the Borcherds superalgebra $\mathscr{B}$ with a tensor hierarchy algebra.

[Cederwall, Palmkvist: 1711.07694, work in progress ...]

The tensor hierarchy algebra is a Lie superalgebra that can be constructed from the same Dynkin diagram as $\mathscr{B}$, but with modified generators and relations: $(i = 1, 2, \ldots, r)$

$$
\begin{array}{ccc}
f_{-1} & \rightarrow & f_{(-1)i} \\
[h_0, f_{-1}] = f_{-1} & \rightarrow & [h_0, f_{(-1)i}] = f_{(-1)i} \\
[e_{-1}, f_{-1}] = h_{-1} & \rightarrow & [e_{-1}, f_{(-1)i}] = h_i
\end{array}
$$

- The simple root α_{-1} has multiplicity 1 as usual, but its negative has multiplicity r.
- The bracket $[e_i, f_{(-1)j}]$ may be nonzero. Not only positive and negative roots, but also mixed ones appear.

[Palmkvist: 1305.0018] [Carbone, Cederwall, Palmkvist: 1802.05767]

To be better understood:

- The tensor hierarchy algebras ...
- The gauge structure when ancillary transformations appear, first when $\mathfrak{g}_r$ is finite-dimensional, second when $\mathfrak{g}_r$ is infinite-dimensional ...
- The dynamics: Under control when $\mathfrak{g}_{r+1}$ is affine. Maybe also when $\mathfrak{g}_r$ itself is affine and $\mathfrak{g}_{r+1}$ hyperbolic? (Henning's talk)

Obvious direction for further research: towards $\mathfrak{g}_r = E_{11}$