LMS Durham Symposium:

Homogenization in disordered media

Multipole expansion in random media

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Effective multipoles in a random medium

Predicting effective multipoles based on "moments" of localized charge distribution using higher-order correctors

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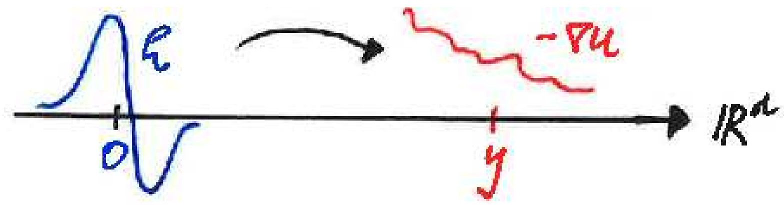
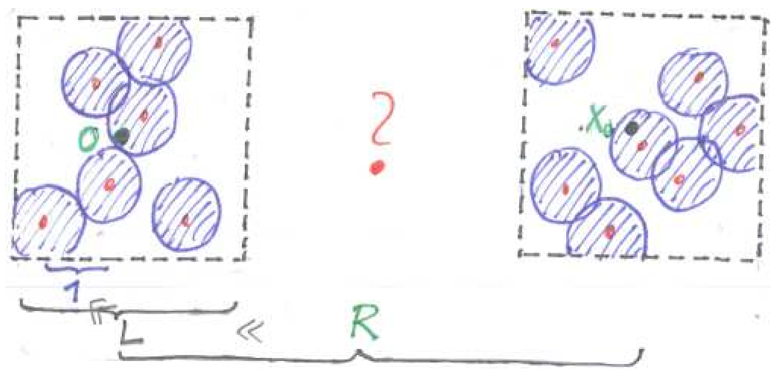
Far-field generated by charge distribution

$\rho \neq 0$ only in $B_L(0)$, $\quad \int \rho = 0$,

$-\nabla \cdot a \nabla u = \rho$, $\quad -\nabla u$ in $B_L(y_0)$





Possible to predict
"effective" multipole expansion
without knowing the medium's
realization in between?

with overwhelming probability in $L \gg 1$

As we shall see for $d = 3$:
Dipole: *yes*; quadrupoles: *yes*; octupoles: *no*

The higher the dimension d, the more effective multipoles do not depend on realization of medium away from source.

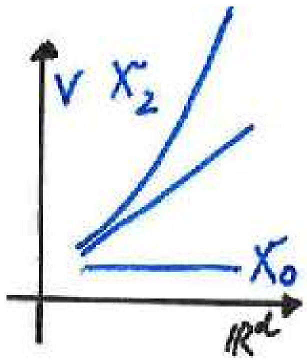
Homogeneous media:
A systematic view of multipole expansion

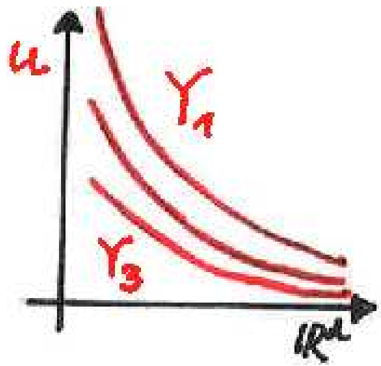
Consider: $\bar{a}$ positive definite $d \times d$ coefficient,
$\bar{X}_k := \{\bar{a}^*\text{-harmonic polynomials degree } \le k\}$

Liouville principle: $\bar{X}_k = \{\, \bar{a}^*\text{-harmonic } \bar{v} \,|\, \bar{v} = O(|x|^k)\};$
$:= \{\, \bar{a}^*\text{-harmonic } \bar{v} \text{ in } \mathbb{R}^d \,|\, \limsup_{R\uparrow\infty} \frac{1}{R^{k-1}}(\frac{1}{R^d}\int_{B_R}|\nabla\bar{v}|^2)^{\frac{1}{2}} < \infty\,\};$

define in analogy: $\bar{Y}_m := \{\, \bar{a}\text{-harmonic } \bar{u} \,|\, \bar{u} = O(|x|^{2-d-m})\,\}$
$:= \{\, \bar{a}\text{-harmonic } \bar{u} \text{ in } \mathbb{R}^d - \{0\} \,|\, \limsup_{R\uparrow\infty} R^{d-1+m}(\frac{1}{R^d}\int_{B_R^c}|\nabla\bar{u}|^2)^{\frac{1}{2}} < \infty\,\}.$

$$\begin{array}{ccccccccc} \bar{v} & \in & \bar{X}_0 & \subset & \bar{X}_1 & \subset & \bar{X}_2 & \subset \\ \bar{u} & \in & \bar{Y}_0 & \supset & \bar{Y}_1 & \supset & \bar{Y}_2 & \supset \end{array}$$





Quotient spaces have nice structure

Recall: $\bar{X}_k = \{\, \bar{a}^*\text{-harmonic } \bar{v} \text{ in } \mathbb{R}^d \,|\, \nabla\bar{v} = O(|x|^{k-1})\,\}$,
$\bar{Y}_m = \{\, \bar{a}\text{-harmonic } \bar{u} \text{ in } \mathbb{R}^d - \{0\} \,|\, \nabla\bar{u} = O(|x|^{-d+1-m})\,\}$.

Natural representation of quotient spaces:
$\bar{X}_k/\bar{X}_{k-1} \cong \{\bar{a}^*\text{-harmonic polynom. degree } = k\}$
$\bar{Y}_m/\bar{Y}_{m+1} \cong \text{span}\{\, \partial^\alpha \bar{G} \,|\, |\alpha| = m,\ \bar{G} \text{ fundamental sol. of } -\nabla\cdot\bar{a}\nabla\,\}$

harmonic polynom.:	$\bar{X}_1/\bar{X}_0$,	$\bar{X}_2/\bar{X}_1$,	$\cdots$
multipoles:	$\bar{Y}_1/\bar{Y}_2$,	$\bar{Y}_2/\bar{Y}_3$,	$\cdots$
dimension in $d=3$:	$d=3$	$\frac{d(d+1)}{2}-1=5$,	$\cdots$

No surprise: Natural isomorphism $\bar{Y}_k/\bar{Y}_{m+1} \cong (\bar{X}_m/\bar{X}_{k-1})^*$.

Isomorphism of quotients provided by pairing B

Recall: $\bar{X}_k = \{\, \bar{a}^*\text{-harmonic } \bar{v} \text{ in } \mathbb{R}^d \,|\, \nabla\bar{v} = O(|x|^{k-1}) \,\}$,
$\bar{Y}_m = \{\, \bar{a}\text{-harmonic } \bar{u} \text{ in } \mathbb{R}^d - \{0\} \,|\, \nabla\bar{u} = O(|x|^{-d+1-m}) \,\}$.

Note: $\bar{v}, \bar{u}$ harmonic in $\mathbb{R}^d - B_r$

$\Longrightarrow$ $q := \bar{u}\bar{a}^*\nabla\bar{v} - \bar{v}\bar{a}\nabla\bar{u}$ is $\nabla\cdot$-free (Green's formula),

$\Longrightarrow$ $\bar{B}(\bar{u}, \bar{v}) := \int_{\partial B_R} \nu \cdot q$ does not depend on $R > r$.

Obvious: $\bar{B}$ bilinear, vanishes on $\bar{Y}_k \times \bar{X}_{k-1}$, hence

$\bar{Y}_k/\bar{Y}_{m+1} \ni [\bar{u}] \mapsto \bar{B}(\bar{u}, \cdot) \in (\bar{X}_m/\bar{X}_{k-1})^*$ well-defined.

Folklore: It is isomorphism.

Isomorphism via B encodes multipole expansion

$\bar{B}(\bar{u}, \bar{v}) := \int_{\partial B_R} \nu \cdot (\bar{u}\bar{a}^* \nabla \bar{v} - \bar{v}\bar{a}\nabla\bar{u})$

Consider $-\nabla \cdot \bar{a}\nabla\bar{u} = \rho$,

$\Longrightarrow \bar{B}(\bar{u}, \bar{v}) = \int \bar{v}\rho$ for all $\bar{a}^*$-harmonic $\bar{v}$.

Suppose 0^{th} *moment* $\int \rho$ *vanishes*

$\Longrightarrow \bar{B}(\bar{u}, \cdot)$ vanishes on $\bar{X}_0$;

suppose 1^{st}&2^{nd} *moments* $\int x\rho, \int x \otimes x\rho$ *known*

$\Longrightarrow \bar{B}(\bar{u}, \cdot) \in \bar{X}_2{}^*$ known;

together $\bar{B}(\bar{u}, \cdot) \in (\bar{X}_2/\bar{X}_0)^*$ known.

By isomorphism $\bar{Y}_1/\bar{Y}_3 \ni [\bar{u}] \mapsto \bar{B}(\bar{u}, \cdot) \in (\bar{X}_2/\bar{X}_0)^*$,

this determines $\bar{u} \in \bar{Y}_1$ up to $\bar{Y}_3$, that is,

monopole vanishes, dipole known, quadrupole known.

Heterogeneous medium: corrector & flux corrector

λ-Uniformly elliptic coefficient field a on $\mathbb{R}^d$:
$\lambda|\xi|^2 \leq \xi\cdot a(x)\xi, \ \ |a(x)\xi|^2 \leq \xi\cdot a(x)\xi$ for all $x\in\mathbb{R}^d$, $\xi\in\mathbb{R}^d$.

Cartesian coordinate direction $i=1,\cdots,d$, e_i unit vector
Corrector ϕ_i modifies Cartesian coordinate x_i to a-harmonic:

$$\nabla\cdot a(e_i+\nabla\phi_i)=\nabla\cdot a\nabla(x_i+\phi_i)=0.$$

Flux $a(e_i+\nabla\phi_i)$ is $\nabla\cdot$-free, thus can be written as
$a(e_i+\nabla\phi_i)=\bar{a}e_i+\nabla\cdot\sigma_i$ with $\bar{a}$ constant, σ_i skew.

$\sigma_i=\{\sigma_{ijk}\}_{j,k=1,\cdots,d}$, $\sigma_{ikj}=-\sigma_{ijk}$,

$(\nabla\cdot\sigma_i)_j=\partial_k\sigma_{ijk}$ (Einstein's summation convention)

(Flux) corrector and two-scale expansion

Recall: $a(e_i+\nabla\phi_i) = \bar{a}e_i + \nabla\cdot\sigma_i$ with $\bar{a}$ constant, σ_i skew.

Merit of (ϕ_i, σ_i):

Relates variable-coefficient operator $-\nabla\cdot a\nabla$

to constant-coefficient operator $-\nabla\cdot\bar{a}\nabla$

via two-scale expansion $(1+\phi_i\partial_i)\bar{u}$: For any $\bar{u}$

$$-\nabla\cdot a\nabla(1+\phi_i\partial_i)\bar{u} = -\nabla\cdot\bar{a}\nabla\bar{u} + \nabla\cdot(\phi_i a\text{-}\sigma_i)\nabla\partial_i\bar{u};$$

σ_i brings error in divergence-form.

(ϕ^*, σ^*) denote objects related to dual coefficient a^*.

Spaces $\bar{X}_k$ and X_k isomorphic

Assumption: *slightly quantified sublinear growth*:

$$\int_1^\infty \frac{1}{R}\Big(\fint_{B_R} |(\phi^*,\sigma^*) - \fint_{B_R}(\phi^*,\sigma^*)|^2\Big)^{\frac{1}{2}}\frac{dR}{R} < \infty$$

Recall $\bar{X}_k = \{\, \bar{a}^*\text{-harmonic } \bar{v} \,|\, \bar{v} = O(|x|^k)\}$

$= \{\, \bar{a}^*\text{-harmonic } \bar{v} \,|\, \limsup_{R\uparrow\infty} \frac{1}{R^{k-1}}(\fint_{B_R} |\nabla\bar{v}|^2)^{\frac{1}{2}} < \infty \,\}.$

$X_k := \{\, a^*\text{-harmonic } v \,|\, \limsup_{R\uparrow\infty} \frac{1}{R^{k-1}}(\fint_{B_R} |\nabla v|^2)^{\frac{1}{2}} < \infty \,\}.$

Note $X_0 \supset \text{span}\{1\}$, $X_1 \supset \text{span}\{1, x_1+\phi_1^*, \cdots x_d+\phi_d^*\}$

Proposition (Fischer-O. CPDE'16).

$[\bar{v}] \in \bar{X}_k/\bar{X}_{k-1} \cong X_k/X_{k-1} \ni [v]$ via two-scale expansion

$$\limsup_{R\uparrow\infty} \frac{1}{R^{k-1}}\Big(\fint_{B_R} |\nabla v - \partial_i\bar{v}(e_i+\nabla\phi_i^*)|^2\Big)^{\frac{1}{2}} = 0.$$

Consequence $X_0 = \text{span}\{1\}$, $X_1 = \text{span}\{1, x_1+\phi_1^*, \cdots, x_d+\phi_d^*\}$

Quotient spaces of Y_m and X_k isomorphic

Recall $\bar{Y}_m := \{\, \bar{a}\text{-harmonic } \bar{u} \,|\, \bar{u} = O(|x|^{2-d-m}) \,\}$

$:= \{\, \bar{a}\text{-harmonic } \bar{u} \text{ in } \mathbb{R}^d - \{0\} \,|\, \limsup_{R\uparrow\infty} R^{d-1+m}(\frac{1}{R^d}\int_{B_R^c}|\nabla\bar{u}|^2)^{\frac{1}{2}} < \infty \,\}.$

Now $Y_m :=$

$\{\, a\text{-harm. } \bar{u} \text{ in } \mathbb{R}^d\text{-}B_r \,|\, \limsup_{R\uparrow\infty} R^{d-1+m}(\frac{1}{R^d}\int_{B_R^c}|\nabla u|^2)^{\frac{1}{2}} < \infty \,\}.$

Recall $X_k := \{\, a^*\text{-harmonic } v \,|\, \limsup_{R\uparrow\infty} \frac{1}{R^{k-1}}(⨍_{B_R}|\nabla v|^2)^{\frac{1}{2}} < \infty \,\}.$

Recall $\bar{B}$. Now $B(u, v) := \int_{\partial B_r} \nu \cdot (u a^* \nabla v - v a \nabla u).$

Proposition Bella-Giunti-O. '17.

$Y_k/Y_{m+1} \ni [u] \mapsto B(u, \cdot) \in (X_m/X_{k-1})^*$ is isomorphism.

Consequence: $Y_m/Y_{m+1} \cong \bar{Y}_m/\bar{Y}_{m+1}$

Based on large-scale regularity theory

Avellaneda&Lin '87, Kenig&Lin&Shen '12, Armstrong&Smart '14

Proposition (Fischer-O' 16). $\bar{X}_k/\bar{X}_{k-1} \overset{\text{two-scale}}{\cong} X_k/X_{k-1}$.

Proposition (Bella-Giunti-O. '17). $Y_k/Y_{m+1} \overset{B}{\cong} (X_m/X_{k-1})^*$.

Proposition (Fischer-O. '16)

For every $r_* \leq r \leq R$ and a^*-harmonic u in B_R:

$$\inf_{v\in X_k} \Big(\fint_{B_r} |\nabla(u-v)|^2\Big)^{\frac{1}{2}} \lesssim (\tfrac{r}{R})^k \Big(\fint_{B_R} |\nabla u|^2\Big)^{\frac{1}{2}}.$$

Intrinsic $C^{k,1}$-theory from radius r_* onwards.

2^{nd}-order corrector, two-scale expansion

Recall defining property of first-order objects $(\phi_i, \sigma_i, \bar{a}_i := \bar{a}e_i)$:

$$a(e_i + \nabla\phi_i) = \bar{a}_i + \nabla\cdot\sigma_i.$$

Recall property of two-scale expansion $(1+\phi_i\partial_i)\bar{u}$: for any $\bar{u}$

$$-\nabla\cdot a\nabla(1+\phi_i\partial_i)\bar{u} = -\nabla\cdot\bar{a}\nabla\bar{u} + \nabla\cdot(\phi_i a - \sigma_i)\nabla\partial_i\bar{u}.$$

Now for $i, j = 1, \cdots, d$ (ϕ'_{ij} scalar, σ'_{ij} skew, $\bar{a}'_{ij}$ constant):

$$a\nabla\phi'_{ij} + (\phi_i a - \sigma_i)e_j = \bar{a}'_{ij} + \nabla\cdot\sigma'_{ij}.$$

2^{nd}-order two-scale expansion $(1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})\bar{u}$:

$$-\nabla\cdot a\nabla(1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})\bar{u}$$
$$= -\nabla\cdot(\bar{a}\nabla\bar{u} + \bar{a}'\nabla^2\bar{u}) + \nabla\cdot(\phi'_{ij}a - \sigma'_{ij})\nabla\partial_{ij}\bar{u}.$$

In particular $(1+\phi^*_i\partial_i+{\phi^*_{ij}}'\partial_{ij})\bar{v} \in X_2$ for $\bar{v} \in \bar{X}_2$.

Stochastic bounds on higher-order correctors

Class of ensembles $\langle\cdot\rangle$ of λ-uniformly elliptic coefficient fields a:
$a(x)=A(g(x))$ with A Lipschitz, g Gaussian field
centered, stationary, integrable covariance.

Y. Gu '17

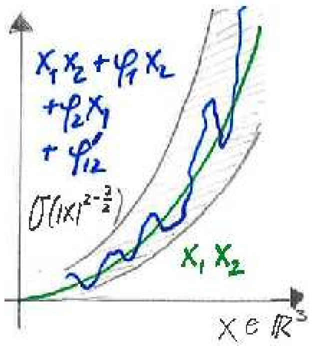
Proposition (Gloria-O. AoP '11, Bella-Fehrman-Fischer-O. PTRF).
For $d=3$: (ϕ,σ) is stationary,
$\big(\int_{B_1(x)}|(\phi,\sigma)|^2\big)^{\frac{1}{2}}$ has stretched exponential moments,
$|y-x|^{-\frac{1}{2}}\big(\int_{B_1(y)}|(\phi',\sigma')-\fint_{B_1(x)}(\phi',\sigma')|^2\big)^{\frac{1}{2}}$
has stretched exponential moments.

$\frac{1}{2}=2$ (from 2^{nd} order) $-\frac{d=3}{2}$, cf. Brownian motion

A canonical isomorphism $\bar{X}_2/\bar{X}_0 \cong X_2/X_0$

Normalize ϕ^* through $\langle\phi^*\rangle = 0$, possible since stationary.
No *canonical* way to normalize $\phi^{*\prime}$, since typ. not stationary.
However may normalize $\nabla\phi^{*\prime}$ through $\langle\nabla\phi^{*\prime}\rangle = 0$;
recall $\nabla\cdot(a^*\nabla\phi_{ij}^{*\prime} + (\phi_i^* a^* - \sigma_i^*)e_j) = 0$.

Thus two-scale expansion $v = (1 + \phi_i^*\partial_i + \phi_{ij}^{*\prime}\partial_{ij})\bar{v}$ provides
canonical isomorphism $[\bar{v}] \in \bar{X}_2/\bar{X}_0 \cong X_2/X_0 \ni [v]$
via $\nabla v = \nabla(1+\phi_i^*\partial_i)\bar{v} + \partial_{ij}\bar{v}\nabla\phi_{ij}^{*\prime}$.



A canonical isomorphism $Y_1/Y_3 \cong \bar{Y}_1/\bar{Y}_3$

Recall: ϕ, σ, $\nabla\phi'$ *canonically* defined, $\bar{a}'_{ij} = \langle a\nabla\phi'_{ij} + (\phi_i a - \sigma_i)e_j\rangle$.

Fix $1 < \beta < \frac{d=3}{2}$

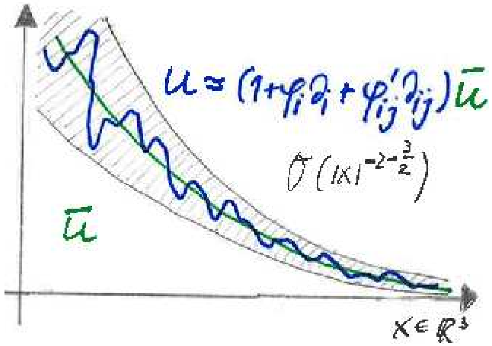
Proposition (Bella-Giunti-O. '17)

Two-scale expansion yields *canonical* $Y_1/Y_3 \cong \bar{Y}_1/\bar{Y}_3$ via

$\forall\, u \in Y_1 \quad \exists\, \bar{u} \in \bar{Y}_1$ unique up to $\bar{Y}_3$ s. t. $\lim_{|y|\uparrow\infty} |y|^{\beta+3}$

$$\Big(\int_{B_1(y)}\big|\nabla u - \nabla(1+\phi_i\partial_i)\bar{u} - \partial_{ij}\bar{u}\nabla\phi'_{ij} - \partial_i\bar{u}'(e_i+\nabla\phi_i)\big|^2\Big)^{\frac{1}{2}} = 0,$$

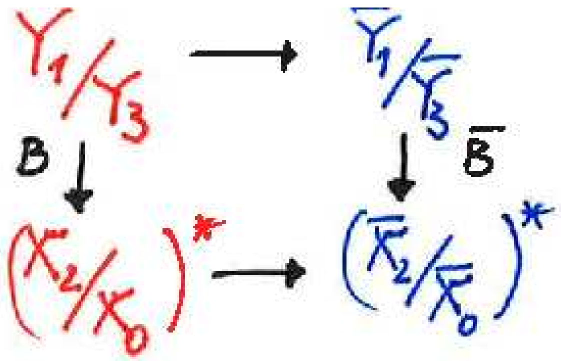
where $\bar{u}'$ is determined by $\bar{u}$ via $\nabla\cdot(\bar{a}\nabla\bar{u}' + \bar{a}'\nabla^2\bar{u}) = 0$



A commuting diagram

Theorem (Bella-Giunti-O. '17)

The following diagram commutes



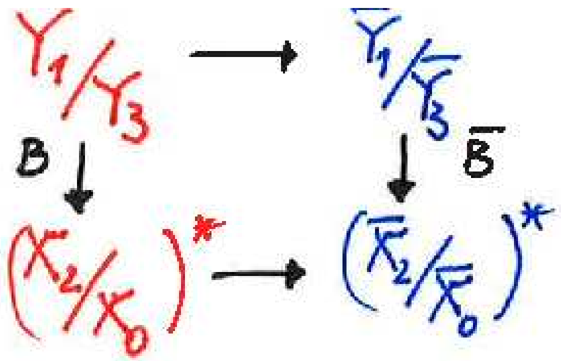
More precisely, for $(\bar v, \bar u) \in \bar X_2 \times \bar Y_1$ and $(v, u) \in X_2 \times Y_1$ related by $\nabla v = \nabla(1+\phi_i^*\partial_i)\bar v + \partial_{ij}\bar v \nabla \phi_{ij}^{*\prime}$ and $\lim_{|y|\uparrow\infty} |y|^{\beta+3} \left(\int_{B_1(y)} \left|\nabla u - \nabla(1+\phi_i\partial_i)\bar u - \partial_{ij}\bar u \nabla\phi'_{ij} - \partial_i \bar u'(e_i+\nabla\phi_i)\right|^2\right)^{\frac{1}{2}} = 0$ we have $\bar B(\bar u, \bar v) = B(u, v)$.

Multipole expansion in random media made explicit

For ρ with $\int \rho = 0$

consider $-\nabla \cdot a\nabla u = \rho$.

Have $B(u, v) = \int v\rho$.



By Proposition $\exists!$ $[\bar u] \in \bar Y_1/\bar Y_3$ s. t. $\lim_{|y|\uparrow\infty} |y|^{\beta+3}$
$\left(\int_{B_1(y)} \left|\nabla u - \nabla(1+\phi_i\partial_i)\bar u - \partial_{ij}\bar u\nabla\phi'_{ij} - \partial_i\bar u'(e_i+\nabla\phi_i)\right|^2\right)^{\frac{1}{2}} = 0.$

How to determine $\bar u$?

By assumption $\int \rho = 0$, so that $B(u, \cdot) \in (X_2/X_0)^*$.

By Theorem, $\bar B(\bar u, \bar v) = B(u, (1+\phi_i^*\partial_i+{\phi_{ij}^*}'\partial_{ij})\bar v)$.

In view of $\bar Y_1/\bar Y_3 \overset{\bar B}{\cong} (\bar X_2/\bar X_0)^*$,

$[\bar u]$ is determined by $\bar B(\bar u, \cdot) \in (\bar X_2/\bar X_0)^*$.

Multipole expansion in random media

Recall that u is well described by multipole $\bar{u}$

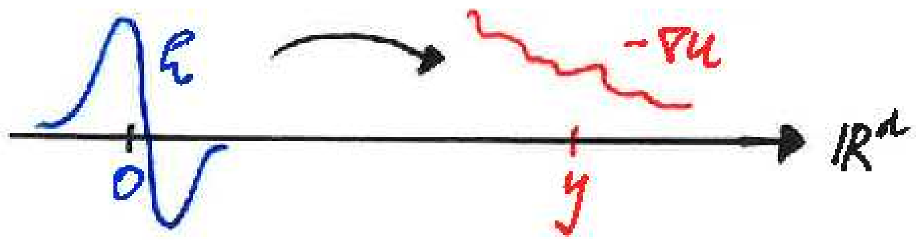
$$\lim_{|y|\uparrow\infty} |y|^{\beta+3}\Big(\int_{B_1(y)}\Big|\nabla u - \nabla(1+\phi_i\partial_i)\bar{u} - \partial_{ij}\bar{u}\nabla\phi'_{ij} - \partial_i\bar{u}'(e_i+\nabla\phi_i)\Big|^2\Big)^{\frac{1}{2}} = 0.$$

and that $\bar{u}$ is determined in $\bar{Y}_1$ up to $\bar{Y}_3$ by "moments"

$$\int \rho(1+\phi_i^*\partial_i+\phi_{ij}^{*\prime}\partial_{ij})\bar{v} \quad \text{for} \quad v \in \bar{X}_2.$$

The latter depends only on $(\phi^*, \nabla\phi^{*\prime})$ restricted to suppρ.

Hence approximation of u near y only requires $(\phi^{(*)}, \nabla\phi^{(*)\prime})$ in Representative-Volume-Element neighborhoods of 0 and y.



Yet another good behavior of two-scale expansion

Recall $d=3$, for simplicity a symmetric.
2^{nd} order two-scale expansion $E\bar u := (1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})\bar u$.
Given $\bar a$-harmonic $\bar v$ and $\bar u$ with $\bar v = O(|x|^2)$, $\bar u = O(|x|^{-2})$.

Claim: $\lim_{R\uparrow\infty}\int_{\partial B_R}\nu\cdot(E\bar v\, a\nabla E\bar u - E\bar u\, a\nabla E\bar v)$
$= \int_{\partial B_R}\nu\cdot(\ \bar v\,\bar a\nabla\ \bar u - \ \bar u\,\bar a\nabla\ \bar v)$.

Step 1. Under $\lim_{R\uparrow\infty}\int_{\partial B_R}\nu\cdot$, $\qquad E\bar v a\nabla E\bar u$
$\rightsquigarrow \bar v(\bar a\nabla\bar u + \bar a'_i\nabla\partial_i\bar u) + \phi_k\partial_k\bar v\,\bar a\nabla\bar u - \partial_k\bar v\,\partial_i\bar u\,\sigma_i(e_k+\nabla\phi_k)$.

Flux representation via 1^{st} and 2^{nd} order flux correctors: $a\nabla E\bar u$
$= \bar a\nabla\bar u + \bar a'_i\nabla\partial_i\bar u + (\phi'_{ij}a - \sigma'_{ij})\nabla\partial_{ij}\bar u + \nabla\cdot(\partial_i\bar u\,\sigma_i + \partial_{ij}\,\bar u\sigma'_{ij})$.
Quenched estimates: $\phi_i,\sigma_i = O(1)$, $\phi'_{ij},\sigma'_{ij} = O(|x|^{\frac12})$.

Two-scale exp. commutes with Green's formula

Claim: $\lim_{R\uparrow\infty}\int_{\partial B_R}\nu\cdot(E\bar v\, a\nabla E\bar u - E\bar u\, a\nabla E\bar v)$

$$= \int_{\partial B_R}\nu\cdot(\ \bar v\,\bar a\nabla\ \ \bar u - \ \ \bar u\,\bar a\nabla\ \ \bar v).$$

Step 1. Under $\lim_{R\uparrow\infty}\int_{\partial B_R}\nu\cdot$, $\quad E\bar v a\nabla E\bar u$

$\rightsquigarrow \bar v(\bar a\nabla\bar u + \bar a_i'\nabla\partial_i\bar u) + \phi_k\partial_k\bar v\,\bar a\nabla\bar u - \partial_k\bar v\,\partial_i\bar u\,\sigma_i(e_k+\nabla\phi_k)$.

Step 2. Under $\lim_{R\uparrow\infty}\int_{\partial B_R}\nu\cdot$, $E\bar v\, a\nabla E\bar u - E\bar u\, a\nabla E\bar v$

$\rightsquigarrow \bar v\,\bar a\nabla\bar u - \bar u\,\bar a\nabla\bar v + \bar v\ \bar a_i'\nabla\partial_i\bar u - \bar u\ \bar a_k'\nabla\partial_k\bar v - \partial_i\bar u\ \bar a_i'\nabla\bar v$.

Use formula $\bar a_{ijk}' = \langle\sigma_{ik\ell}\partial_\ell\phi_j - \sigma_{jk\ell}\partial_\ell\phi_i\rangle$,

normalization $\langle\phi_i\rangle = 0$ (essential), $\langle\sigma_i\rangle = 0$ (convenient).

Two-scale exp. commutes with Green's formula

Claim: $\lim_{R\uparrow\infty}\int_{\partial B_R}\nu\cdot(E\bar v\, a\nabla E\bar u - E\bar u\, a\nabla E\bar v)$

$$= \int_{\partial B_R}\nu\cdot(\ \bar v\,\bar a\nabla\ \ \bar u - \ \ \bar u\,\bar a\nabla\ \ \bar v).$$

Step 1. Under $\lim_{R\uparrow\infty}\int_{\partial B_R}\nu\cdot$, $\quad E\bar v a\nabla E\bar u$

$\rightsquigarrow \bar v(\bar a\nabla\bar u + \bar a_i'\nabla\partial_i\bar u) + \phi_k\partial_k\bar v\,\bar a\nabla\bar u - \partial_k\bar v\partial_i\bar u\,\sigma_i(e_k+\nabla\phi_k).$

Step 2. Under $\lim_{R\uparrow\infty}\int_{\partial B_R}\nu\cdot$, $E\bar v\, a\nabla E\bar u - E\bar u\, a\nabla E\bar v$

$\rightsquigarrow \bar v\,\bar a\nabla\bar u - \bar u\,\bar a\nabla\bar v + \bar v\ \bar a_i'\nabla\partial_i\bar u - \bar u\ \bar a_k'\nabla\partial_k\bar v - \partial_i\bar u\ \bar a_i'\nabla\bar v.$

Step 3. Under $\lim_{R\uparrow\infty}\int_{\partial B_R}\nu\cdot$,

$\bar v\ \bar a_i'\nabla\partial_i\bar u - \bar u\ \bar a_k'\nabla\partial_k\bar v - \partial_i\bar u\ \bar a_i'\nabla\bar v$

$\rightsquigarrow\ \bar v\ \bar a_i^{sym'}\nabla\partial_i\bar u - \bar u\ \bar a_k^{sym'}\nabla\partial_k\bar v - \partial_i\bar u\ \bar a_i^{sym'}\nabla\bar v,$

where $\bar a^{sym'}$ is symmetrization of 3-tensor $\bar a'$,

which vanishes because skew in first two indices.