Min-max formulas for nonconvex effective Hamiltonians and application to stochastic homogenization

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Main goals

Main goals: **qualitative properties, representation formulas** of the effective Hamiltonian in homogenization theory (in both periodic and general stationary ergodic settings).

Homogenization theory aims to identify and study macroscopic behavior of PDEs which typically have high oscillations in the space (or time-space) variables (cf. e.g (1)) for instance. Basic problems include (I) well-posedness: obtaining the existence of limiting effective equations (cf. e.g. (1)) as $\epsilon \to 0$; (II) understanding finer properties of the limiting process and the effective equation. PDEs are usually set in self-averaging (periodic, almost periodic or random) environments. In the periodic setting, (I) is quite well established for some nonlinear PDEs such as first-order and second-order Hamilton-Jacobi equations, fully nonlinear elliptic equations. However, very little is known about (II) because of the nonlinear nature in these equations. We propose to develop some new tools and directions to study (II) in various aspects systematically.

Periodic homogenization of Hamilton-Jacobi equations

For each $\epsilon > 0$, let $u^\epsilon \in C(\mathbb{R}^n \times [0,\infty))$ be the viscosity solution to the following Hamilton-Jacobi equation

$$\begin{cases} u^\epsilon_t + H(Du^\epsilon) - V\left(\frac{x}{\epsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\epsilon(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1}$$

Here the Hamiltonian is of separable form with $H \in C(\mathbb{R}^n)$, which is coercive, and $V \in C(\mathbb{R}^n)$, which is $\mathbb{Z}^n$-periodic.

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It was known (Lions-Papanicolaou-Varadhan, 1987), that u^ϵ, as $\epsilon \to 0$, converges locally uniformly to u, the solution of the effective equation,

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{2}$$

$\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is called the effective Hamiltonian.

The effective Hamiltonian $\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is determined by the cell problems as follows. For any $p \in \mathbb{R}^n$, there exists a UNIQUE number $\overline{H}(p)$ such that the following equation

$$H(p + Dv) - V(y) = \overline{H}(p) \quad \text{in } \mathbb{R}^n, \tag{3}$$

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However, in general, deep properties of $\overline{H}$ are not known so much, especially in the nonconvex setting.

Two questions for today:

1. Qualitative properties of $\overline{H}$ (e.g., if H is even, then is $\overline{H}$ even?).
2. Representation formulas for $\overline{H}$ and applications?

Convex/Quasiconvex setting

When H is convex/quasiconvex, then $\overline{H}$ is also convex/quasiconvex.
Representation formula: inf-max formula

$$\overline{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{y \in \mathbb{T}^n} \left(H(p + D\phi(y)) - V(y) \right).$$

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Hence, if H is even, then $\overline{H}$ is even as

$$\begin{aligned}
\overline{H}(p) &= \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{y \in \mathbb{T}^n} \left(H(p + D\phi(y)) - V(y) \right) \\
&= \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{y \in \mathbb{T}^n} \left(H(-p - D\phi(y)) - V(y) \right) \\
&= \inf_{\psi \in C^1(\mathbb{T}^n)} \max_{y \in \mathbb{T}^n} \left(H(-p + D\psi(y)) - V(y) \right) = \overline{H}(-p).
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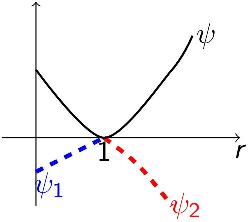
Do we have the evenness property holds true in general?

New results - Simplest case



Let $H(p) = \psi(|p|)$, $H_1(p) = \psi_1(|p|)$, and $H_2(p) = \psi_2(|p|)$, Let $\overline{H}, \overline{H}_1, \overline{H}_2$ be the effective Hamiltonians corresponding to $H - V$, $H_1 - V$, $H_2 - V$, respectively.

Theorem 1 (Qian - T. - Yu)

Assume the above, and $V \in C(\mathbb{T}^n)$ *with* $\min V = 0$. *Then*

$$\overline{H} = \max\left\{\overline{H}_1, \overline{H}_2, 0\right\}.$$

Important consequences

- $\overline{H}$ is even as $\overline{H}_1, \overline{H}_2$ are even.
- In case that $\min V = 0$ and $\max V \geq \psi(0)$, then $\overline{H}_2 \leq 0$. We then get

$$\overline{H} = \max\left\{\overline{H}_1, 0\right\},$$

and thus, $\overline{H}$ is quasiconvex. Strong V makes $\overline{H}$ better.

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- In case that $\min V = 0$ and $\max V \geq \psi(0)$. Let $K : \mathbb{R}^n \to \mathbb{R}$ such that $K(p) = H(p) = \psi(|p|)$ for $|p| \geq 1$, and $0 \leq K(p) \leq H(p)$ for $|p| \leq 1$. Let $\overline{K}$ be the effective Hamiltonian corresponding to $K - V$. Then we always have

$$\overline{K} = \overline{H}.$$

Some ideas in the proof

• It is clear that $\overline{H} \geq \overline{H}_i$ as $H \geq H_i$ for $i = 1, 2$. Also, $\overline{H} \geq 0$ and $\overline{H}(p) = 0$ for $|p| = 1$. Therefore, $\overline{H} \geq \max\{\overline{H}_1, \overline{H}_2, 0\}$.

• Pick $p \in \mathbb{R}^n$ such that $\overline{H}_1(p) \geq \max\{\overline{H}_2(p), 0\}$. We show

$$\overline{H}_1(p) \geq \overline{H}(p).$$

As $\overline{H}_1$ is even, $\overline{H}_1(-p) = \overline{H}_1(p)$. Let $v(y, -p)$ be a solution to

$$H_1(-p + Dv(y, -p)) - V(y) = \overline{H}_1(-p) = \overline{H}_1(p) \quad \text{in } \mathbb{T}^n.$$

Let $w(y) = -v(y, -p)$. If $q \in D^+w(y)$, then $-q \in D^-v(y, -p)$ and

$$0 \leq \overline{H}_1(p) = H_1(-p - q) - V(y) = H_1(p + q) - V(y) = H(p + q) - V(y).$$

Thus, w is a subsolution, and the conclusion follows.

Some ideas in the proof

• Pick $p \in \mathbb{R}^n$ such that $\overline{H}_2(p) \geq \max\{\overline{H}_1(p), 0\}$. This is similar to the above, but we just need to choose $w(y) = v(y, p)$.

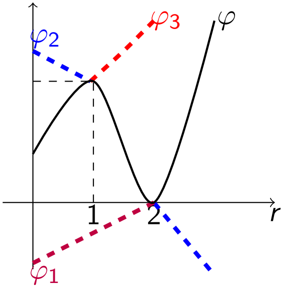
• **Gluing step.** Assume $\max\{\overline{H}_1(p), \overline{H}_2(p)\} < 0$. We show $\overline{H}(p) = 0$. For $\sigma \in [0,1]$, let $\overline{H}^\sigma, \overline{H}_i^\sigma$ be effective Hamiltonians corresponding to $H(p) - \sigma V(y)$, $H_i(p) - \sigma V(y)$, respectively. It is clear that

$$0 \leq \overline{H} = \overline{H}^1 \leq \overline{H}^\sigma \quad \text{for all } \sigma \in [0,1].$$

WLOG, assume $|p| > 1$. As $H_1(p) = \overline{H}_1^0(p) > 0$ and $\overline{H}_1(p) = \overline{H}_1^1(p) < 0$, we can find $s \in (0,1)$ such that $\overline{H}_1^s(p) = 0$. It is clear that $\overline{H}_2^s(p) \leq H_2(p) < 0$ there. Thus,

$$\max\{\overline{H}_1^s(p), \overline{H}_2^s(p)\} = 0 \Rightarrow \overline{H}^s(p) = 0 \Rightarrow \overline{H}(p) = 0.$$

New results - A bit more general case

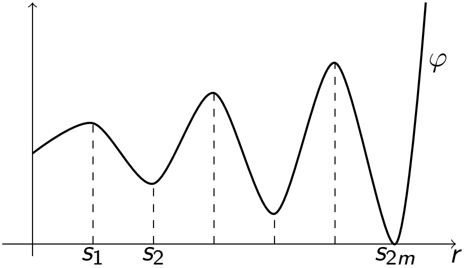


Theorem 2 (Qian-T.-Yu)

Let $H(p) = \varphi(|p|)$ *and* $V \in C(\mathbb{T}^n)$. *Then,*

$$\overline{H} = \max\left\{0, \overline{H}_1, \min\left\{\overline{H}_2, \overline{H}_3, \varphi(1) - \max V\right\}\right\}.$$

New results - Most general case



Theorem 3 (Qian-T.-Yu)

Let $H(p) = \varphi(|p|)$. *Then we have a representation formula*

$$\overline{H} = \max\ \min\ \max\ \min \dots$$

Related results: Armstrong-T.-Yu (2013, 2014), Gao (2015, 2018).

Application: Random homogenization of first-order HJ eqn

- Convex/quasi-convex case: Souganidis (1999) and Rezakhanlou-Tarver (2000). Davini-Siconolfi (2009), Armstrong-Souganidis (2013).

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• **Major open problem.** Random homogenization of nonconvex case?
- Specific n-D case: $H(p) = (|p|^2 - 1)^2$ (Armstrong -T.-Yu (2013)).
- One dimension: Armstrong -T.-Yu (2014) for general separable case $H(p) - V(x)$, Gao (2015) for general non-separable case.
- Counter-example: If H has strict saddle point, then there exists V such that homogenization does not hold (Ziliotto (2016), Feldman-Souganidis (2016)).
- In i.i.d setting: Armstrong-Cardaliaguet (2015), Feldman-Souganidis (2016) for k-positively homogeneous H.
- Nonconvex viscous case: Seems harder. Results in one dimension by Davini-Kosygina (2017), Kosygina-Yilmaz-Zeitouni (2017).

Application: Random homogenization - New results

Theorem 4 (Qian-T.-Yu)

Random homogenization holds for all H mentioned in Theorems 1–3. Moreover, representation formulas also hold true.
Furthermore, homogenization holds for K mentioned in the consequences of Theorem 1 too. That means that even if K has saddle points of any kind, if V has large enough oscillation, it is fine.

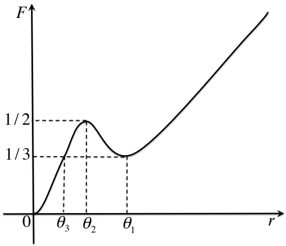
Key philosophy. Knowledge on $\overline{H}$ (in periodic setting) helps to recover same formulas in random setting, and overcome the lack of compactness.

Counterexamples of Ziliotto (2016), Feldman-Souganidis (2016) require V to have small oscillation to see the local structure of the saddle points. When the oscillation of V is large enough, we do not see such local structure and hence we still have the averaging effect in certain cases.

More or less, the whole set of new developments gives rather clear answers to the open question in the general stationary ergodic setting.

Lost of evenness

Let $H : \mathbb{R} \to [0, +\infty)$ be such that $H(p) = F(|p|)$.



Take V such that $\min V = 0$, and $\max V = 1$, then $\overline{H}$ is not even. This is because of hysteresis and the non-symmetric gradient jump structures.
We cannot hope for the same kind of representation formula as earlier...
We still have that $\overline{H}$ is quasiconvex here! Is $\overline{H}$ quasiconvex in multi-dimensional setting?

One numerical example

Let $n = 2$. We consider the following setting

$$H(p) = \min\left\{4\sqrt{p_1^2 + p_2^2},\ 2\left|\sqrt{p_1^2 + p_2^2} - 1\right| + 1\right\} \quad \text{for } p = (p_1, p_2) \in \mathbb{R}^2,$$

$$V(x) = S \cdot (1 + \sin(2\pi x_1))(1 + \sin(2\pi x_2)) \quad \text{for } x = (x_1, x_2) \in \mathbb{T}^2.$$

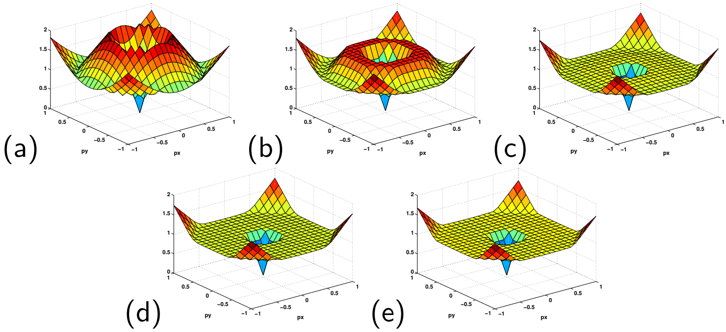


Figure: (a) $S = 0$. (b) $S = 0.125$. (c) $S = 0.250$. (d) $S = 0.30$. (e) $S = 0.50$.

THANK YOU