Nuclear Physics B315 (1989) 193–221
North-Holland, Amsterdam

SUPERCONFORMAL ALGEBRAS AND STRING COMPACTIFICATION ON MANIFOLDS WITH SU(n) HOLONOMY

Tohru EGUCHI and Hirosi OOGURI*

Department of Physics, University of Tokyo, Tokyo, Japan

Anne TAORMINA

CERN, Geneva, Switzerland

Sung-Kil YANG

Research Institute for Fundamental Physics, Kyoto University, Kyoto, Japan

Received 12 September 1988

We discuss string compactifications on manifolds with SU(n) holonomy by making use of representation theories of extended superconformal algebras. In particular, string compactification on K_3 surfaces is discussed in detail. We calculate loop-space indices and show that all $c = 6$ superconformal field theories describe string propagation on manifolds with SU(2) holonomy. We study Gepner's models based on the tensoring of $N = 2$ minimal series and point out that some of these models are identified as orbifolds. We also discuss $c = 9$ superconformal field theories and their relation to Calabi–Yau manifolds.

1. Introduction

Recently, extended superconformal algebras have received much attention in connection with string compactifications with space-time supersymmetry [1–6]. It is now well-known that the $N = 2$ or 4 extended superconformal algebra on the world-sheet is necessary for the presence of space-time supersymmetry after string compactification [2, 3]. In fact, Neveu–Schwarz (NS) and Ramond (R) sectors are isomorphic in these algebras, and are mapped onto each other by the spectral flow which is equivalent to the space-time supersymmetry transformation [7, 8]. Thus, space-time bosons and fermions are necessarily paired up into supergravity multiplets.

On the other hand, from the study of low-energy supergravity theory, it is well-known that the compactifying spaces of the string theory must be Ricci flat and Kähler manifolds [9, 10]. In fact, an n-dimensional complex manifold with SU(n)

* Address after September 1, 1988; School of Natural Sciences, The Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540, USA.

holonomy possesses a unique covariantly constant spinor field which generates the $N = 1$ space-time supersymmetry transformation. Thus, we have the fundamental relationships among the extended superconformal algebras, space-time supersymmetry and manifolds with SU(n) holonomy in the analysis of string compactifications.

Manifolds of SU(n) holonomy, however, in general have a considerable amount of complexity and in cases of practical interest, Calabi–Yau manifolds ($n = 3$) or K_3 surface ($n = 2$), no examples of metrics are known explicitly. Thus, so far the orbifold limits of these manifolds have been studied extensively [11, 12].

Recently, extended superconformal algebras have received much attention in connection with string compactifications with space-time supersymmetry [1–6]. It is now well-known that the $N = 2$ or 4 extended superconformal algebra on the $c_1 = 0$ (vanishing first-Chern-class) manifolds by calculating the massless spectra of his models.

Geometrical aspects of the compactifying manifolds are explicit in orbifold models, while the superconformal symmetry is manifest in Gepner's models. Thus, they are somehow complementary to each other. It turns out, however, that they sometimes coincide as we shall see in the following discussions.

In this article we make an extensive study of string compactification on $c_1 = 0$ manifolds by using the representation theory of $N = 2$ and $N = 4$ superconformal algebras. In particular, we will concentrate on the case of the K_3 surface where we can use the results of $N = 4$ representation theory which has recently been worked out [13, 14]. We shall calculate the loop-space index [15–17] and show that all $c = 6$ superconformal field theories describe string propagation on K_3 manifolds.

In sect. 2 we make use of Gepner's models and describe how to construct modular-invariant partition functions of the non-linear σ-model on the K_3 surface.

We calculate loop-space indices in sect. 3 and show how the theory reproduces known topological invariants of K_3 manifolds. In sect. 4 we discuss heterotic string compactification and its massless particle spectra. We make a detailed comparison of the orbifold and Gepner's models in sect. 5 and point out that models based on the tensoring of $N = 2$ minimal series with levels $k \leqslant 4$ are all identified as orbifolds. In particular, the 2^4 model is identified both as a Z_2 and a Z_4 orbifold. Similarly, the $1^2 4^2$ model is identified as a Z_3 and a Z_6 orbifold. In sect. 6 we discuss $c = 9$ superconformal field theories and discuss their relation to Calabi–Yau manifolds. Some basic formulas of $N = 2$ and $N = 4$ representation theories are summarized in appendices A and B.

2. Non-linear σ-model on K_3 surface

Let us discuss string propagation on $c_1 = 0$ manifolds using the representation theory of $N = 2$ and $N = 4$ algebras. We will concentrate on the case of the K_3 surface. In this section we motivate our discussions making use of Gepner's models based on the tensoring of $N = 2$ minimal theories. Results described below, however, do not depend on the details of the $N=2$ models but hold for generic K_3 surfaces. Specific examples of the tensoring of $N=2$ minimal series will be discussed in detail in sect. 5.

In Gepner's method, one considers a tensor product $k_1^{m_1} k_2^{m_2} \dots k_l^{m_l}$ ($k_i, m_i \in \boldsymbol{N}$) of the $N=2$ discrete series with levels $k_1, \dots, k_l$ in such a way that the central charge adds up to $3n$ (n is the complex dimensionality of the $c_1=0$ manifold)

$$c = \sum m_i \frac{3k_i}{k_i+2} = 3n\,; \tag{2.1}$$

$n=2$ for the K_3 surface and 3 for the Calabi–Yau manifolds. In ref. [5] sixteen possibilities of eq. (2.1) with $c=6$ are listed, which describe string propagation on K_3 surfaces with a variety of complex structures. For a catalogue of Calabi–Yau cases, $c=9$, see ref. [18].

A special feature of the $N=2$ algebra is the isomorphism of the algebra under a continuous shift of the moding of the supercharge operators [8]. One can check that the algebra remains invariant under the transformation

$$\begin{aligned} L_n &\to L_n + \eta J_n + \tfrac{1}{6} c \eta^2 \delta_{n,0}\,, \\ J_n &\to J_n + \tfrac{1}{3} c \eta\, \delta_{n,0}\,, \\ G_r &\to G_{r+\eta}\,, \\ \bar{G}_r &\to \bar{G}_{r-\eta}\,. \end{aligned} \tag{2.2}$$

Here L_n, J_n, G_r and $\bar{G}_r$ are the Virasoro, U(1) current and supercharge generators, respectively, and η is an arbitrary real parameter. Thus, in particular, the R and NS sectors are isomorphic to each other ($\eta = \frac{1}{2}$ and $r \in \boldsymbol{Z}$ or $\boldsymbol{Z} + \frac{1}{2}$). The shift $\eta \to \eta \pm \frac{1}{2}$ corresponds to the space-time supersymmetry transformation [7]. In fact, when $c=3n$, the ground state $h=Q=0$ (h and Q are the eigenvalues of L_0 and J_0) of the NS sector is mapped onto the states with $h=\frac{1}{8}n$, $Q=\pm\frac{1}{2}n$ in the R sector, which correspond to the covariantly constant spinor fields on the Calabi–Yau or K_3 manifold.

On the other hand, under a shift $\eta \to \eta \pm 1$, the theory comes back to its original sector. The highest weight states of the algebra are, however, transformed onto different highest weight states. The ground state of the NS sector is now mapped onto the states with $h=\frac{1}{2}n$, $Q=\pm n$ in the NS sector which correspond to the holomorphic or anti-holomorphic n-form of the Calabi–Yau or K_3 manifold. The generators of the transformation $\Delta\eta = \pm 1$ are conformal fields with $h=\frac{1}{2}n$ and $Q=\pm n$, and in the case of $n=2$, i.e. K_3 surface, they are nothing but the SU(2) currents $J^{\pm}$ (note the factor 2 difference in the U(1) charge and the third component of the isospin $J_0 = 2J_0^3$). When the $N=2$ algebra is extended by the addition of the flow generators $J^{\pm}$, one obtains the $N=4$ superconformal algebra. Thus, the string compactifications on the K_3 surface are described by the representation theory of $N=4$ algebra. On the other hand, when $n=3$, i.e. the case of Calabi–Yau manifolds, the flow generators are fermionic ($h=\frac{3}{2}$ and $Q=\pm 3$) and their addition to $N=2$ gives a new algebra which will be discussed in sect. 6.

Partition functions of Gepner's models are expressed in terms of the character functions of the $N=2$ algebra [4,25]. The $N=2$ characters are defined by $\mathrm{tr}\, q^{L_0 - c/24} \mathrm{e}^{i\theta J_0}$ and the angle θ keeps track of the U(1) charge of the representation contents. The isomorphism of the $N=2$ algebra (2.2) manifests itself in the quasi-periodicity of θ in the character formulas. In fact, the shift $\eta \to \eta + \frac{1}{2}$ corresponds to $\theta \to \theta + \pi\tau$ and we have

$$\mathrm{ch}_{l,m}^{\mathrm{NS},k}(\tau; \theta + \pi\tau) = q^{-c/24} \mathrm{e}^{-ic\theta/6} \mathrm{ch}_{l,m}^{\mathrm{R},k}(\tau; \theta), \tag{2.3}$$

where l, m label the representations of the level-k minimal theories ($0 \leqslant l \leqslant k$, $-l \leqslant m \leqslant l$, $l-m \equiv 0 \pmod 2$). Under a "full" shift $\eta \to \eta + 1$ or $\theta \to \theta + 2\pi\tau$

$$\mathrm{ch}_{l,m}^{k}(\tau; \theta + 2\pi\tau) = q^{-c/6} \mathrm{e}^{-ic\theta/3} \mathrm{ch}_{l,m-2}^{k}(\tau; \theta), \tag{2.4}$$

which holds in both NS and R sectors. Explicit forms of the $N=2$ discrete characters and their modular properties are described in the appendix A.

Let us now concentrate on the case of the K_3 surface and describe a method of constructing the modular-invariant partition functions. For the sake of illustration, we consider the 1^6 model defined by taking 6 copies of the $k=1$ minimal theory. There exist three representations, $l=m=0$, $l=m=1$, $l=-m=1$, in the $k=1$ theory and we denote their characters (in the NS sector) as A, B and C, respectively. Under the spectral flow (2.4) A, B and C are cyclically permuted among each other.

Now we introduce a flow-invariant combination

$$\mathrm{NS}_1 = \mathrm{A}^6 + \mathrm{B}^6 + \mathrm{C}^6, \tag{2.5}$$

which we call the "graviton" orbit. Eq. (2.5) contains the identity operator ($h = Q = 0$) which generates the graviton multiplet in the heterotic string compactification.

SU(2) symmetry acts on the flow-invariant orbit (2.5) and hence the $N=2$ symmetry is enhanced to $N=4$. Therefore NS_1 can be decomposed into the representations of $N=4$ algebra.

Highest weight states of the $N=4$ algebra are parametrized by the conformal dimension h and isospin l. Unitarity puts a restriction $h \geqslant l$ in the NS sector and $h \geqslant \frac{1}{4}$ in the R sector (when $c=6$). There exist two distinct classes of representations of the $N=4$ algebra [13]; massless and massive representations. The massless representations exist at the unitarity bound

$$\begin{cases} h=l=0, & \\ h=l=\frac{1}{2}, & \text{in NS sector}, \end{cases} \tag{2.6}$$

$$\begin{cases} h=\frac{1}{4}, \quad l=0, & \\ h=\frac{1}{4}, \quad l=\frac{1}{2}, & \text{in R sector}, \end{cases} \tag{2.7}$$

Ground states of the R sector carry a non-zero Witten index in these representations and they possess unbroken $N=4$ world-sheet supersymmetry. The massless representations keep track of the non-trivial topology of the K_3 surface. On the other hand, the massive representations exist in the range

$$\begin{aligned} &h>0, \quad l=0, \quad \text{in NS sector}, \\ &h>\tfrac{1}{4}, \quad l=\tfrac{1}{2}, \quad \text{in R sector}, \end{aligned} \tag{2.8}$$

and have ground states with the equal number of bosons and fermions, and thus have vanishing Witten index. They describe the degrees of freedom of deformation of the K_3 surface. Under the spectral flow, a NS representation with isospin l is mapped onto a R representation with isospin $\frac{1}{2}-l$.

The graviton orbit (2.5) contains the $l=0$ massless character and an infinite sum of massive characters

$$\begin{aligned} NS_1(\tau;z) &= ch_0^{NS}(l=0;\tau;z) + \sum_{n=1}^{\infty} f_n^{(1)} ch^{NS}(h=n;\tau;z) \\ &= ch_0^{NS}(l=0;\tau;z) + F_1(\tau) ch^{NS}(h=0;\tau;z), \end{aligned} \tag{2.9}$$

$$F_1(\tau) = \sum_{n=1}^{\infty} f_n^{(1)} q^n. \tag{2.10}$$

(For the explicit form of $N=4$ characters, see appendix B.)

Under the modular transformation S: $\tau \to -1/\tau$, NS_1 transforms into a family of new orbits

$$\begin{aligned} NS_2 &= A^3B^3 + B^3C^3 + C^3A^3, \\ NS_3 &= A^2B^2C^2, \\ NS_4 &= A^4BC + B^4CA + C^4AB. \end{aligned} \tag{2.11}$$

The matrix S_{ij} of the S-transformation

$$\mathrm{NS}_i(\tau;\theta) = \sum S_{ij}\,\mathrm{NS}_j\left(-\frac{1}{\tau};\frac{\theta}{\tau}\right)\mathrm{e}^{-i\theta^2/2\pi\tau} \tag{2.12}$$

can be computed from the S-transformation of the $N=2$ subtheories (see appendix A). In the case of 1^6 theory S_{ij} is given by

$$S_{ij} = \frac{1}{27}\begin{pmatrix} 3 & 60 & 270 & 90 \\ 3 & -21 & 27 & 9 \\ 1 & 2 & 9 & -6 \\ 3 & 6 & -54 & 9 \end{pmatrix}. \tag{2.13}$$

There are, in general, three types of NS orbits in the K_3 compactification. They all possess integral values of the U(1) charge.

(1) Graviton orbit:
NS_1 is the only trajectory containing the ground state $h=Q=0$.

(2) Massless matter orbits:
NS_i $(i=2,\dots,d)$ contain states $h=\frac{1}{2}$, $Q=\pm 1$ and are rewritten as

$$\mathrm{NS}_i(\tau;z) = \mathrm{ch}_0^{\mathrm{NS}}\left(l=\tfrac{1}{2};\tau;z\right) + F_i(\tau)\mathrm{ch}^{\mathrm{NS}}(h=0;\tau;z),$$
$$F_i(\tau) = \sum_{n=1}^{\infty} f_n^{(i)} q^n. \tag{2.14}$$

(In some cases, states $h=\frac{1}{2}$, $Q=\pm 1$ appear more than once in one orbit. Then $\mathrm{ch}_0^{\mathrm{NS}}(l=\frac{1}{2})$ in eq. (2.14) must be multiplied by the multiplicity. We ignore this complication in the following.)

(3) Massive orbits:
NS_j $(j=d+1,\dots,d+d')$ contain massive characters only

$$\mathrm{NS}_j(\tau;z) = F_j(\tau)\mathrm{ch}^{\mathrm{NS}}(h=0;\tau;z),$$
$$F_j(\tau) = q^{r_j}\sum_{n=0}^{\infty} f_n^{(j)} q^n, \quad 0<r_j\in \boldsymbol{Q}. \tag{2.15}$$

In eqs. (2.10), (2.14) and (2.15) the expansion coefficients $f_n^{(m)}$ are non-negative integers. The number of orbits, d and d', and the functions F_m depend on the tensoring of subtheories.

In the case of 1^6, $d=2$, $d'=2$ and

$$F_1(\tau)=5q+29q^2+80q^3+\dots,$$
$$F_2(\tau)=5q+26q^2+85q^3+\dots,$$
$$F_3(\tau)=q^{2/3}(1+5q+20q^2+59q^3+\dots),$$
$$F_4(\tau)=q^{1/3}(1+16q+38q^2+127q^3+\dots). \tag{2.16}$$

These $d+d'$ trajectories enter into the modular-invariant partition functions.

Using the symmetry property of the S-matrix of subtheories, it is easy to show that the S-matrix of orbits (2.12) is symmetrizable by a diagonal matrix D with integral elements D_i

$$D_iS_{ij}=D_jS_{ji} \qquad (\text{no sum on } i,\, j), \tag{2.17}$$

with

$$D_i=\frac{S_{1i}}{S_{i1}}, \qquad i=1,\dots,d+d' \tag{2.18}$$

(D is normalized as $D_1=1$). In the case of 1^6 theory $D_i=(1,20,270,30)$. D_i's are essentially the combinatorial factors in the tensoring of representations

$$D_i=\frac{(\text{combinatorial factor of orbit } i)\times(\text{standard length of orbits})}{(\text{length of orbit } i)}. \tag{2.19}$$

The standard length of orbits of $k_1^{m_1}\dots k_l^{m_l}$ is the least common multiple of $k_1+2,\dots,k_l+2$.

The matrix D is the key ingredient in the construction of modular-invariant partition functions. Indeed, it is easy to check, using (2.17) and $S^2=1$, that

$$\sum_{i=1}^{d+d'} D_i(\mathrm{NS}_i)^*(\mathrm{NS}_i) \tag{2.20}$$

is S-invariant. The sum of D_i for massless matter orbits always adds up to 20

$$\sum_{i=2}^{d} D_i=20, \tag{2.21}$$

in K_3 compactification. This is the Hodge number $h^{1,1}$ and it gives the multiplicity of massless spinors in the $\underline{56}$ of E_7 in heterotic string compactification. We will derive eq. (2.21) in the next section.

The structure of the trajectories in the other sectors is determined by the spectral flow. By shifting θ by $\pi\tau$ and $\pi\tau+\pi$ in eq. (2.12), we find that

$$\mathrm{R}_i(\tau;\theta)=\sum S_{ij}\widetilde{\mathrm{NS}}_j\left(-\frac{1}{\tau};\frac{\theta}{\tau}\right)\mathrm{e}^{-i\theta^2/2\pi\tau}, \tag{2.22}$$

$$\tilde{\mathrm{R}}_i(\tau;\theta)=-\sum S_{ij}\tilde{\mathrm{R}}_j\left(-\frac{1}{\tau};\frac{\theta}{\tau}\right)\mathrm{e}^{-i\theta^2/2\pi\tau}, \tag{2.23}$$

where $\widetilde{\mathrm{NS}}$ and $\tilde{\mathrm{R}}$ are NS and R sectors with $(-1)^F$ insertion and $\tilde{\mathrm{R}}_i(\theta)$ gives the Witten index I_i at $\theta=0$. Since $I_1=-2$, $I_i=1$ $(i=2,\dots,d)$, $I_j=0$ $(j=d+1,\dots,d+d')$ (see appendix B), the S-matrix has an eigenvector $(-2,1,\dots,1,0,\dots,0)$ with eigenvalue -1

$$\sum S_{ij}I_j=-I_i. \tag{2.24}$$

The modular-invariant partition function of the non-linear σ-model on the K_3 surface is then given by (in the case of A-type invariant)

$$Z_\sigma=\tfrac{1}{2}\sum_{i=1}^{d+d'}D_i\left\{|\mathrm{NS}_i|^2+|\widetilde{\mathrm{NS}}_i|^2+|\mathrm{R}_i|^2+|\tilde{\mathrm{R}}_i|^2\right\}. \tag{2.25}$$

The Euler number is equal to

$$\chi=\sum_{i=1}^{d+d'}D_iI_i^2=4+\sum_{i=2}^{d}D_i=24. \tag{2.26}$$

3. Loop-space index

Functions $F_i(\tau)$ $(i=1,\dots,d+d')$ depend on the tensoring of $N=2$ subtheories and thus are dependent on the complex structure of the K_3 surface. In order to characterize general aspects of K_3 compactifications, it is convenient to introduce topological invariants which are independent of the complex structure or the moduli of the K_3 surface. In this section we consider the loop-space indices [15–17] (or elliptic genera) which are string-theoretic generalizations of classical topological invariants.

We introduce

$$\Phi(\hat{A})\equiv\mathrm{tr}_{\mathrm{NS}\times\mathrm{R}}\,q^{L_0-1/4}(-1)^{F_{\mathrm{L}}}\bar{q}^{\bar{L}_0-1/4}, \tag{3.1}$$

where the trace is taken by imposing the NS and R (with $(-1)^F$ insertion) boundary conditions on the right- and left-moving sectors of the theory, respectively. $\Phi(\hat{A})$ is the elliptic genus generalizing the Dirac index $\hat{A}$ [16]. Eq. (3.1) may be easily evaluated making use of the non-linear σ-model. If we use the representations of $N=4$ massless characters, (B.8) and (B.9), we find that

$$\Phi(\hat{A}) = \sum_i D_i(\mathrm{NS}_i)(\tilde{\mathrm{R}}_i),$$

$$= \left\{ \frac{q^{-1/8}}{\eta} \left(-2 + \sum_{i=1}^{d} D_i l_i F_i(\tau) \right) + \sum_{i=1}^{d} D_i l_i^2 h_3(\tau) \right\} \left(\frac{\vartheta_3(\tau)}{\eta(\tau)} \right)^2. \quad (3.2)$$

We can explicitly compute (3.2) by using any of the tensorings of $N=2$ models and find that they all give the same result. Thus $\Phi(\hat{A})$ is in fact common to all K_3 compactifications. Calculation is easy in the 2^4 theory and we obtain

$$\Phi(\hat{A}) = 2\frac{(\vartheta_2^4 - \vartheta_4^4)}{\eta^4} \left(\frac{\vartheta_3}{\eta} \right)^2$$

$$= -q^{-1/4}(2 - 40q^{1/2} - 124q + \dots), \quad (3.3)$$

eq. (3.3) may also be derived directly from (3.1); we note that the boundary condition of $\Phi(\hat{A})$ is invariant under the transformations S and T^2 (T: $\tau \to \tau + 1$) and thus $\Phi(\hat{A})$ is a modular form invariant under Γ_2, the level-2 principal congruence subgroup. This fact uniquely determines $\Phi(\hat{A})$ up to an overall constant (this constant is fixed by comparing the first terms in the q-expansion). Eq. (3.3) agrees with the calculation of the elliptic genus for K_3 by using the theory of characteristic classes.

If we compare the q-expansions of eqs. (3.2) and (3.3), we find that $\Sigma_i D_i l_i^2 = 24$ (eq. (2.26)) and the theory reproduces the Euler number of the K_3 surface. Thus the $c=6$ superconformal field theory describes the string propagation on K_3 manifolds. Our only assumption in sect. 2 is the absence of the mixture of $l=0$ and $l=\frac{1}{2}$ massless representations. If there is a contribution of the $l=\frac{1}{2}$ representation in the graviton orbit, then the interference term $\mathrm{ch}_0(l=0)^*\mathrm{ch}_0(l=\frac{1}{2})$ gives primary fields with conformal dimension $\bar{h}=0$, $h=\frac{1}{2}$. These are nothing but free (complex) spinor fields and in this case the theory describes string propagation on the product of complex tori $T\times T$. This is actually what happens in Gepner's models $1^3 2^2$, $1^2 2^1 (10')^1$ and $1^1(10')^2$ ($10'$ means the use of the E_6 type invariant for the $k=10$ subtheory).

In this context we shall note that the function $F_1(\tau)$ in the graviton orbit generates (anti-) holomorphic fields of type $(h, \bar{h}) = (n,0)$ or $(0,n)$, $n=1,2,\dots$. In particular, if $f_1^{(1)} \neq 0$, it generates extra U(1) gauge fields in the heterotic string compactification (see sect. 4).

Instead of eq. (3.1) we may impose the R boundary condition in the right-moving sector and define

$$\Phi(\sigma)=\operatorname{tr}_{\mathrm{R}\times\mathrm{R}}q^{L_0-1/4}(-1)^{F_{\mathrm{L}}}\bar{q}^{\bar{L}_0-1/4}. \tag{3.4}$$

Eq. (3.4) can also be evaluated using the non-linear σ-model

$$\begin{aligned}\Phi(\sigma)&=\sum_i D_i(\mathrm{R}_i)(\tilde{\mathrm{R}}_i)\\&=\left\{\frac{q^{-1/8}}{\eta}\left(-2+\sum_{i=1}^{d}D_iI_iF_i(\tau)\right)+\chi h_3(\tau)\right\}\left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^2\\&\quad+\chi\left(\frac{\vartheta_4(\tau)}{\vartheta_3(\tau)}\right)^2.\end{aligned} \tag{3.5}$$

Comparing eq. (3.5) with eq. (3.2) we obtain

$$\Phi(\sigma)=2\frac{\left(\vartheta_3^4+\vartheta_4^4\right)}{\eta^4}\left(\frac{\vartheta_2}{\eta}\right)^2=16(1+34q+\dots). \tag{3.6}$$

$\Phi(\sigma)$ is the elliptic genus corresponding to the Hirzebruch signature σ.

Finally, if we insert $(-1)^{F_{\mathrm{R}}}$ into eq. (3.4), we obtain the Euler characteristic

$$\begin{aligned}\Phi(\chi)&=\operatorname{tr}_{\mathrm{R}\times\mathrm{R}}(-1)^{F_{\mathrm{R}}+F_{\mathrm{L}}}q^{L_0-1/4}\bar{q}^{\bar{L}_0-1/4}\\&=\sum D_iI_i^2=24.\end{aligned} \tag{3.7}$$

Actually, these three genera may be combined into a single function

$$\Phi(\theta)=\operatorname{tr}_{\mathrm{NS}\times\mathrm{R}}q^{L_0-1/4}\,\mathrm{e}^{i\theta J_0}(-1)^{F_{\mathrm{L}}}\bar{q}^{\bar{L}_0-1/4}. \tag{3.8}$$

$\Phi(\hat{A})$, $\Phi(\sigma)$ and $\Phi(\chi)$ are given by $\Phi(\theta)$ at $\theta=0$, $\pi\tau$ and $\pi\tau+\pi$, respectively. Thus, the classical topological invariants are nicely "unified" in the superconformal field theory.

By making use of the elliptic genus, it is possible to separate the partition function into two parts; one of them, Z_{top}, is the topological part which is independent of the moduli of the K_3 surface. The other piece, Z_{deform}, depends on the moduli and varies as one deforms the complex structure of the K_3 manifold.

After some algebra we find that

$$Z_\sigma = Z_{\text{top}} + Z_{\text{deform}}, \tag{3.9}$$

$$Z_{\text{top}} = 24 \times \tfrac{1}{2} \sum_i \left| \frac{\vartheta_i}{\vartheta_3} \right|^4 + \tfrac{1}{2}\left(\Phi(\hat{A}) - \text{c.c.}\right)\left(\left(\frac{\vartheta_2 \vartheta_4^*}{|\eta|^2} \right)^2 - \text{c.c.} \right) \left| \frac{\eta}{\vartheta_3} \right|^4$$

$$= \tfrac{1}{2}\left(\left| \frac{\vartheta_2\vartheta_3}{\eta^2} \right|^4 + \left| \frac{\vartheta_3\vartheta_4}{\eta^2} \right|^4 + \left| \frac{\vartheta_2\vartheta_4}{\eta^2} \right|^4 - \left| \frac{\vartheta_2}{\eta} \right|^8 - \left| \frac{\vartheta_4}{\eta} \right|^8 \right) \times \tfrac{1}{2} \sum_i \left| \frac{\vartheta_i}{\eta} \right|^4, \tag{3.10}$$

$$Z_{\text{deform}} = \left(\left| (1+F_1)\frac{q^{-1/8}}{\eta} - 2h_3 \right|^2 + \sum_{i=2}^{d} D_i \left| F_i \frac{q^{-1/8}}{\eta} + h_3 \right|^2 + \sum_{j=d+1}^{d+d'} D_j \left| F_j \frac{q^{-1/8}}{\eta} \right|^2 \right)$$

$$\times \tfrac{1}{2} \sum_i \left| \frac{\vartheta_i}{\eta} \right|^4. \tag{3.11}$$

Here the sum over i is $i = 2, 3, 4$. Terms involving the function h_3 may be separated in Z_{deform} using eq. (3.2), and it may be rewritten as

$$Z_{\text{deform}} = \left(-24|h_3|^2 + \left\{ h_3^* \Phi(\hat{A}) \left(\frac{\eta}{\vartheta_3} \right)^2 + \text{c.c.} \right\} \right) \tfrac{1}{2} \sum_i \left| \frac{\vartheta_i}{\eta} \right|^4$$

$$+ \left(\left| (1+F_1)\frac{q^{-1/8}}{\eta} \right|^2 + \sum_{i=2}^{d+d'} D_i \left| F_i \frac{q^{-1/8}}{\eta} \right|^2 \right) \tfrac{1}{2} \sum_i \left| \frac{\vartheta_i}{\eta} \right|^4. \tag{3.12}$$

At generic points in the moduli space, holomorphic factorization does not take place. The second term of eq. (3.12) is replaced by a more general structure

$$\sum c_{nm} \frac{q^{h_n - 1/8} \bar{q}^{\bar{h}_m - 1/8}}{|\eta|^2} \frac{1}{2} \sum_i \left| \frac{\vartheta_i}{\eta} \right|^4, \tag{3.13}$$

where c_{nm} are non-negative integers and $h_n, \bar{h}_m \geqslant 0,\ h_n - \bar{h}_m \in \mathbf{Z}$.

4. Heterotic string compactification

Let us now turn to the discussion on the heterotic string compactification on the K_3 surface. In this case we must take into account the degrees of freedom of the uncompactified six-dimensional Minkowski space and the internal space in the left-moving sector.

In the right sector of the theory, $N=4$ characters are multiplied by the characters of the SO(4) Kac–Moody algebra which is generated by the four (transverse) spinor fields of the uncompactified Minkowski space (four uncompactified bosons generate an additional $\eta(\tau)^{-4}$). The orbits are given by

$$X_{\mathrm{R},i}(z)=\chi_v^{\mathrm{SO}(4)}(z)\mathrm{NS}_i^+(z)+\chi_b^{\mathrm{SO}(4)}(z)\mathrm{NS}_i^-(z)$$
$$-\chi_c^{\mathrm{SO}(4)}(z)\mathrm{R}_i^-(z)-\chi_s^{\mathrm{SO}(4)}(z)\mathrm{R}_i^+(z). \tag{4.1}$$

Here $\mathrm{NS}^{\pm}\equiv\frac{1}{2}(\mathrm{NS}\pm\widetilde{\mathrm{NS}})$, $\mathrm{R}^{\pm}\equiv\frac{1}{2}(\mathrm{R}\pm\tilde{\mathrm{R}})$ and b, v, s, c are the conjugacy classes of the level-1 representations. Eq. (4.1) is written as

$$X_{\mathrm{R},i}(z)=\frac{1}{2}\left\{\left(\frac{\vartheta_3(z)}{\eta}\right)^2\mathrm{NS}_i(z)-\left(\frac{\vartheta_4(z)}{\eta}\right)^2\widetilde{\mathrm{NS}}_i(z)-\left(\frac{\vartheta_2(z)}{\eta}\right)^2\mathrm{R}_i(z)\right.$$
$$\left.+\left(\frac{\vartheta_1(z)}{\eta}\right)^2\tilde{\mathrm{R}}_i(z)\right\}, \tag{4.2}$$

where we have used

$$\chi_b^{\mathrm{SO}(2n)}+\chi_v^{\mathrm{SO}(2n)}=\left(\frac{\vartheta_3}{\eta}\right)^n, \qquad \chi_b^{\mathrm{SO}(2n)}-\chi_v^{\mathrm{SO}(2n)}=\left(\frac{\vartheta_4}{\eta}\right)^n,$$
$$\chi_s^{\mathrm{SO}(2n)}+\chi_c^{\mathrm{SO}(2n)}=\left(\frac{\vartheta_2}{\eta}\right)^n, \qquad \chi_s^{\mathrm{SO}(2n)}-\chi_c^{\mathrm{SO}(2n)}=\left(\frac{-i\vartheta_1}{\eta}\right)^n. \tag{4.3}$$

On the other hand, in the left sector the $N=4$ characters are multiplied by those of the E_8 and SO(12) Kac–Moody algebras which describe the degrees of freedom of the internal space. Note that the standard $\mathrm{E}_8\times\mathrm{E}_8$ gauge symmetry of the heterotic string is broken down to $\mathrm{E}_8\times\mathrm{E}_7$ in the K_3 compactification. The SU(2)′ gauge symmetry in

$$\mathrm{E}_8\supset\mathrm{SU}(2)'\times\mathrm{E}_7\supset\mathrm{SU}(2)'\times\mathrm{SU}(2)\times\mathrm{SO}(12) \tag{4.4}$$

is lost due to the holonomy of the K_3 surface while the SU(2) symmetry of $N=4$ algebra is combined with SO(12) and generates the E_7 gauge group.

The orbits in the left sector are given by

$$X_{\mathrm{L},i}(z)=\big(\chi_b^{\mathrm{SO}(12)}(z)\mathrm{NS}_i^+(z)+\chi_v^{\mathrm{SO}(12)}(z)\mathrm{NS}_i^-(z)$$
$$+\chi_c^{\mathrm{SO}(12)}(z)\mathrm{R}_i^+(z)+\chi_s^{\mathrm{SO}(12)}(z)\mathrm{R}_i^-(z)\big)\chi_1^{\mathrm{E}_8}(z), \tag{4.5}$$
$$=\frac{1}{2}\left[\left(\frac{\vartheta_3(z)}{\eta}\right)^6\mathrm{NS}_i(z)+\left(\frac{\vartheta_4(z)}{\eta}\right)^6\widetilde{\mathrm{NS}}_i(z)\right.$$
$$\left.+\left(\frac{\vartheta_2(z)}{\eta}\right)^6\mathrm{R}_i(z)+\left(\frac{\vartheta_1(z)}{\eta}\right)^6\tilde{\mathrm{R}}_i(z)\right]\frac{1}{2}\sum_i\left(\frac{\vartheta_i(z)}{\eta}\right)^8. \tag{4.6}$$

$X_{R,i}$ and $X_{L,i}$ are constructed in such a way that they transform under S as in equation (2.12) with the same S-matrix. GSO projections in eqs. (4.1) and (4.5) ensure the correct spin-statistics connection. Modular invariants are formed as

$$Z = \frac{\text{const.}}{(\operatorname{Im}\tau)^2 |\eta|^8} \sum_i D_i(X_{R,i})(X_{L,i}^*). \tag{4.7}$$

It is easy to see that the right-moving orbits (4.2) actually vanish and hence the theory has zero cosmological constant. We first note that the $N=4$ massive characters are proportional to the squares of elliptic theta-functions (B.11)

$$\mathrm{ch}^{\mathrm{NS}}(z) \propto \frac{1}{\eta}\left(\frac{\vartheta_3(z)}{\eta}\right)^2, \quad \mathrm{ch}^{\widetilde{\mathrm{NS}}}(z) \propto \frac{1}{\eta}\left(\frac{\vartheta_4(z)}{\eta}\right)^2,$$

$$\mathrm{ch}^{\mathrm{R}}(z) \propto \frac{1}{\eta}\left(\frac{\vartheta_2(z)}{\eta}\right)^2, \quad \mathrm{ch}^{\tilde{\mathrm{R}}}(z) \propto \frac{1}{\eta}\left(\frac{\vartheta_1(z)}{\eta}\right)^2. \tag{4.8}$$

Thus, the contributions of the massive representations vanish in each orbit (4.2) due to the Jacobi identity. It is easy to see that also the contributions of the massless representations cancel in the right sector.

On the other hand, in the left-moving sector of the theory, the massive representations in $X_L(z)$ are combined into E_8 characters

$$\sum_i \vartheta_i(z)^8 \chi_1^{E_8}(z) \propto \left(\chi_1^{E_8}(z)\right)^2. \tag{4.9}$$

Thus, the massive sector of the theory does not feel the holonomy of the K_3 surface and retains the original $E_8 \times E_8$ symmetry. On the other hand, massless components of each orbit are expressed as a sum of E_7 characters, and we have

$$X_{L,1} = \left\{ A_{1,1}(\tau)\chi_1^{E_7}(z) + A_{1,2}(\tau)\chi_{56}^{E_7}(z) + \frac{q^{-1/8}}{\eta} F_1(\tau)\chi_1^{E_8}(z) \right\} \chi_1^{E_8}(z), \tag{4.10}$$

$$X_{L,i} = \left\{ A_{2,2}(\tau)\chi_{56}^{E_7}(z) + A_{2,1}(\tau)\chi_1^{E_7}(z) + \frac{q^{-1/8}}{\eta} F_i(\tau)\chi_1^{E_8}(z) \right\} \chi_1^{E_8}(z). \tag{4.11}$$

Here $A_{2l+1,2l'+1}(\tau)$ are the branching functions of $N=4$ massless characters into those of SU(2) (see appendix B) and we have used

$$\chi_1^{E_7}(z) = \chi_b^{SO(12)}(z)\chi_1^{SU(2)}(z) + \chi_s^{SO(12)}(z)\chi_2^{SU(2)}(z), \tag{4.12}$$

$$\chi_{56}^{E_7}(z) = \chi_v^{SO(12)}(z)\chi_2^{SU(2)}(z) + \chi_c^{SO(12)}(z)\chi_1^{SU(2)}(z). \tag{4.13}$$

(Indices of the characters represent the multiplicity of the highest weight state.)

The massless spectra of the theory are easily read off from eqs. (4.10), (4.11) and (B.14). Besides the standard gravity, E_7 gauge multiplets and 20 spinors of $\underline{56}$ of E_7, there exist $\Sigma_{i=2}^{d} D_i(2+f_1^{(i)})$ gauge singlets coming from the massless matter orbits. If $f_1^{(1)} \neq 0$, there also appear additional $f_1^{(1)}$ U(1) gauge fields from the graviton orbit. At generic points in the moduli space of the K_3 surface, $\Sigma D_i(2+f_1^{(i)}) = 130$, $f_1^{(1)} = 0$, while in $N=2$ and orbifold models there always exists an extra U(1) symmetry ($f_1^{(1)} > 0$) and an excess of gauge singlets $\Sigma D_i(2+f_1^{(i)}) > 130$. The elliptic genus $\Phi(\hat{A})$ predicts, however, that their difference $\Sigma D_i(2+f_1^{(i)}) - 2f_1^{(1)}$ must be always equal to 130.

5. Orbifolds

We now turn to the orbifold models of K_3 compactification and discuss their relation to the models of Gepner. It is known [20] that there are four possible types of orbifold limits of the K_3 surface with the symmetry group Z_l, $l = 2$, 3, 4 and 6. They are defined by dividing the product of complex tori $T_1 \times T_2$ by the action of Z_l

$$z_1 \to e^{2\pi i/l} z_1, \qquad z_2 \to e^{-2\pi i/l} z_2, \qquad l = 2, 3, 4, 6. \tag{5.1}$$

Conical singularities appear at the fixed points of the transformation (5.1).

Note that a holomorphic and anti-holomorphic 2-form, $dz_1 \wedge dz_2$ and $d\bar{z}_1 \wedge d\bar{z}_2$, exist on orbifolds (invariant under (5.1)) and together with the harmonic (1,1)-form $dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2$ they generate SU(2) symmetry. Thus, the $N=4$ algebra acts on Z_l orbifolds. Z_l orbifold with $l = 3, 4, 6$ has an additional harmonic form $dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$ which generates an extra U(1) gauge symmetry in heterotic string compactification. Z_2 orbifold, on the other hand, has extra $U(1)^3$ symmetry corresponding to the forms $dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$, $dz_1 \wedge d\bar{z}_2$ and $dz_2 \wedge d\bar{z}_1$.

Basic building blocks of the orbifold partition functions are given by (in the NS sector)

$$f_{r,s}(\tau,\theta) = \frac{\vartheta_3(\theta + (s+r\tau)/l)\,\vartheta_3(\theta - (s+r\tau)/l)}{\vartheta_1((s+r\tau)/l)^2},$$

$$r, s = 0, 1, \ldots, l-1 \quad ((r,s) \neq (0,0)). \tag{5.2}$$

Eq. (5.2) describes the product of partition functions of a pair of complex fermions and bosons twisted by ω^r and ω^s ($\omega = e^{2\pi i/l}$) in the spatial and temporal directions, respectively. θ is the U(1) angle as before. Note that two complex fermions have opposite Z_l charges (see eq. (5.1)). The partition function of orbifold models (of the

NS sector) is then given by

$$Z^{\mathrm{NS}} = \sum_{r,s}{}' n_{r,s} |f_{r,s}(\theta)|^2 + Z^{\mathrm{NS}}_{\mathrm{lattice}}, \tag{5.3}$$

where coefficients $n_{r,s}$ are defined by

$$n_{0,s} = \frac{1}{l}\left(2\sin\frac{\pi s}{l}\right)^4, \tag{5.4}$$

and the symmetry relations

$$n_{r,s} = n_{r,s+r}, \qquad n_{r,s} = n_{s,l-r}. \tag{5.5}$$

Σ' means the sum over r, s with $(r, s) \neq (0,0)$. One can check that for each value of r the sum over s in eq. (5.3) gives a projection onto states with zero Z_l charge. The lattice part of the partition function is given by

$$Z^{\mathrm{NS}}_{\mathrm{lattice}} = \frac{1}{l} \sum_{w_{\mathrm{R}} w_{\mathrm{L}}} q^{w_{\mathrm{R}}^2/2} \bar{q}^{w_{\mathrm{L}}^2/2} \frac{1}{|\eta|^8} \left| \frac{\vartheta_3(\theta)}{\eta} \right|^4, \tag{5.6}$$

where $w_{\mathrm{R}}, w_{\mathrm{L}}$ are weight vectors of a (real) four-dimensional lattice where the action of Z_l symmetry is well defined. Explicitly, they are parametrized as

$$w_{\mathrm{R}} = \left(\frac{m_i}{2R_i} - b_{ik} R_k n_k\right) e_i^* + n_i R_i e_i, \tag{5.7}$$

$$w_{\mathrm{L}} = \left(\frac{m_i}{2R_i} - b_{ik} R_k n_k\right) e_i^* - n_i R_i e_i, \tag{5.8}$$

where m_i, n_i $(i = 1, \ldots, 4)$ are integers and R_i $(i = 1, \ldots, 4)$ are the radii. Vectors e_i, e_i^* are dual to each other, $e_i e_i^* = \delta_{ij}$. The metric of the lattice is given by $g_{ij} = e_i e_j$ (e_i^2 is normalized to 1) and b_{ij} is the anti-symmetric tensor field. There are restrictions on the possible forms of g_{ij}, b_{ij} due to Z_l symmetry.

The partition functions in other sectors are again given by the flow of θ. It is convenient to rewrite $f_{r,s}(\theta)$ as

$$f_{r,s}(\theta) = \left(\frac{\vartheta_1(\theta)}{\vartheta_3(0)}\right)^2 + \left[\frac{\vartheta_3((s+r\tau)/l)}{\vartheta_1((s+r\tau)/l)}\right]^2 \left(\frac{\vartheta_3(\theta)}{\vartheta_3(0)}\right)^2. \tag{5.9}$$

The elliptic genus $\Phi(\hat{A})$ of orbifolds can then be easily evaluated as

$$\Phi(\hat{A}) = -\bar{q}^{1/4}\sum_{r,s}{}' n_{r,s} f_{r,s}(\theta=0) f_{r,s}^{*}(\theta=\pi\tau+\pi) \tag{5.10}$$

$$= -\sum_{r,s}{}' n_{r,s}\left[\frac{\vartheta_3((s+r\tau)/l)}{\vartheta_1((s+r\tau)/l)}\right]^2. \tag{5.11}$$

We can express the ratio $(\vartheta_3/\vartheta_1)^2$ in terms of the Weierstrass $\mathscr{P}$-function and the sum over r, s (5.11) then becomes l-independent and agrees with eq. (3.3),

$$-\sum_{r,s}{}' n_{r,s}\left[\frac{\vartheta_3((s+r\tau)/l)}{\vartheta_1((s+r\tau)/l)}\right]^2 = 2\left(\frac{\vartheta_2^4-\vartheta_4^4}{\eta^4}\right)\left(\frac{\vartheta_3}{\eta}\right)^2. \tag{5.12}$$

Thus, the orbifold models in fact describe compactifications on the K_3 surface. The parameters involved in the specification of their lattices constitute the moduli of orbifold models. The number of moduli is 8 for Z_3, Z_4 and Z_6, and 16 for Z_2 orbifold.

Let us now turn to the discussion on the relation between Gepner's models and orbifolds. We first consider 2^4 and $1^1 2^2 4^1$ theories and compare them with Z_2 orbifolds. After somewhat lengthy algebra which makes use of $N=2$ discrete characters, we find the following partition functions

$$Z(1^1 2^2 4^1) = \tfrac{1}{2}\left(\sum_{\mathrm{SU(3)}\times\mathrm{SU(2)}^2\ \mathrm{lattice}} q^{w_R^2/2}\bar{q}^{w_L^2/2}\frac{1}{|\eta|^8} + \left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^4 + \left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^4 + \left|\frac{\vartheta_4\vartheta_2}{\eta^2}\right|^4\right) \times \frac{1}{2}\sum_i\left|\frac{\vartheta_i}{\eta}\right|^4, \tag{5.13}$$

$$Z(2^4) = \tfrac{1}{2}\left(\sum_{\mathrm{SO(8)}\ \mathrm{lattice}} q^{w_R^2/2}\bar{q}^{w_L^2/2}\frac{1}{|\eta|^8} + \left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^4 + \left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^4 + \left|\frac{\vartheta_4\vartheta_2}{\eta^2}\right|^4\right) \times \frac{1}{2}\sum_i\left|\frac{\vartheta_i}{\eta}\right|^4. \tag{5.14}$$

We recognize the familiar structure of the Z_2 orbifold [21]; $\frac{1}{2}\Sigma_{\mu>\nu}|\vartheta_\mu\vartheta_\nu/\eta^2|^4$ gives the sum over Z_2 twisted sectors.

In eq. (5.13), SU(3) lattice is defined by $\sqrt{2}\,\boldsymbol{e}_i = \boldsymbol{\alpha}_i$ (simple roots of SU(3), normalized as $\boldsymbol{\alpha}_i^2 = 2$) and $R_i = 1/\sqrt{2}$ $(i=1,2)$. The anti-symmetric tensor b_{ij} is put to $b_{12} = -b_{21} = -\frac{1}{2}$. It turns out that the lattice sum equals the sum over level-1 representations of SU(3)

$$Z^{SU(3)} = \sum_{SU(3)\text{ lattice}} q^{w_R^2/2} \bar{q}^{w_L^2/2} \frac{1}{|\eta|^4}$$

$$= |\chi_1^{SU(3)}|^2 + |\chi_3^{SU(3)}|^2 + |\chi_{3^*}^{SU(3)}|^2 . \tag{5.15}$$

Here, $\chi^{SU(3)}$'s are the SU(3) characters ($\chi_1^{SU(3)} = (\Theta_{0,3}\Theta_{0,1} + \Theta_{3,3}\Theta_{1,1})/\eta^2$, $\chi_3^{SU(3)} = \chi_{3^*}^{SU(3)} = (\Theta_{2,3}\Theta_{0,1} + \Theta_{1,3}\Theta_{1,1})/\eta^2$). Similarly the SU(2) lattice of eq. (5.13) is defined as $\sqrt{2}\,\boldsymbol{e} = \boldsymbol{\alpha}$ (simple root of SU(2)) and $R = 1/\sqrt{2}$. $SU(2)^2$ gives a (complex) 1-dimensional square lattice. The lattice sum again equals the sum over level-1 representations

$$Z^{SU(2)} = \sum_{SU(2)\text{ lattice}} q^{w_R^2/2} \bar{q}^{w_L^2/2} \frac{1}{|\eta|^2}$$

$$= |\chi_1^{SU(2)}|^2 + |\chi_2^{SU(2)}|^2 . \tag{5.16}$$

When the radius of the lattice equals $1/\sqrt{2}$, there occurs symmetry enhancement due to the Frenkel–Kac mechanism. In fact, $1^1 2^2 4^1$ theory has $SU(2)^2 \times U(1)^2$ extra gauge group [5]. It turns out that the partition function of the theory $2^1 4^1 (10')^1$ is identical to that of $1^1 2^2 4^1$ theory (5.13). Furthermore, their lattice sum, $Z^{SU(3)} \times (Z^{SU(2)})^2$, agrees with the partition functions of the models $1^3 2^2$ and $1^2 2^1 (10')^1$ which describe string propagation on the product of tori. Thus $1^1 2^2 4^1 = 2^1 4^1 (10')^1$ models are the Z_2 twisting of the theories $1^3 2^2 = 1^2 2^1 (10')^1$

$$\left\{1^1 2^2 4^1 = 2^1 4^1 (10')^1\right\} = Z_2 \text{ twisting of } \left\{1^3 2^2 = 1^2 2^1 (10')^1\right\} . \tag{5.17}$$

On the other hand, SO(8) lattice of eq. (5.14) is defined by $\sqrt{2}\,\boldsymbol{e}_i = \boldsymbol{\alpha}_i$ (simple roots of SO(8), $i = 1, \dots, 4$) but with the radius $R_i = 1$ $(i = 1, \dots, 4)$ $(b_{ij} \equiv 0)$. The lattice sum is given by

$$Z^{SO(8)} = \tfrac{1}{2}\left[\left(|\chi_1^{SU(2)}|^2 + |\chi_2^{SU(2)}|^2\right)^4 + \left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^4 + \left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^4 + \left|\frac{\vartheta_4\vartheta_2}{\eta^2}\right|^4\right] . \tag{5.18}$$

We note a relation [26]

$$Z^{SO(8)} = \tfrac{1}{2}\left(\left(Z^{SU(2)}\right)^4 + \sum_{\mu > \nu} \left|\frac{\vartheta_\mu\vartheta_\nu}{\eta^2}\right|^4\right) . \tag{5.19}$$

Eq. (5.19) leads to a remarkable phenomenon; the 2^4 theory can also be identified as a Z_4 orbifold. In fact

$$Z(2^4)=\tfrac{1}{4}\Bigg[\sum_{\mathrm{SU}(2)^4\ \mathrm{lattice}} q^{w_R^2/2}\bar{q}^{w_L^2/2}\frac{1}{|\eta|^8} + 3\left(\left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^4+\left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^4+\left|\frac{\vartheta_4\vartheta_2}{\eta^2}\right|^4\right)\Bigg]\times\frac{1}{2}\sum_i\left|\frac{\vartheta_i}{\eta}\right|^4, \tag{5.20}$$

and $\frac{3}{4}\Sigma_{\mu>\nu}|\vartheta_\mu\vartheta_\nu/\eta^2|^4$ gives precisely the sum over Z_4 twisted sectors. Thus, Gepner's model 2^4 sits at the intersection of the moduli space of Z_2 and Z_4 orbifolds (see fig. 1a).

We have made further identifications of Gepner's models (see fig. 1b). From both analytical and numerical analyses, we find that

$$1^6 \text{ theory} = 1^4 4^1 \text{ theory} = Z_3 \text{ orbifold with } \mathrm{SU}(3)\times\mathrm{SU}(2)^2 \text{ lattice}, \tag{5.21}$$

$$1^2 4^2 \text{ theory} = Z_6 \text{ orbifold with } \mathrm{SU}(3)^2 \text{ lattice}, = Z_3 \text{ orbifold with } \mathrm{SU}(3)\times\mathrm{SU}(3)' \text{ lattice}, \tag{5.22}$$

$$4^3 \text{ theory} = Z_6 \text{ orbifold with } \mathrm{SU}(3)\times\mathrm{SU}(3)' \text{ lattice}. \tag{5.23}$$

Here SU(3)′ lattice is the SU(3) lattice with the radius R_i replaced by $R_i = 1$. Note that $1^2 4^2$ is also identified as two different types of orbifolds, Z_3 and Z_6 at the same time. The above list, eqs. (5.13), (5.14) and (5.17) and eqs. (5.21)–(5.23) exhausts all $N = 2$ models consisting of subtheories with the level $k \leqslant 4$.

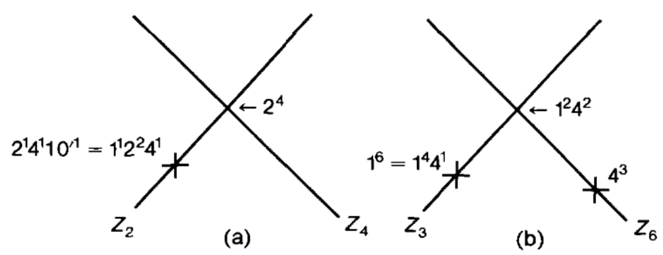

Fig. 1. The relation between Gepner's models and the orbifolds. Moduli spaces of orbifolds are represented symbolically by straight lines.

6. $c=9$ superconformal field theories

In this section we discuss $c=9$ superconformal field theories and their relation to Calabi–Yau manifolds. As we have mentioned in sect. 2, for a systematic treatment of $c=9$ theories we need to study an enlarged version of the $N=2$ algebra extended by the addition of the flow generators. The flow generators, denoted as X, $\bar{X}$, have $h=\frac{3}{2}$ and $Q=\pm 3$ and their commutators with $G, \bar{G}$ generate additional operators $Y, \bar{Y}$ with $h=2$, $Q=\pm 2$. These generators, together with L, J, generate an algebra which contains bilinear terms in the right-hand side of commutation relations. This is a non-Lie algebra of the type introduced by Zamolodchikov [19] and we call it as the $c=9$ algebra.

Commutation relations of the $c=9$ algebra have recently been worked out [22]. Among its commutators, important ones are given by

$$\{X_r, \bar{X}_s\} = \left(r^2-\tfrac{1}{4}\right)\delta_{r+s,0} + (r-s)J_{r+s} + (J^2)_{r+s}, \tag{6.1}$$

$$[X_r, \bar{Y}_m] = \left(r+\tfrac{1}{2}\right)G_{r+m} + (JG)_{r+m}, \tag{6.2}$$

$$[\bar{X}_r, Y_m] = \left(r+\tfrac{1}{2}\right)\bar{G}_{r+m} - (J\bar{G})_{r+m}, \tag{6.3}$$

$$\begin{aligned}[Y_n, \bar{Y}_m] = {} & \tfrac{1}{2}n(n^2-1)\,\delta_{n+m,0} + \tfrac{1}{2}\left(n(n+1)+m(m+1)\right)J_{n+m} \\ & + \tfrac{1}{4}(n-m)(J^2)_{n+m} - (m+1)L_{n+m} + (JL)_{n+m} \\ & - \tfrac{1}{2}(G\bar{G})_{n+m}.\end{aligned} \tag{6.4}$$

We also record

$$[(J^2)_n, X_r] = 3(n-2r)X_{r+n}. \tag{6.5}$$

Here the bilinear forms of operators are defined with the normal ordering; $(AB)_n \equiv \sum_{p\leqslant -h_A} A_p B_{n-p} + \sum_{p>-h_A}(-1)^{AB} B_{n-p}A_p$ (h_A is the conformal dimension of A). Inside the $c=9$ algebra, $L, J, G, \bar{G}$ form the standard $N=2$ algebra with $c=9$ and $\frac{1}{6}J^2$, $\frac{1}{3}J$, $1/\sqrt{3}\,X$ and $1/\sqrt{3}\,\bar{X}$ form an additional $N=2$ algebra with $c=1$. We note that the latter is isomorphic to the algebra of Waterson [28].

The $c=9$ algebra is again invariant under a transformation which shifts the moding of the operators G, $\bar{G}$, X and $\bar{X}$, and thus the NS and R sectors of the algebra are isomorphic to each other.

It follows from eqs. (6.1) and (6.5) that the allowed values of h and Q of the highest weight states are given by (in the NS sector).

Massless representations:

$$\begin{cases} h=0, & Q=0, \\ h=\frac{1}{2}, & Q=1, \\ h=\frac{1}{2}, & Q=-1. \end{cases} \tag{6.6}$$

Massive representations:

$$\begin{cases} h>0, & Q=0, \\ h>\frac{1}{2}, & Q=\pm 1. \end{cases} \tag{6.7}$$

Under the spectral flow, representations (6.6) and (6.7) are mapped onto the R representations.

Massless representations:

$$\begin{cases} h=\frac{3}{8}, & Q=\pm\frac{3}{2}, \\ h=\frac{3}{8}, & Q=\frac{1}{2}, \\ h=\frac{3}{8}, & Q=-\frac{1}{2}. \end{cases} \tag{6.8}$$

Massive representations:

$$\begin{cases} h>\frac{3}{8}, & Q=\pm\frac{3}{2}, \pm\frac{1}{2}, \\ h>\frac{3}{8}, & Q=\pm\frac{1}{2}. \end{cases} \tag{6.9}$$

Although precise character formulas of the $c=9$ algebra have not yet been worked out, their dependence on the U(1) angle θ can be described by the functions

$$f_Q(\theta) \equiv \frac{1}{\eta} \sum_n q^{3(n+Q/3)^2/2} \, \mathrm{e}^{3i(n+Q/3)\theta}. \tag{6.10}$$

In fact these are the only functions whose θ-dependence is consistent with the spectral flow

$$f_Q(\theta \pm \pi\tau) = q^{-3/8} \, \mathrm{e}^{\mp i3\theta/2} f_{Q\pm 3/2}(\theta). \tag{6.11}$$

The f_Q, with $Q=0, \pm 1$ ($\pm 3/2, \pm 1/2$), give characters of the NS(R) representations (6.6) and (6.7) ((6.8) and (6.9)) up to factors depending only on τ. We note that

$$f_{Q=0}(\theta=\pi\tau+\pi)=0, \tag{6.12}$$

$$\begin{aligned} f_{Q=1}(\theta=\pi\tau+\pi) &= -f_{Q=-1}(\theta=\pi\tau+\pi) \\ &= q^{-3/8}. \end{aligned} \tag{6.13}$$

Let us now repeat our analysis of sect. 2 and construct modular-invariant partition functions of $c=9$ superconformal theories. If we consider, for instance, 1^9 theory, the graviton orbit is given by

$$\mathrm{NS}_1(z)=\mathrm{A}(z)^9+\mathrm{B}(z)^9+\mathrm{C}(z)^9. \tag{6.14}$$

An essential difference from the case of the K_3 surface is that the Witten index of eq. (6.14) now vanishes

$$\tilde{\mathrm{R}}_1(\theta=0)\propto \mathrm{NS}_1(\theta=\pi\tau+\pi)=0. \tag{6.15}$$

This is due to the cancellation of contributions from the $h=\frac{3}{8}$, $Q=\pm\frac{3}{2}$ states in the R sector. (This is related to the fact that the charge-conjugation operator (anti-) commutes with the helicity operator in $4m(+2)$ dimensions. Covariantly constant spinors $h=\frac{1}{8}n$, $Q=\pm\frac{1}{2}n$ on an n-dimensional complex manifold with SU(n) holonomy have the same (opposite) helicities when $n=$ even (odd).)

NS_1 is then expanded as

$$\mathrm{NS}_1(\tau;z)=G_1(\tau)(f_1(z)+f_{-1}(z))+H_1(\tau)f_0(z), \tag{6.16}$$

$$G_1(\tau)=\sum_{n=1} g_n^{(1)}q^n, \qquad H_1(\tau)=q^{-1/3}\Big(1+\sum_{n=1} h_n^{(1)}q^n\Big). \tag{6.17}$$

Other orbits are again generated by the S transformation of the graviton trajectory. In the $c=9$ case, there are four types of NS orbits.

(1) Graviton orbit:

NS_1 is the only trajectory containing the ground state $h=Q=0$.

(2) Massless matter orbits:

NS_i ($i=2,\dots,d$) contains a state $h=\frac{1}{2}$, $Q=1$ and is rewritten as

$$\mathrm{NS}_i(\tau;z)=f_1(z)+G_i(\tau)(f_1(z)+f_{-1}(z))+H_i(\tau)f_0(z), \tag{6.18}$$

$$G_i(\tau)=\sum_{n=1} g_n^{(i)}q^n, \qquad H_i(\tau)=q^{-1/3}\sum_{n=1} h_n^{(i)}q^n. \tag{6.19}$$

Orbit i is paired with its conjugate orbit $i^*\equiv i+d-1$. $\mathrm{NS}_{i^*}(\tau;\theta)$ is given by $\mathrm{NS}_i(\tau;\theta)$ with θ replaced by $-\theta$

$$\mathrm{NS}_{i^*}(\tau;z)=f_{-1}(z)+G_i(\tau)(f_1(z)+f_{-1}(z))+H_i(\tau)f_0(z), \tag{6.20}$$

NS_{i^*} contains a state $h=\frac{1}{2}$, $Q=-1$.

(In some cases, a state $h=\frac{1}{2}$, $Q=1$ or -1 appears more than once in each orbit.

Then, the first terms in the right-hand-side of eqs. (6.18) and (6.20) must be multiplied by the multiplicity.)

(3) Self-conjugate massless orbits:
NS_j ($j = 2d,\dots,2d+d'-1$) contains both states $h=\frac{1}{2}$ and $Q=\pm 1$ and is self-conjugate. It is rewritten as

$$\mathrm{NS}_j(\tau;z) = G_j(\tau)(f_1(z)+f_{-1}(z)) + H_j(\tau)f_0(z), \tag{6.21}$$

$$G_j(\tau) = 1 + \sum_{n=1} g_n^{(j)}q^n, \qquad H_j(\tau) = q^{-1/3}\sum_{n=1} h_n^{(j)}q^n. \tag{6.22}$$

(4) Massive orbits:
NS_m ($m = 2d+d',\dots,2d+d'+d''$) does not contain any of the states $h=Q=0$, $h=\frac{1}{2}$, $Q=\pm 1$. It is written as

$$\mathrm{NS}_m(\tau;z) = G_m(\tau)(f_1(z)+f_{-1}(z)) + H_m(\tau)f_0(z), \tag{6.23}$$

$$G_m(\tau) = q^{r_m}\sum_{n=0} g_n^{(m)}q^n, \qquad H_m(\tau) = q^{r'_m}\sum_{n=0} h_n^{(m)}q^n, \tag{6.24}$$

$$r_m, r'_m \in \boldsymbol{Q}, \quad 0 < r_m, \quad -\tfrac{1}{3} < r'_m.$$

These $2d+d'+d''$ trajectories enter into the modular-invariant partition functions. In the 1^9 case, $d=2$, $d'=0$, $d''=3$.

As in the case of $c=6$ theories, modular invariants are formed by using the D-coefficients which symmetrize the matrix of S-transformation

$$Z_\sigma = \tfrac{1}{2}\sum_{i=1}^{2d+d'+d''} D_i\left\{|\mathrm{NS}_i|^2 + |\widetilde{\mathrm{NS}}_i|^2 + |\mathrm{R}_i|^2 + |\tilde{\mathrm{R}}_i|^2\right\}. \tag{6.25}$$

Note that $D_{i^*} = D_i$. The Euler number is given by

$$\chi = -2\sum_{i=2}^{d} D_i. \tag{6.26}$$

(There exists a sign ambiguity in deriving the Euler number by using the Witten index. In eqs. (2.26) and (6.26), we have defined the states $h=\frac{1}{2}n$, $Q=\pm\frac{1}{2}n$ in the R sector to be bosonic (fermionic) when $n=$ even (odd)). In the case of $c=9$ theories, however, a set of new modular invariants can be constructed from each "A-type" invariant (6.25). These new invariants have Euler numbers which differ from (6.26) by $4\times$ integers as we shall see later.

Let us now consider partition functions of the heterotic string theory in the $c=9$ case. They are constructed by multiplying SO(2), and SO(10) and E_8 characters to the orbits in the right and left-moving sectors, respectively. The cosmological constant again vanishes in heterotic string compactification. Orbits in the right sector are proportional to

$$\vartheta_3 f_Q - \vartheta_4 \tilde{f}_Q - \vartheta_2 f_{Q+3/2} = 0, \qquad Q = 0, \pm 1, \tag{6.27}$$

where $\tilde{f}_Q \equiv f_Q(\theta = \pi)$. Eq. (6.27) can be shown to be true by using the product formula of theta functions. On the other hand, orbits in the left sector are reexpressed in terms of E_6 characters

$$\chi_1^{E_6} = \frac{1}{2}\left[\left(\frac{\vartheta_3}{\eta}\right)^5 f_0 + \left(\frac{\vartheta_4}{\eta}\right)^5 \tilde{f}_0 + \left(\frac{\vartheta_2}{\eta}\right)^5 f_{3/2}\right], \tag{6.28}$$

$$\chi_{27}^{E_6} = \frac{1}{2}\left[\left(\frac{\vartheta_3}{\eta}\right)^5 f_{-1} + \left(\frac{\vartheta_4}{\eta}\right)^5 \tilde{f}_{-1} + \left(\frac{\vartheta_2}{\eta}\right)^5 f_{1/2}\right], \tag{6.29}$$

$$\chi_{27^*}^{E_6} = \frac{1}{2}\left[\left(\frac{\vartheta_3}{\eta}\right)^5 f_1 + \left(\frac{\vartheta_4}{\eta}\right)^5 \tilde{f}_1 + \left(\frac{\vartheta_2}{\eta}\right)^5 f_{-1/2}\right]. \tag{6.30}$$

Partition functions are again formed as

$$Z = \frac{\text{const.}}{(\operatorname{Im}\tau)|\eta|^4} \sum D_i (X_{R,i})(X^*_{L,i}). \tag{6.31}$$

Eq. (6.31) contains the graviton and E_6 gauge multiplets. There also exist massless scalar multiplets of $\underline{27}$ and $\underline{27}^*$ of E_6 which come from the combinations $|h = \frac{1}{2}, Q = +1\rangle \otimes |\bar{h} = \frac{1}{2}, \bar{Q} = +1\rangle$ and $|h = \frac{1}{2}, Q = +1\rangle \otimes |\bar{h} = \frac{1}{2}, \bar{Q} = -1\rangle$, respectively. Their multiplicities are

$$n_{27} = \sum_{i=2}^{d} D_i + \sum_{j=2d}^{2d+d'-1} D_j, \tag{6.32}$$

$$n_{27^*} = \sum_{j=2d}^{2d+d'-1} D_j. \tag{6.33}$$

In Calabi–Yau compactifications they are identified with the Hodge numbers, $n_{27} = h^{2,1}$ and $n_{27^*} = h^{1,1}$, and are related to the Euler characteristic as $\chi = 2(n_{27^*} - n_{27})$.

set of invariants with Euler numbers by varying m_i, n_i

$$|\chi|, |\chi| - 4, \dots, -|\chi| + 4, -|\chi| . \tag{6.40}$$

In forming D-type invariants (6.35), the left–right pairing of states $h = \frac{1}{2}$, $Q = \pm 1$ is reshuffled and this corresponds to interchanging the roles of $h^{2,1}$ and $h^{1,1}$. Such a procedure does not have a well-defined geometrical significance and the modular invariants (6.38) could not all describe string propagation on Calabi–Yau manifolds. It is important to check the consistency of the operator interpretation of D-type modular invariants.

We thank Prof. A. Hattori and Dr. M. Furuta for discussions on elliptic genera. We also thank Prof. K. Saito for his lectures on K_3 surfaces.

Note added in proof

Quite recently Vafa-Warner (Harvard preprint, Nov. 1988) and Martinec (Chicago preprint, Nov. 1988) have proposed an interpretation of Gepner's models as describing string propagation on algebraic varieties in (weighted) complex projective spaces. According to this interpretation 2^4 and 4^3 models, for instance, describe strings compactified on K_3 surfaces defined by $z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$ (Fermat surface) and $z_1^6 + z_2^6 + z_3^6 + z_4^2 = 0$, respectively (association of 2^4 model with Fermat's surface is originally due to ref. [5]).

Our results in sect. 5 then imply that partition functions of strings compactified on some algebraic varieties coincide with those on orbifolds. Thus our analysis gives rise to examples of different points in the moduli space of K_3 surface being described by identical conformal field theories.

Appendix A

DISCRETE REPRESENTATIONS OF $N = 2$ ALGEBRA

Unitary representations of $N = 2$ superconformal algebra in the region $0 < c < 3$ exist at [1, 23]

$$c = \frac{3k}{k+2}, \qquad k = 1, 2, 3, \dots . \tag{A.1}$$

Their highest weight states are characterized by the conformal dimension h and the U(1) charge Q. In the NS sector their allowed values are

$$h_{l,m} = \frac{l(l+2) - m^2}{4(k+2)}, \tag{A.2}$$

$$Q_{l,m} = \frac{m}{k+2}, \tag{A.3}$$

where $0 \leqslant l \leqslant k$, $-l \leqslant m \leqslant l$, $l - m \equiv 0 \bmod 2$. $N = 2$ discrete representations are intimately related to the parafermion algebra [24]. Their character formulas are expressed in terms of the $A_1^{(1)}$ string function $c_{l,m}^{(k)}$ [4, 25]

$$\mathrm{ch}_{l,m}^{\mathrm{NS},k}(\tau;\theta) = \sum_{m'=-k+1}^{k} c_{l,m'}^{(k)}(\tau)\,\Theta_{(k+2)m'-mk,\,k(k+2)}\left(\tfrac{1}{2}\tau, \frac{\theta}{k+2}\right). \tag{A.4}$$

Here $\Theta_{m,k}$ is the level-k theta-function

$$\Theta_{m,k}(\tau,\theta) = \sum_n q^{k(n+m/2k)^2}\,\mathrm{e}^{ik\theta(n+m/2k)}. \tag{A.5}$$

Character formulas in the Ramond sector are obtained by the flow from the NS sector

$$\mathrm{ch}_{l,m}^{\mathrm{NS}}(\tau;\theta+\pi\tau) = q^{-c/24}\,\mathrm{e}^{-ic\theta/6}\,\mathrm{ch}_{l,m}^{\mathrm{R}}(\tau;\theta). \tag{A.6}$$

Under modular transformation, string functions behave as [27]

$$c_{l,m}^{(k)}\left(-\frac{1}{\tau}\right) = \left(\frac{1}{k(k+2)}\right)^{1/2}\left(\frac{i}{\tau}\right)^{1/2} \sum_{l',m'} \mathrm{e}^{i\pi m m'/k} \sin\left(\pi\frac{(l+1)(l'+1)}{k+2}\right) c_{l',m'}^{(k)}(\tau), \tag{A.7}$$

$$c_{l,m}^{(k)}(\tau+1) = \exp\left[2\pi i\left(\frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k} - \frac{c}{24}\right)\right] c_{l,m}^{(k)}(\tau). \tag{A.8}$$

Eqs. (A.7) and (A.8) determine modular properties of the $N = 2$ discrete characters.

For the unitary representations of the $N = 2$ algebra in the continuum range $c > 3$ and their connection to $N = 4$ algebra, see ref. [14].

Appendix B

UNITARY REPRESENTATIONS OF $N = 4$ ALGEBRA [13, 14]

The $N = 4$ algebra with the central charge $6k$ contains a level-k affine SU(2) algebra. $N = 4$ highest-weight states are parametrized by h and isospin l. When $c = 6$, $k = 1$, possible unitary representations are:

(1) massless representations

$$h = l, \quad l = 0, \tfrac{1}{2}, \quad \text{NS sector},$$

$$h = \tfrac{1}{4}, \quad l = 0, \tfrac{1}{2}, \quad \text{R sector};$$

(2) massive representations

$$h > 0, \quad l = 0, \quad \text{NS sector},$$

$$h > \tfrac{1}{4}, \quad l = \tfrac{1}{2}, \quad \text{R sector}.$$

Character formulas are given by:
(1) massless representations

$$\mathrm{ch}_0^{\mathrm{NS}}(l=0;\tau;z) = \sum q^{m^2/2-1/4} z^m \frac{zq^{m-1/2}-1}{1+zq^{m-1/2}}$$

$$\times \frac{\prod_{n=1}(1+zq^{n-1/2})(1+z^{-1}q^{n-1/2})}{\prod_{n=1}(1-q^n)^2}, \tag{B.1}$$

$$\mathrm{ch}_0^{\mathrm{NS}}(l=\tfrac{1}{2};\tau;z) = \sum q^{m^2/2-1/4} z^m \frac{1}{1+zq^{m-1/2}}$$

$$\times \frac{\prod_{n=1}(1+zq^{n-1/2})(1+z^{-1}q^{n-1/2})}{\prod_{n=1}(1-q^n)^2}; \tag{B.2}$$

(2) massive representations

$$\mathrm{ch}^{\mathrm{NS}}(h;\tau;z) = q^{h-1/4} \frac{\prod_{n=1}(1+zq^{n-1/2})^2(1+z^{-1}q^{n-1/2})^2}{\prod_{n=1}(1-q^n)}. \tag{B.3}$$

Note that from eqs. (B.1)–(B.3)

$$\mathrm{ch}^{\mathrm{NS}}(h=0) = \mathrm{ch}_0^{\mathrm{NS}}(l=0) + 2\,\mathrm{ch}_0^{\mathrm{NS}}(l=\tfrac{1}{2}). \tag{B.4}$$

R characters are related to NS characters by

$$\mathrm{ch}_0^{\mathrm{NS}}(l;zq^{1/2}) = q^{-1/4}z^{-1}\,\mathrm{ch}_0^{\mathrm{R}}(\tfrac{1}{2}-l;z). \tag{B.5}$$

The same relation also holds for the massive characters. The Witten index is given by the Ramond character at $z=-1$

$$\mathrm{ch}_0^{\mathrm{R}}(l;\tau;z=-1) = \begin{cases} -2, & l=0, \\ 1, & l=\tfrac{1}{2}. \end{cases} \tag{B.6}$$

$$\mathrm{ch}^{\mathrm{R}}(l=\tfrac{1}{2};\tau{:}\ z=-1) = 0. \tag{B.7}$$

A convenient parametrization of massless characters is given by

$$\mathrm{ch}_0^{\mathrm{NS}}(l=0;\tau;z)=2\left(\frac{\vartheta_1(\theta)}{\vartheta_3(0)}\right)^2+\left(\frac{q^{-1/8}}{\eta(\tau)}-2h_3(\tau)\right)\left(\frac{\vartheta_3(\theta)}{\eta(\tau)}\right)^2, \tag{B.8}$$

$$\mathrm{ch}_0^{\mathrm{NS}}(l=\tfrac{1}{2};\tau;z)=-\left(\frac{\vartheta_1(\theta)}{\vartheta_3(0)}\right)^2+h_3(\tau)\left(\frac{\vartheta_3(\theta)}{\eta(\tau)}\right)^2, \tag{B.9}$$

where $h_3(\tau)$ is defined by

$$h_3(\tau)=\frac{1}{\eta(\tau)\vartheta_3(0)}\sum_m\frac{q^{m^2/2-1/8}}{1+q^{m-1/2}}. \tag{B.10}$$

The massive character (B.3) may also be rewritten as

$$\mathrm{ch}^{\mathrm{NS}}(h;\tau;z)=\frac{q^{h-1/8}}{\eta(\tau)}\left(\frac{\vartheta_3(\theta)}{\eta(\tau)}\right)^2. \tag{B.11}$$

The $N=4$ massless characters can be expanded into those of the SU(2) Kac–Moody algebra

$$\mathrm{ch}_0^{\mathrm{NS}}(l=0;\tau;z)=A_{1,1}(\tau)\chi_1^{\mathrm{SU(2)}}(\tau;z)+A_{1,2}(\tau)\chi_2^{\mathrm{SU(2)}}(\tau;z), \tag{B.12}$$

$$\mathrm{ch}_0^{\mathrm{NS}}(l=\tfrac{1}{2};\tau;z)=A_{2,1}(\tau)\chi_1^{\mathrm{SU(2)}}(\tau;z)+A_{2,2}(\tau)\chi_2^{\mathrm{SU(2)}}(\tau;z). \tag{B.13}$$

Branching functions $A_{2l+1,2l'+1}$ have the following q-expansions

$$A_{1,1}(\tau)=\frac{q^{2/24}}{\eta(\tau)}q^{-1/4}(1-q+3q^3+\dots),$$

$$A_{1,2}(\tau)=\frac{q^{2/24}}{\eta(\tau)}(2q+4q^5+\dots),$$

$$A_{2,1}(\tau)=\frac{q^{2/24}}{\eta(\tau)}q^{3/4}(2+2q+2q^2+\dots),$$

$$A_{2,2}(\tau)=\frac{q^{2/24}}{\eta(\tau)}(1+3q^2+4q^3+\dots). \tag{B.14}$$

References

[1] W. Boucher, D. Friedan and A. Kent, Phys. Lett. B172 (1986) 316
[2] A. Sen, Nucl. Phys. B278 (1986) 287; B284 (1987) 423
[3] T. Banks, L. Dixon, D. Friedan and E. Martinec, Nucl. Phys. B299 (1988) 601;
T. Banks and L. Dixon, Nucl. Phys. B307 (1988) 93
[4] D. Gepner, Nucl. Phys. B296 (1988) 757
[5] D. Gepner, Phys. Lett. 199B (1987) 380
[6] N. Seiberg, Nucl. Phys. B303 (1988) 286
[7] D. Friedan, A. Kent, S. Shenker and E. Witten, unpublished (1987)
[8] A. Schwimmer and N. Seiberg, Phys. Lett. B184 (1987) 191
[9] P. Candelas, G. Holowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46
[10] M. Green, J. Schwarz and E. Witten, Superstring Theory 2 (Cambridge University Press, Cambridge, 1987
[11] L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 678; B274 (1986) 285
[12] S. Hamidi and C. Vafa, Nucl. Phys. B279 (1987) 465;
L. Dixon, D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B279 (1987) 13
[13] T. Eguchi and A. Taormina, Phys. Lett. B196 (1987) 75; B200 (1988) 315
[14] T. Eguchi and A. Taormina, Phys. Lett. B210 (1988) 125
[15] A. Schellekens and N. Warner, Phys. Lett. B177 (1986) 317; B181 (1986) 339;
O. Alvarez, T. Killingback, M. Mangano and P. Windey, Berkeley preprint LBL-23044 (1987)
[16] E. Witten, Commun. Math. Phys. 109 (1987) 525
[17] D. Zagier, Note on the Landweber–Stong elliptic genus, Bonn preprint (1987);
C. Taubes, S^1 action and elliptic genera, Harvard preprint (1988)
[18] C. Lütken and G. Ross, Taxonomy of heterotic superconformal field theories, CERN preprint (1988);
K. Higashijima, unpublished (1988)
[19] A.B. Zamolodchikov, Theor. Math. Phys. 65 (1986) 1205
[20] M. Walton, Phys. Rev. D37 (1988) 377
[21] S. Elitzur, E. Gross, E. Rabinovici and N. Seiberg, Nucl. Phys. B283 (1987) 431
[22] S. Odake, private communication
[23] P. DiVecchia, J.L. Petersen and H.B. Zheng, Phys. Lett. B174 (1986) 280;
S. Nam, Phys. Lett. B172 (1986) 323
[24] V.A. Fateev and A.B. Zamolodchikov, Sov. Phys. JETP 62 (1985) 215;
D. Gepner and Z. Qiu, Nucl. Phys. B285 [FS19] (1987) 432
[25] F. Ravanini and S.-K. Yang, Phys. Lett. B195 (1987) 202;
Z. Qiu, Phys. Lett. B198 (1987) 497
[26] K. Kobayashi and M. Sakamoto, Orbifold-compactified models in torus-compactified string theory, Kyoto preprint RIFP-729 (1987)
[27] V. Kac and D. Peterson, Adv. Math. 53 (1984) 125
[28] G. Waterson, Phys. Lett. B171 (1986) 77
[29] A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. B280 (1987)