Stochastic averaging: effectives

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Introduction

Diffusion Operators

In local coordinates,

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k(x)\frac{\partial}{\partial x_k}.$$

Here $A(x) := (a_{i,j}(x))$ is a $n \times n$ symmetric non-negative matrix. If Lipschitz continuous, there exist a family of vector fields $X_1, \ldots, X_m$, $m \geq n$, s.t.

$$\mathcal{L}f = \frac{1}{2}\sum_{k=1}^{m} X_k(X_k f) + X_0 f.$$

- $\mathcal{L}$ is **elliptic** if and only if $X(x) : \mathbf{R}^m \to T_x M$ is a surjection and so determines a Riemannian metric. An elliptic operator is $\frac{1}{2}\Delta$ plus drift for some Riemannian metric. A strong Markov process with generator $\frac{1}{2}\Delta$ is a **BM**.

SDEs

Given $\mathcal{L} = \frac{1}{2}\sum(X_i)^2 + X_0$, define

$$dx_t = \sum_{i=1}^{m} X_i(x_t) \circ dB_t^i + X_0(x_t)dt.$$

The solutions are diffusions (strong Markov processes) with generator $\mathcal{L}$.

Stochastic slow fast systems

$$\begin{cases} dx_t^\epsilon = \sum_{k=1}^{m_1} X_k(x_t^\epsilon, y_t^\epsilon) \circ dB_t^k + X_0(x_t^\epsilon, y_t^\epsilon)\, dt, \\ dy_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{m_2} Y_k(x_t^\epsilon, y_t^\epsilon) \circ dW_t^k + \frac{1}{\epsilon} Y_0(x_t^\epsilon, y_t^\epsilon)\, dt. \end{cases}$$

Problem. Take $\epsilon \to 0$, show x_t^ϵ converges weakly.

$$\frac{1}{\epsilon}\left(\overbrace{\frac{1}{2}Y_i^2(x,\cdot) + Y_0(x,\cdot)}^{\mathcal{L}_0 \equiv \mathcal{L}_0^x}\right) + \overbrace{\left(\frac{1}{2}X_k(\cdot,y)^2 + X_0(\cdot,y)\right)}^{\mathcal{L}_1 \equiv \mathcal{L}_1^y}.$$

$$\begin{cases} \frac{d}{dt}u^\epsilon(t,x,y) = (\frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1)u^\epsilon(t,x,y) \\ u^\epsilon(0,x,y) = f(x) \end{cases}$$

Uniform LLN (uniform Birkhoff)

- Suppose for each x, $\mathcal{L}^x$ has a unique invariant measure. Then $\mathcal{L}^x$ is said to satisfy a **locally uniform** law of large numbers if
  - $x \to \mu^x$ is locally Lipschitz continuous.
  - For every $f \in L^2 \cap C^r$, there exists a locally bounded $C(x)$ such that

$$\left| \frac{1}{T} \int_t^{t+T} f(y_r^x)\, dr - \int_G f(y)\mu^x(dy) \right|_{L_2(\Omega)} \le C(x)c(f)\frac{1}{\sqrt{T}}.$$

–This is useful for estimating speed of convergence.
–Not trivial: consider $dy_t = \sigma(x)dB_t + \nabla h(x, y_t)dt$.
–Proved in case G is compact, $\sum Y_i$ satisfies Hörmander's condition+ bounds [xml18, Abel Symp].

Zero index Fredholm operators

-To solve $\mathcal{L}^x f = v$, v must satisfy several independent constraints. The dimension of the solutions minus the dimension of the independent constraints is the 'index'.
-If $\mathcal{L}$ satisfies Hörmander's conditions, it is Fredholm from its domain to L^2: $\mathcal{L}$ has closed range,

$$\dim(\ker \mathcal{L}^x) < \infty, \quad \text{dim}(\ker(\mathcal{L}^x)^*) < \infty$$

-If the Fredholm index =0, define $\Pi_x : L_2 \to \ker(\mathcal{L}^x)$, functioning as an averaging operator.

Open Problem.

$$\left| \frac{1}{T} \int_t^{t+T} f(y_r^x)\, dr - \Pi_x f \right| \le C(x)c(f)\beta(T)?$$

Note: Given $\mathcal{L}^x : E \to F$, smooth in x, $\dim(\ker(\mathcal{L}^x))$ may not be continuous in x. Their index is [M. Atiyah]. Caution: $\mathrm{Dom}(\mathcal{L}^x)$ may vary with x.

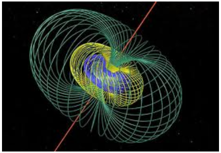
Effective Motions

Effective motions

Effective motions coming from averaging is associated with a first integral or a conserved map. Effective motions typically live in a reduced space: a quotient of the original space and an action space.
When the unperturbed motion has a full range of symmetries, the quotient space (or orbit space) is a smooth manifold. The classification will rely on algebra and differential geometry. The reduced space is often a foliation or a graph.
When this is a graph, the identification of the effective motion is associated with exit laws of Markov processes. [Brin, Freidlin, Wentzell, Bhatin, Borodin, Koralov, ...]

A baby model on Hopf fibration

$$S^3 \sim SU(2) = \left\{ \begin{pmatrix} z & w \\ -\bar{w}, & \bar{z} \end{pmatrix} : z, w \in \mathbb{C} \right\}.$$



- The Pauli matrices :

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

- Berger's spheres is S^3 with $\{\frac{1}{\sqrt{\epsilon}}X_1, X_2, X_3\}$ o.n.b. The spectra of Berger's spheres, i.e. $\frac{1}{\epsilon}(X_1)^2 + (X_2)^2 + (X_3)^2$, converges.
- **Problem.**

$$\mathcal{L}^\epsilon = \frac{1}{2\epsilon}(X_1)^2 + X_2.$$

What information can we extract from $\mathcal{L}^\epsilon$, when ϵ is taken to zero? Look at $dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}}\, g_t^\epsilon X_1 \circ dB_t + g_t^\epsilon X_2\, dt$.

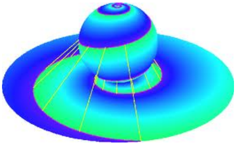
a baby theorem

Take a unit vector $Y_0 \in \langle X_2, X_3 \rangle$.

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} g_t^\epsilon X_1 dB_t + g_t^\epsilon Y_0 dt.$$

$\pi(z, w) = (\frac{1}{2}(|w|^2 - |z|^2), z\bar{w})$ is the Hopf map, mapping S^3

to $S^2 = SU(2)/S^1$, and $x_t^\epsilon = \pi(g_t^\epsilon)$.



Theorem. [xml'18 JJMS]

- As $\epsilon \to 0$, $x_t^\epsilon := \pi(g_t^\epsilon) \to \pi(g_0)$
- $x_{\frac{t}{\epsilon}}^\epsilon$ converges in law to the BM on $S^2(\frac{1}{2})$ scaled by $\lambda = \frac{1}{2}$.
- The horizontal lift, $(\tilde{x}_t^\epsilon)$, of (x_t^ϵ), converges weakly to the hypoelliptic diffusion with generator $\bar{\mathcal{L}} = \frac{1}{2}\Delta^{hor}$.

Using symmetries

Suppose that H is a compact subgroup of a Lie group G with a left invariant metric.

- Then $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{h}^\perp$ and there is an $ad(H)$-invariant orthogonal splitting :

  $$\mathfrak{h}^\perp = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_l,$$

  $\mathfrak{m}_0$ is the space of $\mathrm{Ad}(H)$ invariant vectors.
- Take $A_k \in \mathfrak{h}$, and $Y \in \mathfrak{h}^\perp$.

  $$dg_t = \sum_{k=1}^{p} \gamma A_k(g_t) \circ dB_t^k + \delta Y(g_t)dt.$$

  The solutions interpolate between translates of the one parameter group on G and diffusions on H.
- We will take $\gamma \to \infty$ while setting $\delta = 1$. Consider diffusions on the orbit space G/H.

Adiabatic limit on homogeneous spaces

Suppose $\{A_k\}$ and their iterated commutators generate $\mathfrak{h}$.

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^N A_k(g_t^\epsilon) \circ db_t^k + Y(g_t^\epsilon)dt, \quad g_0^\epsilon = g_0,$$

- There exists $\tilde{g}_t^\epsilon$, with $g_t^\epsilon H = \tilde{g}_t^\epsilon H$, converging to the solution of $\frac{\partial}{\partial t}\bar{g} = Y_{\mathfrak{m}_0}(\bar{g})$. Key: $\int_H \mathrm{Ad}(H)(Y)dh = 0$ iff $Y \in \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_l$,
- If $Y \in \mathfrak{m}_k$ is a unit vector, $\tilde{g}_{t/\epsilon}^\epsilon$ converges to a diffusion with generator $\lambda(Y) \sum_{j=1}^{\dim(\mathfrak{m}_k)} (Y_{k,j})^2$. associate matrix eigenvector to eigenfunction of $\mathcal{L}_0$, solve Poisson eq. and use a result of D. Rumynin.
- $\pi(\tilde{g}_{t/\epsilon}^\epsilon) \in G/H$ converges to Markov proces. Paralle translations along $\pi(\tilde{g}_{t/\epsilon}^\epsilon)$, converges to stochastic parallel transports along the limiting diffusions.
- If $\{A_k\}$ is an o.n.b. of $\mathfrak{h}$, $\lambda_k(Y)$ is independent of Y.

Taking the adiabatic limit in geometry

–Taking the adiabatic limit in geometry is popular: Getzler, Bismut, Lebeau,...,
–The theorem follows from a separation of scales, and :

$$\dot{y}_t^\epsilon(\omega) = \sum_{k=1}^m Y_k\left(y_t^\epsilon(\omega)\right)\alpha_k(z_t^\epsilon(\omega)), \qquad y_0^\epsilon(\omega) = y_0. \tag{1}$$

where z_t^ϵ is a $\frac{1}{\epsilon}\mathcal{L}_0$ diffusion, α_k 'averages' to zero w.r.t. the invariant measure of $\mathcal{L}_0$.
Then $y_{\frac{t}{\epsilon}}^\epsilon$ converges to an explicit Markov process with rate $\epsilon^{\frac{1}{4}}$ in Wasserstein distance.
[xml, PTRF'17], Lions, Sougnidis, Papaniclaou, Keller, Varadhan, ...

Hamiltonian systems

Averaging

- Let $x_0 \in T^n$ and $\omega = (\omega_1, \ldots, \omega_n)$ where $\omega_1, \ldots, \omega_n$ are linearly independent real numbers over Q. Then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(x_0 + s\omega) ds = \int_{T^n} f(x)\mu(dx), \quad f \in L^1.$$

- If

$$\dot{I} = \epsilon g(I, \theta), \qquad \dot{\theta} = \omega(I) + \epsilon f(I, \theta),$$

then $I^\epsilon(\frac{t}{\epsilon}) \to \bar{I}(t)$, where

$$\frac{d}{dt}\bar{I}(t) = \int g(\bar{I}(t), \theta)\mu(d\theta).$$

Integrable Hamiltonian

- Darboux's theorem. Given an integrable Hamiltonian system, for almost every point, there is a canonical action-angle coordinates such that the Hamiltonian H is a function of I only. Then $\dot{x}_t = X_H(x_t)$ is equivalent to
$$\dot{I} = 0, \qquad \dot{\theta} = \omega(I).$$
Let us consider a perturbation:
$\dot{x}_t = (\nabla H)^\perp(x_t) + \epsilon V(x_t)$. Then $H(x_t^\epsilon)$ is a slow motion and converges within the canonical charts. [1]
- Shape of the effective limit is a challenge beyond the local coordinates: Non-constant frequencies. more than 1-degree of freedom, product of one dimensional Hamiltonians is most promising. Neshdadt, ...

[1] Early 60's: Bogolyubov-Mitropolskii, Anosov, 70's: V I Arnold, Neishtadt.

Perturbation of 1-dim. Hamiltonians

M. Brin, M. Freidlin, A.D. Wentzell,...:

$$dx_t^{\epsilon,\delta} = \frac{1}{2}\delta\, dB_t + \frac{1}{\epsilon} J\nabla H(x_t^{\epsilon,\delta}).$$

As $\delta \to 0$, $x_t^{\epsilon,\delta}$ converges weakly to a diffusion $\bar{x}_t^\epsilon$ on graph.
Then take $\epsilon \to 0$, $\bar{x}_t^\epsilon$ converges weakly to a motion on graph, deterministic on each edge of the energy graph, random on vertex.
Bhatin, Dolgopyat, Korolov, Kifer, ...

The shallow water equation

Suppose we have a shallow water with height H and free surface $z + H$.

$$\begin{aligned} \frac{\partial u}{\partial t} + u \cdot \nabla u &= \nu \Delta u_t + \xi_1 + \nabla h \\ \frac{\partial h}{\partial t} + \operatorname{div}((z + H)u) &= \alpha\, \Delta h + \xi_2. \end{aligned}$$

We then put this on a rotationary frame: $\dot{\tilde{e}} = R \times e$ and obtain a rotationary shallow water equation. Then any vector $A = \sum_{i=1}^{3} A_i e_i$ evolves by

$$\dot{A} = \sum_{i=1}^{3} \frac{d}{dt}\,(A_i e_i) = \sum_{i=1}^{3} \frac{d}{dt} \tilde{A}_i \dot{\tilde{e}}_i.$$

$$\dot{A}_{rot} := \dot{A}_{int} + R \times A.$$

Shallow water in rotational frame

The shallow water equation in rotational frame has an additional Coriolis force: $2R \times u$ and an additional centrifugal force $R \times R \times u$.

By analyzing the wave forms, Salzman (1962) used a double Fourier series expansion and obtained a set of ODE's. Lorenz (1963) found (also 60) the same equations by brutally truncating the Fourier modes.

$$\begin{aligned}
\dot{u} &= -vw + bvz, \\
\dot{v} &= uw - buz \\
\dot{w} &= -uv \\
\dot{x} &= -\frac{1}{\epsilon} z \\
\dot{z} &= \frac{1}{\epsilon} x + buv
\end{aligned}$$

(u, v, w) represents slow waves in large scale caused by rotation of the planet, (x, z) represents gravity wave (eg surface wave at beach, fast smaller in scale).

Lorenz system

$$\dot{u} = -vw + bvz,$$
$$\dot{v} = uw - buz$$
$$\dot{w} = -uv$$
$$\dot{x} = -\frac{1}{\epsilon}z$$
$$\dot{z} = \frac{1}{\epsilon}x + buv$$

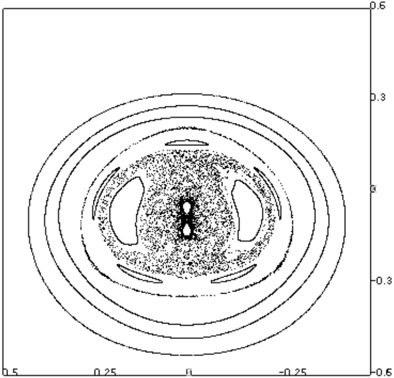
$b = \frac{u_0}{\sqrt{g_0 l_0}}$, $\epsilon =$ Rossby $\frac{b}{\sqrt{1+b^2}}$. We first take $\epsilon = 1, b$ small.
The system has two constants of motion:

$$u^2 + v^2 = C_1, \quad v^2 + w^2 + x^2 + z^2 = C_2.$$

There are no non-trivial solutions such that $C_1 = 0$ or $C_2 = 0$. We restrict it to the energy surfaces, is it chaotic? is it integrable?

Poincare map for hydrodynamic 5d system

The Poincare map for $w = 0$ section on (z, x) plane [2]



$R = 1, b = 0.1, C = 1.$

[2] Acknowledgement: Obtained for me by Alexey Kazakov and Dimitry Turaev.

Hamiltonian

Setting $u = \sqrt{C}\cos\phi'$, $v = \sqrt{C}\sin\phi'$, $\phi' = \phi - \epsilon bx$. The system is in fact a Hamiltonian system in (u, v, z, x). with

$$H = \frac{1}{2}C\sin^2(\phi' + \epsilon bx) + \frac{1}{2}(w^2 + z^2 + x^2),$$

The part chaotic and part integrable nature is characteristic of Hamiltonian systems.
Restricted to a constant energy surface $u^2 + v^2 = C$, It is also equivalent to the nearly integrable system:

$$\begin{aligned}
\dot{\psi} &= w - bz \\
\dot{w} &= -C\sin(2\psi) \\
\dot{z} &= x + bC\sin(2\psi) \\
\dot{x} &= -z
\end{aligned}$$

Stochastic integrable systems

If we consider a time dependent random energy $\sum_{i=1}^n H_i \dot{B}_t^i$ on $\mathbf{R}^{2n}$ (or symplectic manifolds), we are naturally lad to a stochastic Hamiltonian system:

$$dy_t = \sum_{i=1}^n X_{H_i}(y_t) \circ dB_t^i.$$

Consider now a small perturbation

$$dy_t = \frac{1}{\epsilon} \sum_i X_{H_i}(y_t) \circ dB_t^i + K(y_t)dt.$$

Theorem. [xml, nonlinearity 08.] Inside canonical coord.

- $H_i(y_{\frac{s}{\epsilon}})$ converges in L_p to solution of an ODE, speed of convergence is controlled above by $c(t)\epsilon^{\frac{1}{4}}$.
- **Fluctuation from limit.** If K is a Hamiltonian vector field, then $H_i(y_{\frac{s}{\epsilon^2}})$ converges to a Markov process.

Stochastic Lorenz equation

Set $H_1 = \frac{1}{2}w^2 + \sin^2\psi$, $H_2 = \frac{1}{2}(z^2 + x^2)$.

$$\begin{cases} \dot{\psi} = w & -bz \\ \dot{w} = -C\sin(2\psi) & \\ \dot{z} = x & +bC\sin(2\psi) \\ \dot{x} = -z & \end{cases}$$

Consider

$$\begin{cases} d\psi & = & w \circ dB^1 & -bzdt \\ \dot{w} & = & -C\sin(2\psi) \circ dB^1 & \\ \dot{z} & = & x \circ dB^2 & +bC\sin(2\psi)dt \\ \dot{x} & = & -z \circ dB^2 & \end{cases}.$$

Set $H_i^b(t) = H_2(x_{t/b}, y_{t/b})$. Observe that $H_{tot} = H_1 + H_2$ is a first integral, so H_1, H_2 are bounded,

$$\frac{d}{dt}H_2^b(x_{t/b}, y_{t/b}) = C z_{t/b} \sin(2\psi_{t/b}).$$

Product canonical coordinates

$$x = \sqrt{2I_2}\cos\theta_2, \qquad z = \sqrt{2I_2}\sin\theta_2.$$

Let (I_1, θ_1) denote the canonical coordinate for the pendulum,

$$H_1 = \frac{1}{2}w^2 + \sin^2\psi,$$

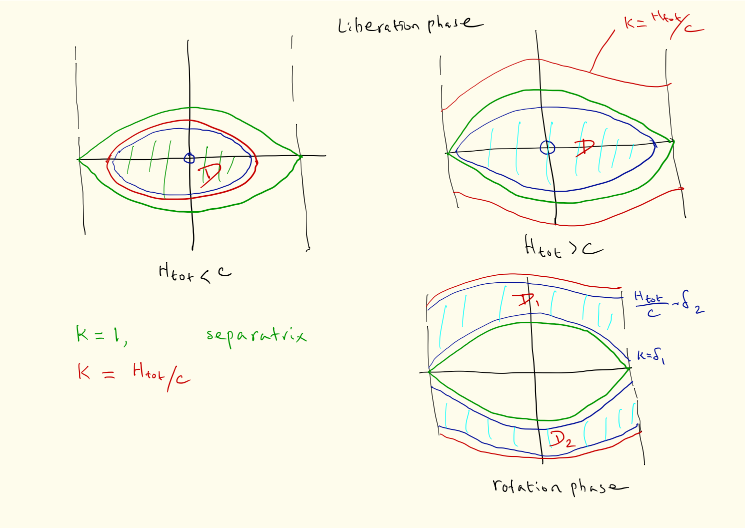
which divides into two phases: $H < C$ and $H > C$. Set $\kappa(I_1) := \frac{\tilde{H}_1(I_1)}{C}$. On $H < C$,

$$I_1 = \frac{4}{2\pi}\int_0^{\sin^{-1}\frac{\tilde{H}_1(I_1)}{C}} \sqrt{2\tilde{H}(I_1) - 2C\sin^2\psi\, d\psi}.$$

$$\theta_1 = \frac{d}{dI_1}\int_0^{\psi} \sqrt{2\tilde{H}_1(I_1) - 2C\sin^2\psi\, d\psi}.$$

We take the product canonical coordinates. Observe that $H_1 + H_2$ is a first integral.

Couple oscillator with pendulum



[xml+ Patching18+].

In canonical local coordinates, the perturbation vector field can be written with elliptic integrals, in 4 lines. The reduced equation in liberation phase is:

$$\begin{aligned} d\theta_1 &= \omega(I_1) dB_t^1 + b\tilde{K}_{\theta_1} dt \\ d\theta_1 &= dB_t^2 + b\tilde{K}_{\theta_2} dt \\ dI_1 &= b\tilde{K}_{I_1} dt \\ dI_2 &= b\tilde{K}_{I_2} dt. \end{aligned}$$

All functions are explicit, involving elliptic integrals. The drifts in the angle components are complicated involving both I_2, I_1 and θ_2. The invariant measures turns out to be the normalized Lebesgue measure. Using a result in [xml'08], [c.f. Stratonovich, A.D. Khasminski, M. Freidlin, D. Athreya 09,...], explicit limits can be obtained, with rate of convergence.

Rough exit time estimates

In the liberation phase,

$$T^b = \inf_{t\geq 0}\{\kappa(I_1(t)) = \delta_1, \text{ or } \kappa(I_1(t)) = (1 \wedge \frac{H_{tot}}{C}) - \delta_2\}.$$

Then, [xml+pathcing 18+]

$$T^b \geq \left(\frac{\sin^{-1}\sqrt{\frac{C}{H_{tot}}\kappa(0)} - \sin^{-1}\sqrt{\frac{C}{H_{tot}}\delta_1}}{b\sqrt{C/2}} \right)$$
$$\wedge \left(\frac{\sin^{-1}\sqrt{\frac{C}{H_{tot}}(1 \wedge \frac{H_{tot}}{C}) - \delta_2} - \sin^{-1}\sqrt{\frac{C}{H_{tot}}\kappa(0)}}{b\sqrt{C/2}} \right)$$

A separate formula is available in the rotation phase.

Effective limits

–Despite that K is not Hamiltonian, there is no visible movement of H_i on $[0, \frac{1}{b}]$. (for the liberating case:

$$\mathbf{E}\left\{\sup_{s\le t}\left|H_1(y_{\frac{s}{b}\wedge T^b}) - H_i(y_0)\right|\Big| T^b > t/b\right\} \le \frac{2c(t)b^{\frac{1}{4}}}{1 - b^{1/4}\frac{c(t)}{\tilde{c}}}.$$

–Within the liberaing phrase, $H_1^b(t/b^2)$ converges to

$$d\zeta_t = \sqrt{a}(\zeta_t)\circ dW_t + \gamma_1(\zeta_t)dt$$

$$a(I_1) = \frac{C\kappa I_2}{2K(\kappa)}\int_0^{u_0(\kappa)} sn(u,k)dn(u,k)\hat{\phi}(u,k)du.$$

$$\hat{\phi}(u;k) = [A_1(\kappa)+F_1(u,\kappa)e^{u/\sqrt{2C}} + [A_2(\kappa)+F_2(u,\kappa)]e^{-u/\sqrt{2C}}$$

$$\gamma_1 = \frac{1}{2}\int_{[0,2\pi]^2} L_K\Theta d\theta.$$

$$\Theta = \sqrt{2KI_2}\hat{\phi}(u;\kappa)\sin(\theta_2)$$