Quantitative homogenization in non-linear elasticity

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Homogenisation in Disordered Media
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Outline

nonlinear elasticity and homogenization

validity of one-cell formula for small strains

quantitative two-scale expansion

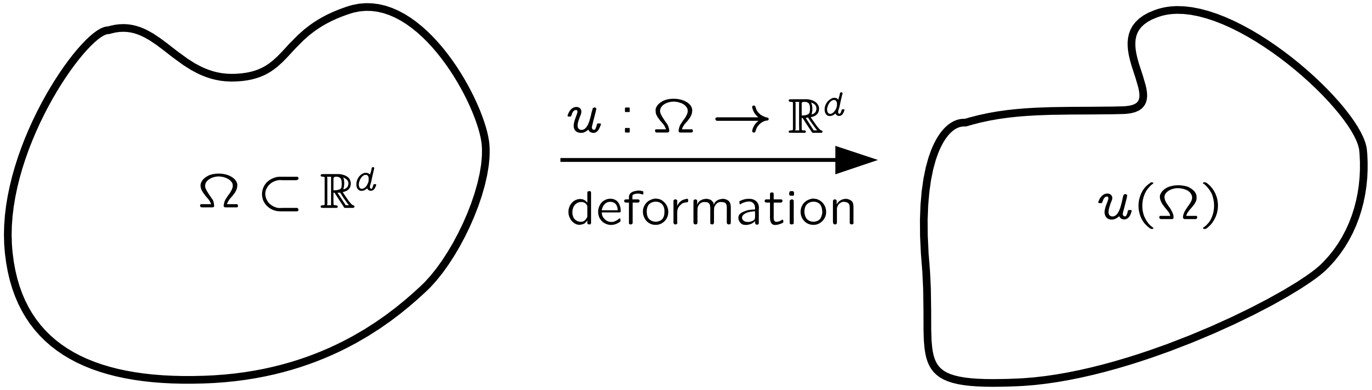
homogenization and linearization at strained equilibrium

uniform Lipschitz estimates

$$\mathcal{I}(u) := \int_\Omega W(\nabla u) - f \cdot u \qquad \text{(elastic energy functional)},$$

where W is a non-convex energy density s.t.

- $W(F) = W(RF)\ \forall F \in \mathbb{R}^{d\times d}$, $R \in SO(d)$ (**frame indifferent**)
- $W(Id) = \min W = 0$ (**reference configuration = natural state**)
- $W(F) \geq \alpha\, \mathrm{dist}^2(F, SO(d))\ \forall F \in \mathbb{R}^{d\times d}$ (**non-degeneracy**)



$$\mathcal{I}_\varepsilon(u) := \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u(x))\, dx$$

Suppose: $W(y,F)$ $\square$-periodic in y; & p-growth ($p > 1$)

Theorem: [Müller ARMA'87, Braides RAN'85]

$\mathcal{I}_\varepsilon$ Γ-converges to $\mathcal{I}_{\text{hom}}(u) := \int_\Omega W_{\text{hom}}(\nabla u(x))\, dx$.

- W **convex** $\Rightarrow$ single-cell hom. formula & **corrector**

$$W_{\text{hom}}(F) = W^{(1)}(F) := \min_{\phi \in W^{1,p}_{\text{per}}(\square)} ⨍_\square W(y, F+\nabla\phi(y)) = ⨍_\square W(y, F+\nabla\phi_F(y))$$

- W **non-convex** $\Rightarrow$ multi-cell hom. formula & **no corrector**

$$W_{\text{hom}}(F) := \inf_{k\in\mathbb{N}} \inf_{\phi \in W^{1,p}_{\text{per}}(k\square)} ⨍_{k\square} W(y, F+\nabla\phi(y))$$

notion of corrector $\nabla\phi_F$ = starting point for
- effective properties analysis
- (large scale) regularity
- quantitative homogenization
- ...

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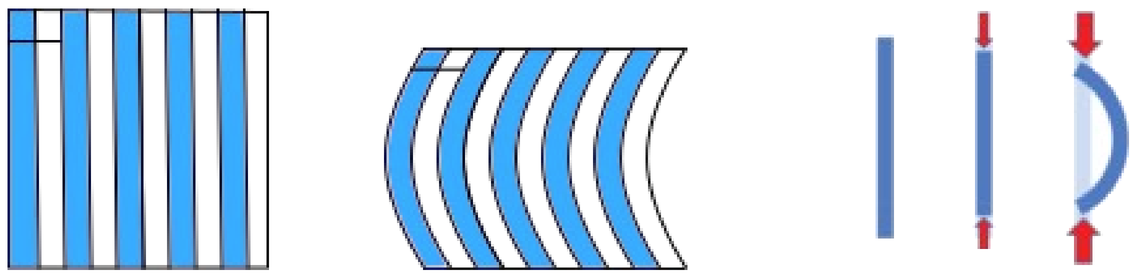
Can we have $W_{\text{hom}}(F) = W^{(1)}(F)$ and a corrector?

☒ **macroscopic instability by buckling of microstructure**

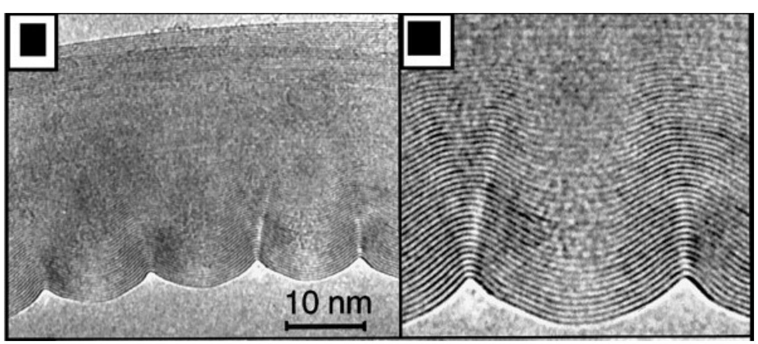
Layered (stiff/soft) elastic two-phase composite

$$\Rightarrow \quad \exists F \text{(compression) s.t.} \qquad W_{\text{hom}}(F) < W^{(1)}(F)$$



[Müller ARMA'87] [Geymonat, Müller, Triantafyllidis: ARMA'93]

Buckling pattern of a bended multi-walled nanotube (TEM image)



[Lourie et al., Phys.Rev.Lett.'98]

[N.: (PhD-Thesis) '10], [Müller & N.: ARMA '11], [Gloria & N.: AIHP'11]

☑ **Homogenization and linearization commute at Id**

On the level of the energy density

$$\begin{aligned} W(y, Id+G) &= Q(y,G) + o(|G|^2) \\ \Rightarrow \qquad W_{\mathrm{hom}}(Id+G) &= Q^{(1)}(G) + o(|G|^2) \end{aligned}$$

On the level of energy functionals

$$\begin{array}{ccc} \frac{1}{h^2}\int W(\frac{x}{\varepsilon}, Id + h\nabla\varphi) & \xrightarrow{\text{lin}} & \int Q(\frac{x}{\varepsilon}, \nabla\varphi) \\ \big\downarrow \text{hom} & & \big\downarrow \text{hom} \\ \frac{1}{h^2}\int W_{\mathrm{hom}}(Id + h\nabla\varphi) & \xrightarrow{\text{lin}} & \int Q^{(1)}(\nabla\varphi) \end{array}$$

Similar statement for $Id \rightsquigarrow F \notin SO(d)$?

Main result

validity of one-cell formula

quantitative two-scale expansion

homogenization and linearization at strained equilibrium

Validity of the single-cell formula for small strains

Assumption **(A)** on $W : \mathbb{R}^d \times \mathbb{R}^{d\times d} \to [0, +\infty]$:

Let $p \geq d$ and $\alpha > 0$. Suppose that

- $W(y, F)$ is $\square$-periodic in y,
- $W(y, RF) = W(y, F)$ $\forall R \in d$ (**frame indifference**),
- $W(y,) = \min W = 0$ (**reference configuration = natural state**),
- $W(y, F) \geq \alpha \,\mathrm{dist}^2(F, d)$ (**non-degeneracy**),

- $W(y, \cdot)$ is C^3 close to $SO(d)$ (**regularity in** F),
- $\alpha |F|^p - \frac{1}{\alpha} \leq W(y, F)$ (**growth from below**)

Theorem 1: (validity of the one-cell formula) [N. & Schäffner ARMA '18]

Suppose **(A)** and a **regularity condition (R)** on microstructure.

Then $\exists \rho > 0$ s.t. for all $F \in \mathbb{R}^{d\times d}$ with $\mathrm{dist}(F, SO(d)) \leq \rho$:

- (single-cell formula and corrector).

$$W_{\mathrm{hom}}(F) = W^{(1)}(F) = \fint_{\Box} W(y, F + \nabla\phi_F)\, dy$$

for a corrector $\phi_F \in W^{1,p}_{\mathrm{per}}(\Box)$ (unique up to a constant).

- (regularity of W_{hom}).
W_{hom} is C^3 in neighborhood to $SO(d)$ and

$$DW_{\mathrm{hom}}(F)[G] = \int_{\Box} DW(y, F + \nabla\phi_F)[G]\, dy$$

$$D^2W_{\mathrm{hom}}(F)[G, G] = \inf_{\psi \in H^1_{\mathrm{per}}(\Box)} \int_{\Box} D^2W(y, F + \nabla\phi_F)[G + \nabla\psi, G + \nabla\psi]\, dy$$

Validity of the single-cell formula for small strains

Theorem 1: (validity of the one-cell formula) [N. & Schäffner ARMA '18]

Suppose **(A)** and a **regularity condition (R)** on microstructure.

Then ∃

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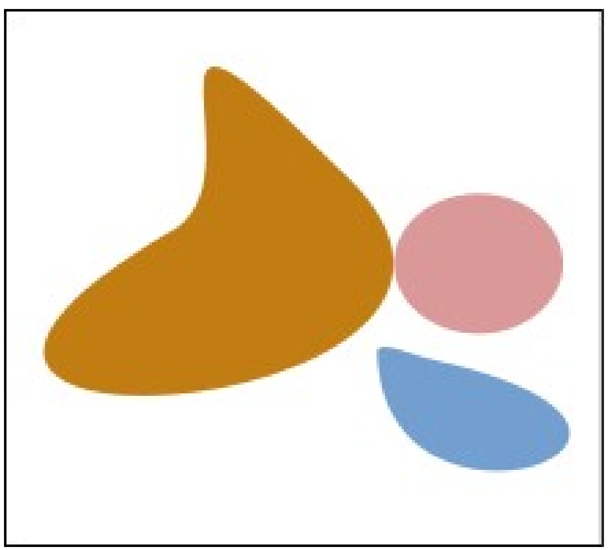
Problem: (R) ⇒ Lipschitz estimate for monotone elliptic systems

We discuss 3 variants for (R) :

(R1) (smooth): W is C^3 in neighbourhood of $\mathbb{R}^d \times SO(d)$.

(R2) (laminate): $W(y, F) = W(y_d, F)$.

(R3) (piecewise smooth composite)



Geometry of microstructure as in [Li & Nirenberg: CPAM'03] (Lipschitz estimates linear systems)

Consider

$$\mathcal{I}_\varepsilon(u) := \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u) - f\cdot u\,dx,$$

$$\mathcal{I}_{\text{hom}}(u) := \int_\Omega W_{\text{hom}}(\nabla u) - f\cdot u\,dx,$$

subject to affine boundary condition $u(x) = Gx$ on $\partial\Omega$ (BC)

Theorem 2: (Quantitative two-scale expansion) [N. & Schäffner ARMA'18]

Let $r > d$. There exists $\bar\rho > 0$. Suppose smallness of the data in form of

$$\Lambda(f,G) := \|f\|_{L^r(\Omega)} + \operatorname{dist}(G, SO(d)) \le \bar\rho.$$

(a) $\mathcal{I}_{\text{hom}}$ admits a unique minimizer $u_0 \in W^{1,p}(\Omega)$ subject to (BC).

(b) Every minimizer $u_\varepsilon \in W^{1,p}_0(\Omega)$ of $\mathcal{I}_\varepsilon$ subject to (BC) satisfies

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} + \|u_\varepsilon - (u_0 + \varepsilon\nabla\phi_{\nabla u_0}(\tfrac{\cdot}{\varepsilon}))\|_{H^1(\Omega)} \lesssim \varepsilon^{\frac{1}{2}}\Lambda(f,G).$$

Consider

$$\mathcal{I}_\varepsilon(u) := \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u) - f \cdot u \, dx,$$

$$\mathcal{I}_{\text{hom}}(u) := \int_\Omega W_{\text{hom}}(\nabla u) - f \cdot u \, dx,$$

subject to $u = g$ on $\partial\Omega$ (BC)

Theorem 2: (Quantitative two-scale expansion) [N. & Schäffner ARMA'18]

Let $r > d$. There exists $\bar\rho > 0$. Suppose smallness of the data,

$$\Lambda(f, g, g_0) := \|f\|_{L^r(\Omega)} + \|g - g_0\|_{W^{2,r}(\Omega)} + \|\operatorname{dist}(\nabla g_0, SO(d))\|_{L^\infty(\Omega)} \leq \bar\rho.$$

where $g_0 \in W^{2,r}(\mathbb{R}^d)$ satisfies $-\operatorname{div} DW_{\text{hom}}(\nabla g_0) = 0$.

(a) $\mathcal{I}_{\text{hom}}$ admits a unique minimizer $u_0 \in W^{1,p}(\Omega)$ subject to (BC).

(b) All $u_\varepsilon \in W^{1,p}_0(\Omega)$ with (BC) satisfy

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} + \|u_\varepsilon - (u_0 + \varepsilon \nabla\phi_{\nabla u_0}(\tfrac{\cdot}{\varepsilon}))\|_{H^1(\Omega)}$$
$$\lesssim \varepsilon^{\frac{1}{2}} \Lambda(f, g, g_0) + (\mathcal{I}_\varepsilon(u_\varepsilon) - \inf \mathcal{I}_\varepsilon)^{\frac{1}{2}}$$
$$+ \varepsilon(1 + \|\nabla^2 g_0\|_{L^r(\Omega)}^{\frac{r}{r-d}})(\|\nabla^2 g_0\|_{L^r(\Omega)} + \Lambda(f, g, g_0)).$$

Consider

$$\mathcal{I}_\varepsilon(u) := \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u) - f \cdot u \, dx,$$

$$\mathcal{I}_{\mathrm{hom}}(u) := \int_\Omega W_{\mathrm{hom}}(\nabla u) - f \cdot u \, dx,$$

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Theorem 2: (Quantitative two-scale expansion) [N. & Schöffner ARMA'18]

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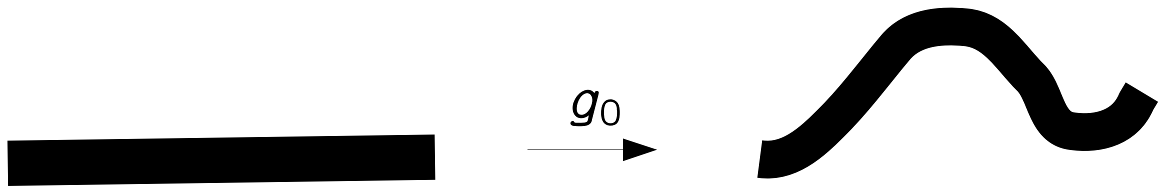


$\Omega := (0,1) \times (-\frac{h}{2}, \frac{h}{2})$

[N. & Schäffner ARMA'18]

Our result implies: For $\operatorname{dist}(F, SO(d)) \ll 1$, $u_\varepsilon(x) := Fx + \varepsilon\phi_F(\frac{x}{\varepsilon})$ is a equilibrium state (with positive energy):

$$-\nabla \cdot DW(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon(x)) = 0 \qquad \mathcal{D}'(\Omega).$$

Motivated by this, rewrite the elastic energy in terms of a displacement $\varphi \in W_0^{1,p}(\Omega)$ relative to u_ε:

$$\begin{aligned}
\mathcal{G}_{\varepsilon,h}(\varphi) &:= \frac{1}{h^2}\int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon + h\nabla\varphi) - W(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon)\,dx\\
\mathcal{G}_{\mathrm{hom},h}(\varphi) &:= \frac{1}{h^2}\int_\Omega W_{\mathrm{hom}}(F + h\nabla\varphi) - W_{\mathrm{hom}}(F)\,dx\\
\mathcal{G}_{\varepsilon,\mathrm{lin}}(\varphi) &:= \int_\Omega \frac{1}{2}D^2W(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon(x))[\nabla\varphi]^2\,dx\\
\mathcal{G}_{\mathrm{hom},\mathrm{lin}}(\varphi) &:= \int_\Omega \frac{1}{2}D^2W_{\mathrm{hom}}(F)[\nabla\varphi]^2\,dx
\end{aligned}$$

[N. & Schäffner ARMA'18]

Corollary 1:

$$\begin{array}{ccc} \mathcal{G}_{\varepsilon,h} & \xrightarrow{\text{lin}} & \mathcal{G}_{\varepsilon,\text{lin}} \\ \big\downarrow \text{hom} & & \big\downarrow \text{hom} \\ \mathcal{G}_{\text{hom},h} & \xrightarrow{\text{lin}} & \mathcal{G}_{\text{hom},\text{lin}} \end{array}$$

commutes w.r.t. $\Gamma(L^2)$-convergence

…ist$(F, SO(d)) \ll 1$, $u_\varepsilon(x) := Fx + \varepsilon\phi_F(\frac{x}{\varepsilon})$
…ith positive energy):

…$(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)) = 0 \qquad \mathcal{D}'(\Omega)$.

…e the elastic energy in terms of a displace-
…ive to u_ε:

$$\begin{aligned}
\mathcal{G}_{\varepsilon,h}(\varphi) &:= \frac{1}{h^2}\int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon + h\nabla\varphi) - W(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon)\,dx \\
\mathcal{G}_{\text{hom},h}(\varphi) &:= \frac{1}{h^2}\int_\Omega W_{\text{hom}}(F + h\nabla\varphi) - W_{\text{hom}}(F)\,dx \\
\mathcal{G}_{\varepsilon,\text{lin}}(\varphi) &:= \int_\Omega \frac{1}{2}D^2W(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon(x))[\nabla\varphi]^2\,dx \\
\mathcal{G}_{\text{hom},\text{lin}}(\varphi) &:= \int_\Omega \frac{1}{2}D^2W_{\text{hom}}(F)[\nabla\varphi]^2\,dx
\end{aligned}$$

Comments on the proof of Theorem 1 (one-cell formula)

- Matching convex lower bound
- Lipschitz estimates for monotone systems

Lemma (matching convex lower-bound) [N. & Schäffner ARMA'18]

If **(A)**, then $\exists \beta, \mu, \rho > 0$ and $V : \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ s.t.

- $V(y,F)$ strongly β-convex in F, periodic in y
- $\beta|F|^2 - \frac{1}{\beta} \leq V(y,F) \leq \frac{1}{\beta}(1+|F|^2)$
- $V(y,\cdot) \in C^3(\mathbb{R}^{d\times d})$
- matching and lower-bound property

$$
\begin{aligned}
W(y,F) + \mu \det(F) &\geq V(y,F) \quad \text{for all } F \in \mathbb{R}^{d\times d} \\
W(y,F) + \mu \det(F) &= V(y,F) \quad \text{for } \operatorname{dist}(F, SO(d)) \leq \rho
\end{aligned}
$$

Variant of [Friesecke & Theil: J.Nonl.Sci.'02], [Conti et al: JEMS'06] (context: atomistic modeling).

$$\det(\cdot) \text{ Null-Lagrangian} \Rightarrow \big(W + \mu \det\big)_{\text{hom}} = W_{\text{hom}} + \mu \det$$

Relate W_{hom} and V_{hom}

- Poly-convex lower bound:

$$W_{\mathrm{hom}}(F) \geq V^{(1)}_{\mathrm{hom}}(F) - \mu \det(F) \quad \text{for all } F \in \mathbb{R}^{d\times d}$$

- Exploit matching property:

$$\|\operatorname{dist}(F + \nabla\phi_F, SO(d))\|_{L^\infty(\square)} < \delta$$
$$\Rightarrow \quad W_{\mathrm{hom}}(F) = V^{(1)}_{\mathrm{hom}}(F) - \mu \det F = W^{(1)}_{\mathrm{hom}}(F).$$

- Energy estimate:

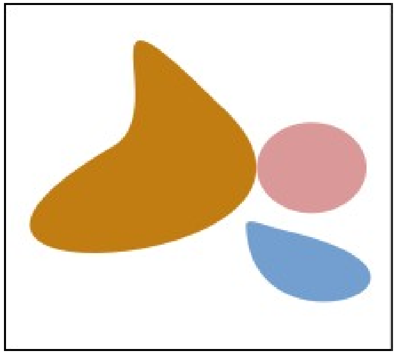
$$\|\operatorname{dist}(F + \nabla\phi_F, SO(d))\|_{L^2(\square)} \lesssim \operatorname{dist}(F, SO(d))$$

(not enough ☒).

- Exploit **regularity condition (R)** to get Lipschitz estimate for ϕ_F:

$$\operatorname{dist}(F, SO(d)) \ll 1 \Rightarrow \|\operatorname{dist}(F + \nabla\phi_F, SO(d))\|_{L^\infty(\square)} \lesssim \operatorname{dist}(F, SO(d)).$$

(R3) (piecewise smooth composite)



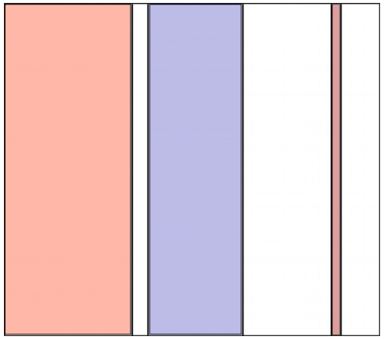
Geometry of microstructure as in
[Li & Nirenberg: CPAM'03]
(Lipschitz estimates linear systems)

[Byun, Ryu & Wang '10, Byun & Kim '16 & '17]
(L^p Gradient estimates for linear systems
and scalar monotone equations)

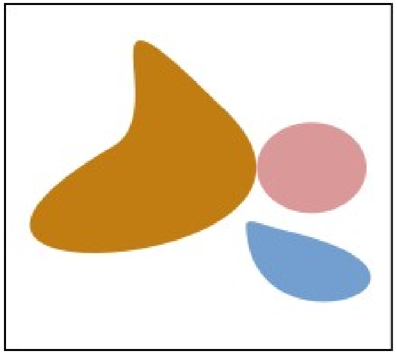
Comments on the proof of Theorem 1 (one-cell formula)

Matching convex lower bound

→ Lipschitz estimates for monotone systems



- $\mathcal{D} := \{D_\ell\}_{\ell\in\mathbb{Z}}$ is a *laminate*, if $D_\ell = \{x \in \mathbb{R}^d \,:\, h_\ell < x\cdot e < h_{\ell+1}\}$ for a direction $e \in \mathbb{R}^d$ and a strictly monotone sequence $\{h_\ell\}_{\ell\in\mathbb{Z}}$



- $\mathcal{D} := \{D_\ell\}_{\ell\in\mathbb{Z}}$ is *(E,s)-regular* (where $0 < s \le 1$ and $E < \infty$), if it is a mutually disjoint partition of $\mathbb{R}^d$ and for all $x \in \mathbb{R}^d$ there exists a laminate $\mathcal{D}_x$ s.t.

$$\sup_{0<r} r^{-s}\Big(|B_r|^{-1}\sum_{\ell\in\mathbb{Z}}\Big|(D_\ell \triangle D'_{x\ell})\cap B_r(x)\Big|\Big)^{\frac{1}{2}} \le E,$$

where $\triangle$ denote the symmetric difference.

Strongly elliptic monotone system of class $\mathcal{A}_\beta$ (with $\beta \in (0, 1]$)
$\boldsymbol{a} \in \mathcal{A}_\beta$ iff $\boldsymbol{a} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ satisfies for all $F, G \in \mathbb{R}^{d\times d}$

$$
\begin{aligned}
\boldsymbol{a}(0) &= 0 \\
\beta|F - G|^2 &\leq \langle \boldsymbol{a}(F) - \boldsymbol{a}(G), F - G\rangle \\
\beta|\boldsymbol{a}(F) - \boldsymbol{a}(G)| &\leq |F - G| \\
\beta|D\boldsymbol{a}(F) - D\boldsymbol{a}(G)| &\leq \omega(|F - G|) \qquad \text{with } \omega(t) = \max\{t, 1\}
\end{aligned}
$$

[N. & Schäffner]

Proposition 1: (Local Lipschitz estimate for monotone systems)
Suppose $\boldsymbol{a}$ is a *(E, s)-regular coefficient field of class $\mathcal{A}_\beta$*
Given $q > d$, $\exists\ \overline{\kappa} > 0$ and $c \in [1, \infty)$ such that:
Suppose $u \in H^1(B_1)$ and $f \in L^q(B_1)$ satisfy

$$-\nabla \cdot \boldsymbol{a}(x, \nabla u) = f \qquad \text{in } \mathscr{D}'(B_1),$$

and the smallness condition

$$\|f\|_{L^q(B_1)} + \|\nabla u\|_{L^2(B_1)} \leq \begin{cases} \infty & \text{if } d = 2 \\ \overline{\kappa} & \text{if } d \geq 3 \end{cases}.$$

Then,

$$\|\nabla u\|_{L^\infty(B_{\frac{1}{2}})} \leq c(\|\nabla u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}).$$

- homogeneous case ($\boldsymbol{a}(x, F) = \boldsymbol{a}(F)$):
  ε-regularity statements: (see e.g. textbooks [Giaquinta], [Giusti])
  $\forall\alpha \in (0,1)$ $\exists\varepsilon > 0$ such that for any u $\boldsymbol{a}$-harmonic:

$$E(\nabla u, B_R) := \min_a \left(\fint_{B_R} |\nabla u - a|^2 \right)^{\frac{1}{2}} \leq \varepsilon \qquad \text{(smallness condition)}$$

$$\Rightarrow E(\nabla u, B_r) \lesssim \left(\frac{r}{R}\right)^{\alpha} E(\nabla u, B_R) \qquad \text{(excess decay)}$$

- e_d-layered case:
  - controll decay of $\nabla' u$ and $\boldsymbol{a}(\cdot, \nabla u) e_d$
  - combine with strong ellipticity $\Rightarrow$ excess decay of ∇u
- (E, s)-regular coefficients:
  ...are locally close to layered coefficients. Perturbation argument in the spirit of [Kuusi & Mingione '12], [Byun & Kim '17]

Application: Uniform Lipschitz estimates

- Lipschitz-estimate (strongly-elliptic, constant-coefficient system):

$$-\nabla\cdot\mathbb{L}\nabla u = 0 \quad \text{in } \mathscr{D}'(B) \quad \Rightarrow \quad \|\nabla u\|^2_{L^\infty(\frac{1}{2}B)} \le c \fint_B |\nabla u|^2$$

- [Avellaneda, Lin '87]
  Consider $\mathbb{L} \in C^{0,\alpha}(\mathbb{R}^d, \mathbb{R}^{d^4})$ is periodic & uniformly elliptic.

  Homogenization: $(-\nabla\cdot\mathbb{L}(\frac{\cdot}{\varepsilon})\nabla)^{-1} \to (-\nabla\cdot\mathbb{L}_{\mathrm{hom}}\nabla)^{-1}$

  Philosophy: **Lift good regularity of $\mathbb{L}_{\mathrm{hom}}$ to $\mathbb{L}_\varepsilon$.**

  Uniform estimate: $\exists c < \infty$ such that for all $\varepsilon \in (0,1)$:

$$-\nabla\cdot(\mathbb{L}_\varepsilon\nabla u) = 0 \quad \text{in } \mathscr{D}'(B) \quad \Rightarrow \quad \|\nabla u\|^2_{L^\infty(\frac{1}{2}B)} \le c \fint_B |\nabla u|^2$$

- Recent developments: **stochastic homogenization and large-scale regularity**, e.g., [Armstrong, Smart '14], [Gloria, Neukamm, Otto '14], [Armstrong, Mourrat '16], ...

Consider

$$\mathcal{I}_\varepsilon(u) \quad := \quad \int_\square W(\tfrac{x}{\varepsilon}, \nabla u) - f \cdot u \, dx,$$

subject to periodic boundary condition $u \in Gx + W^{1,p}_{\mathrm{per}}(\square)$ (pBC)

Theorem 3: (Uniform Lipschitz estimate) [N. & Schäffner]

Let $q > d$ and $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$. There exists $\bar{\rho} > 0$. Suppose smallness of the data in form of

$$\Lambda(f, G) := \|f\|_{L^q(\square)} + \mathrm{dist}(G, SO(d)) < \bar{\rho}.$$

(a) (Existence & uniqueness) $\mathcal{I}_{\varepsilon_n}$ admits a unique (up to a constant) minimizer $u_\varepsilon \in W^{1,p}(\square)$ subject to (pBC).

(b) (Uniform Lipschitz estimate & Euler-Lagrange equation) Every minimizer $u_{\varepsilon_n} \in W^{1,p}(\square)$ of $\mathcal{I}_{\varepsilon_n}$ subject to (pBC) satisfies

$$\| \mathrm{dist}(\nabla u_{\varepsilon_n}, SO(d)) \|_{L^\infty(\square)} \leq C \, \mathrm{dist}(G, SO(d)) + \|f\|_{L^q(\square)}$$

and

$$-\nabla \cdot \; DW(\tfrac{x}{\varepsilon_n}, \nabla u_{\varepsilon_n}) = f \qquad \text{in } \mathscr{D}'(\square)$$

Suppose

$$\nabla \cdot \boldsymbol{a}(x, \nabla u) = 0 \qquad \text{in } \mathscr{D}'(B_R) \text{ with } \quad 1 \ll R$$

$\exists\ \gamma = \gamma(\beta, d) \in (0,1)$ and $\kappa(\beta, d) > 0$ such that if

$$\tilde{E}(\nabla u, B_R) := \inf_{F \in \mathbb{R}^{d\times d}} \left(\fint_{B_R} |\nabla u - (F + \nabla \phi_F)|^2 \right)^{\frac{1}{2}} \leq \begin{cases} +\infty & \text{if } d = 2, \\ \kappa & \text{if } d \geq 3 \end{cases}$$

Then,

- *Large-scale (intrinsic) excess decay:* for all $\gamma' \in (0, \gamma)$

$$\tilde{E}(\nabla u, B_r) \lesssim_{\gamma'} \left(\frac{r}{R}\right)^{\gamma'} \tilde{E}(\nabla u, B_R) \qquad \text{for all } r \geq 1,$$

- *Large-scale Lipschitz estimate*: $\fint_{B_1} |\nabla u|^2 \lesssim \fint_{B_R} |\nabla u|^2$.
- *Lipschitz estimate (all scales)*:

$$\|\nabla u\|_{L^\infty(B_1)} \lesssim \fint_{B_R} |\nabla u|^2.$$

(here we exploit (E, s)-regularity of $\boldsymbol{a}$)

Summary:

- One-cell formula and Corrector close to $SO(d)$
- Uniform Lipschitz estimate for small data
- Estimate on homogenization error for small data

Outlook:

- quantitative homogenization/linearization close to rotations
- quantitative stochastic homogenization in nonlinear elasticity

References:

- S. Neukamm, M. Schäffner. Quantitative homogenization in non-linear elasticity for small loads, *Archive for Rational Mechanics and Analysis* (online first) *arXiv:1703.07947* (one cell formula / error estimate for smooth / layered coefficients)
- S. Neukamm, M. Schäffner. Lipschitz estimates and existence of correctors for nonlinearly elastic, periodic composites subject to small strains. *(arXiv preprint)*