Scaling limits and homogenization for some stochastic Hamilton-Jacobi equations

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Multiplicative noise: parabolic scaling

$$\partial_t u^\epsilon + H(Du^\epsilon, x/\epsilon)\xi^\epsilon(t) = 0 \text{ in } \mathbb{R}^d \times (0,T],\ u^\epsilon(\cdot,0) = u_0 \in BUC(\mathbb{R}^d)$$

$H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ convex, coercive, locally Lipschitz,

- stationary with finite range dependence, ..., or
- periodic

$\xi : [0,\infty) \to \mathbb{R}$ piecewise C^1, independent of H, stationary, mean zero, uniformly bounded, and "sufficiently mixing"

$$\xi^\epsilon(t) := \frac{1}{\epsilon}\xi\left(\frac{t}{\epsilon^2}\right) \xrightarrow{\epsilon\to 0} dB \text{ in distribution,}$$

B is a Brownian motion.
The equation arises from scaling $\partial_t u + H(Du, y)\xi(t) = 0$ by $u^\epsilon(x,t) \approx \epsilon u(x/\epsilon, t/\epsilon^2)$.

Homogenization result for multiplicative noise

$$\partial_t u^\epsilon + H(Du^\epsilon, x/\epsilon)\xi^\epsilon(t) = 0 \text{ in } \mathbb{R}^d \times (0, T], \; u^\epsilon(\cdot, 0) = u_0 \in BUC(\mathbb{R}^d)$$

Theorem

There exists convex $\overline{H} : \mathbb{R}^d \to \mathbb{R}$ such that

$$u^\epsilon \xrightarrow{\epsilon \to 0} \overline{u} \in BUC(\mathbb{R}^d \times [0, T]) \quad \text{in distribution}$$

where $\overline{u}$ is the stochastic viscosity solution of

$$d\overline{u} + \overline{H}(D\overline{u}) \circ dB = 0 \text{ in } \mathbb{R}^d \times (0, T], \quad \overline{u}(\cdot, 0) = u_0.$$

$\overline{u}$ is defined in the Lions-Souganidis stochastic viscosity sense (unique extension of the solution operator to continuous paths)

Example: Front Propagation

$$\begin{cases} \Gamma_t^\epsilon = \partial\Omega_t^\epsilon \subset \mathbb{R}^d, \quad \text{normal velocity } \dfrac{1}{\epsilon} v\left(n, \dfrac{x}{\epsilon}, \dfrac{t}{\epsilon^2}\right), \\ v(n,y,t) = a\,(n,y)\,\xi(t),\ a \in C^{0,1}(S^{d-1} \times \mathbb{R}^d; \mathbb{R}_+) \\ a \text{ is stationary, finite range dependence, } p \mapsto a(p/|p|, y)|p| \text{ convex} \end{cases}$$

level set method: $\Omega_t^\epsilon = \{u^\epsilon(\cdot, t) > 0\}, \quad \partial_t u^\epsilon = a\left(\dfrac{Du^\epsilon}{|Du^\epsilon|}, \dfrac{x}{\epsilon}\right)|Du^\epsilon|\,\xi^\epsilon(t)$

Then, for some Lipschitz, deterministic $\overline{a} : S^{d-1} \to \mathbb{R}_+$ with $p \mapsto \overline{a}(p/|p|)|p|$ convex, $u^\epsilon \xrightarrow{\epsilon \to 0} \overline{u}$ in distribution

$$d\overline{u} = \overline{a}\left(\frac{D\overline{u}}{|D\overline{u}|}\right)|D\overline{u}| \circ dB(t),$$

describing motion of an interface $\overline{\Gamma}_t$ with normal velocity $\overline{a}(n)dB$

General problem with multiplicative noise

Applies to more general problems:

$$u_t^\epsilon + H\left(Du^\epsilon, x/\epsilon\right)\dot{\zeta}^\epsilon(t) = 0,$$
$$H \in C^{0,1}(\mathbb{R}^d \times \mathbb{R}^d) \text{ convex, coercive}$$
$$C^1([0,T]) \ni \zeta^\epsilon \xrightarrow{\epsilon \to 0} \zeta \in C([0,T]) \text{ uniformly}$$

Solutions of

$$\partial_t U^\epsilon \pm H(DU^\epsilon, x/\epsilon) = 0$$

converge quantifiably to solutions of

$$\partial_t \overline{U} \pm \overline{H}(DU) = 0$$

Additive noise: hyperbolic scaling

$H \in C^{0,1}(\mathbb{R}^d)$ convex, superlinear, $f \in C^2(\mathbb{R}^d)$ stationary-ergodic, nonconstant random field, independent of Brownian motion B
$0 \le \sigma \le 1$,

$$du^\epsilon + H(Du^\epsilon)\,dt = \epsilon^\sigma f(x/\epsilon)\,dB, \quad u^\epsilon(\cdot, 0) = u_0$$

Scaling critical case is $\sigma = 1/2$ (arises from hyperbolic scaling)
If $\sigma < 1/2$, the oscillations are too fast, and

$$u^\epsilon \xrightarrow{\epsilon \to 0} -\infty \quad \text{locally uniformly in distribution}$$

In distribution,

$$u^\epsilon(x,t) \lesssim \epsilon^{\sigma - 1/2} \inf \left\{ \epsilon \int_0^{t/\epsilon} f(\gamma_s) dB_s : \gamma_{t/\epsilon} = x/\epsilon, |\dot{\gamma}| \le M \right\}.$$

There exists such paths γ for which

$$\limsup_{\epsilon \to 0} \epsilon \int_0^{t/\epsilon} f(\gamma_s) dB_s < 0 \quad \text{(law of large numbers)}$$

Future work

Goal: if u^ϵ solves

$$du^\epsilon + H\left(Du^\epsilon\right)dt = \epsilon^{1/2} f\left(x/\epsilon\right)dB, \quad u^\epsilon(\cdot,0) = u_0,$$

then $u^\epsilon \to \overline{u}$ where, for some $\overline{H} > H$,

$$\partial_t \overline{u} + \overline{H}(D\overline{u}) = 0, \quad \overline{u}(\cdot,0) = u_0.$$

If ξ is the mixing field from before,

$$\xi^\epsilon(t) = \frac{1}{\epsilon^{1/2}}\xi(t/\epsilon),$$

and u^ϵ solves

$$\partial_t u^\epsilon + H(Du^\epsilon) = \epsilon^{1/2} f(x/\epsilon)\xi^\epsilon(t),$$

then such a result holds (Schwab 2009, Kosygina and Varadhan 2008, general stochastic homogenization of HJ equations in stationary ergodic spatio-temporal media)

Thank you!