Shapes of local minimizers for the Alt-Caffarelli functional in inhomogeneous media

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Partially based on joint work with Charles Smart (Chicago)

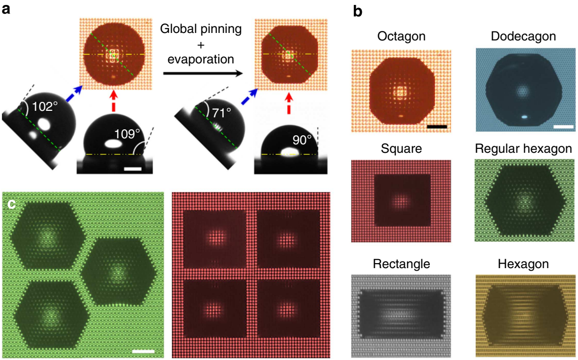
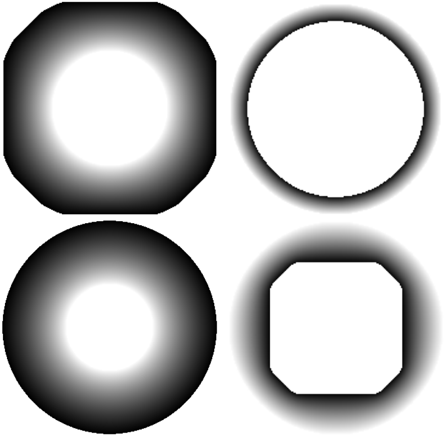
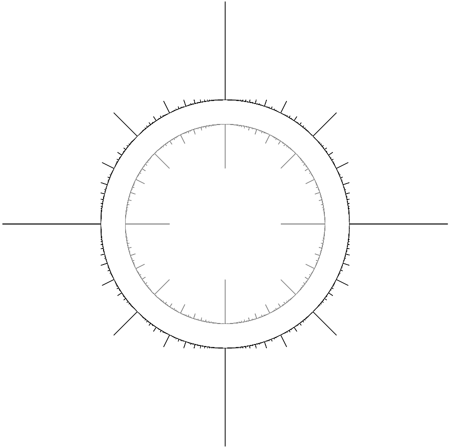


Figure: [Raj, Adera, Enright and Wang (Nat. Commun., 2014)]





Quasi-static problem

(*i*) An initial data u_0 is given and at each $t \geq 0$ the function u_t solves

$$(QS) \qquad \begin{cases} \Delta u_t = 0 & \text{in } \{u_t > 0\} \cap U \\ |\nabla u_t| \in [Q_*(n), Q^*(n)] & \text{on } \partial\{u_t > 0\} \cap U \\ u_t = F(t) & \text{on } \partial U \end{cases}$$

(*ii*) If F is monotone increasing (resp. decreasing) on the the interval $[a, b]$ then u_b is the minimal supersolution of the problem above (resp. maximal subsolution below) u_a.

Theorem (F.)

(1) *For each admissible initial data u_0 and forcing F (finitely many monotonicity changes) there is a unique solution of the quasi-static evolution (QS).*

(2) *Suppose that $K = \mathbb{R}^d \setminus U$ is a compact, convex, inner regular set and $\{u_0 > 0\}$ is convex. Then the solution of (QS) is convex for all $t > 0$.*

(3) *In the convex setting above, call $\Omega_p(t)$ to be the intersection of $\partial\{u_t > 0\}$ with the supporting hyperplane with normal p:*

   (i) *Suppose $F(t)$ is increasing. Then either $\Omega_p(t) = \Omega_p(0)$, $\Omega_p(t)$ is a singleton, or*
   $$Q^*(p) > \min\{Q^*_\ell(p), Q^*_r(p)\}.$$

   (ii) *Suppose $F(t)$ is decreasing. Then either $\Omega_p(t) \subset \Omega_p(0)$, $\Omega_p(t)$ is a singleton, or*
   $$Q_{*,\ell}(p) \neq Q_{*,r}(p).$$

Quasi-static problem

(*i*) An initial data u_0 is given and at each $t \geq 0$ the function u_t solves

$$(\mathrm{QS}_\varepsilon) \qquad \begin{cases} \Delta u_t^\varepsilon = 0 & \text{in } \{u_t^\varepsilon > 0\} \cap U \\ |\nabla u_t^\varepsilon| = Q(x/\varepsilon) & \text{on } \partial\{u_t^\varepsilon > 0\} \cap U \\ u_t^\varepsilon = F(t) & \text{on } \partial U \end{cases}$$

(*ii*) If F is monotone increasing (resp. decreasing) on the the interval $[a, b]$ then u_b is the minimal supersolution of the problem above (resp. maximal subsolution below) u_a.

Theorem (F.)

The following properties holds for the pinning interval endpoints:

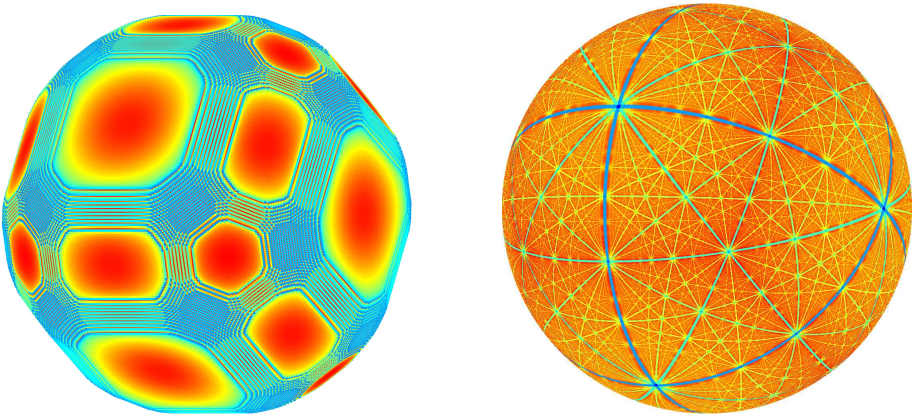
(1) *Let* $e \in S^{d-1}$ *there exist* $Q_*(e) \leq \langle Q^2 \rangle^{1/2} \leq Q^*(e)$, *respectively upper and lower semicontinuous in* e, *such that, for any* $\alpha \in [Q_*(e), Q^*(e)]$ *there exists a global solution of of the cell problem with slope* αe.

(2) *When* $d = 2$, Q^*, Q_* *are continuous at irrational directions* $e \in S^1 \setminus \mathbb{R}\mathbb{Z}^2$.

(3) *When* $d = 2$, *directional limits of* Q^*, Q_* *exist at rational directions* $e \in S^1 \cap \mathbb{R}\mathbb{Z}^2$.

(4) *Given any* k*-dimensional rational subspace,* $1 \leq k \leq d-1$, *there exists* Q *such that* Q^*, Q_* *are discontinuous on that subspace.*

(5) *There exist* Q *such that the pinning interval is nontrivial at every direction,* $\inf_{S^{d-1}}(Q^* - Q_*) \geq \delta > 0$.



Thank you for your attention!