3H NUMERICAL ANALYSIS 1997-98. PROBLEM SHEET 1.

1.1 Let p_3 be the cubic interpolating polynomial based on the function values $f(0)$, $f(1)$, $f(2)$ and $f(3)$. By using the Newton forward difference formula, or otherwise, express $p_3(6)$ in terms of those function values. What is the uncertainty in the value of $p_3(6)$ if the uncertainty in each function value is $\pm\epsilon$?

1.2 Let $x_0, \ldots, x_n$ be fixed points in $[a, b]$. For any function $f \in C[a, b]$ let $\mathcal{L}_n f$ denote the polynomial of degree $\leq n$ which agrees with f at $x_0, \ldots, x_n$.
(a) Show that $\mathcal{L}_n$ is a linear operator: i.e., $\mathcal{L}_n(\lambda f) = \lambda \mathcal{L}_n f$ and $\mathcal{L}_n(f + g) = \mathcal{L}_n f + \mathcal{L}_n g$.
(b) Show that $\mathcal{L}_n f = f$ if and only if f is a polynomial of degree $\leq n$.

1.3 Let $\{l_i(x)\}$ be the set of Lagrange polynomials for a given set of distinct nodes $x_0, \ldots, x_n$. Prove that $\sum_{i=0}^{n} l_i(x) = 1$, and more generally, $\sum_{i=0}^{n} x_i^k l_i(x) = x^k$ for $k = 0, \ldots, n$.
Hint: If $f(x) \equiv 1$ what is the corresponding interpolating polynomial?

1.4 Prove that for every choice of distinct nodes $x_0, x_1, \ldots, x_n$ the set $\{l_i(x)\}$ of Lagrange polynomials is a basis for the space P_n of polynomials of degree not exceeding n; in other words, the set $\{l_i(x)\}$ consists of the appropriate number of linearly independent elements of P_n.

1.5 Let $f \in C^2[0, 1]$, and let the function value $f(x)$ be estimated by linear interpolation using two of the three values $f(0.0) = 0.0$, $f(0.7) = 0.7$ and $f(1.0) = 0.1$. Express the truncation error in terms of f'' and show that, to minimise the magnitude of the multiplying factor at each point of the interval, it is best to base the interpolation on $f(0.0)$ and $f(0.7)$ if $0 \leq x < 0.5$, but it is best to use $f(0.7)$ and $f(1.0)$ if $0.5 < x \leq 1.0$. Deduce that $f(0.5)$ satisfies the inequalities

$$1.1 - 0.05\|f''\|_\infty \leq f(0.5) \leq 0.5 + 0.05\|f''\|_\infty,$$

and hence obtain a lower bound on $\|f''\|_\infty$.

1.6 A function f takes the values tabulated below.

x	2.0	3.0	4.0	5.0	6.0
$f(x)$	2.0	1.0	0.5	1.0	−5.0

Show that

$$\begin{aligned} f(3.5) &= -\frac{1}{16}f(2) + \frac{9}{16}f(3) + \frac{9}{16}f(4) - \frac{1}{16}f(5) + \frac{3}{128}f^{iv}(\theta) \\ &= 0.65625 + 0.0234375 f^{iv}(\theta) \end{aligned}$$

where $2 < \theta < 5$. Similarly, by using $3, 4, 5, 6$ as interpolation nodes show that

$$f(3.5) = 0.15625 - 0.0390625 f^{iv}(\mu)$$

where $3 < \mu < 6$. If $|f^{(iv)}(x)| \leq 10$ on the interval $[2, 6]$ find upper and lower bounds for $f(3.5)$.

1.7 Find the Hermite interpolating cubic for $1/(1+x^2)$, based on the values of this function and its derivative at -1 and 1.

1.8 (a) Construct a cubic polynomial p_3 such that $p_3(0) = 1$, $p_3(1) = 3$, $p_3'(-1) = 4$ and $p_3''(0) = 0$. Is the solution unique?

(b) Is there a unique quadratic p_2 such that $p_2(-1) = 0 = p_2(1)$ and $p_2'(0) = 0$?

(c) Let $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$. Given a function $f \in C^2[-1, 1]$, prove that there is no quintic polynomial p_5 such that

$$p_5(x_i) = f(x_i), \qquad p_5''(x_i) = f''(x_i), \qquad i = 0, 1, 2.$$

1.9 Show that the Lagrange interpolating polynomial for nodes at the zeros of the Chebyshev polynomial $T_{n+1}(x)$ is

$$p_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} \frac{(-1)^k\sqrt{1-x_k^2}\,T_{n+1}(x)}{x-x_k} f(x_k).$$

Hint: Express $\omega_{n+1}(x)$ in terms of $T_{n+1}(x)$ and then use the equation $T_{n+1}(x) = \cos(n+1)\theta$, with $x = \cos\theta$.

1.10 Let $q_n(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be appproximated by $p_{n-1}(x)$, the polynomial of degree $\leq n-1$ which takes the same values as q_n at $x_0, x_1, \ldots, x_{n-1}$. Prove that, when the nodes are chosen so as to minimise the maximum error on $[-1,1]$,

$$p_{n-1}(x) = q_n(x) - 2^{1-n}T_n(x).$$

1.11 By Chebyshev economisation find a quadratic approximation for $\cos x$ with error less than 0.007 on the interval $[-1,1]$.

1.12 Starting with the truncated power series

$$e^{-x} \approx 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

find a quadratic approximation for e^{-x}, and find an upper bound on the truncation error in this approximation on $[-1,1]$. Is this a realistic bound?

1.13 Economise the truncated power series

$$\tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}$$

reducing the degree of the polynomial as much as possible while ensuring that the error incurred *in the process of economisation* does not exceed 0.003 on the interval $[-1,1]$.

1.14 The Bessel function of order zero may be expressed as

$$J_0(2x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(k!)^2}.$$

Find a fourth-degree polynomial approximation for $J_0(2x)$ which has a truncation error of magnitude no greater than 0.003 for $-1 \leq x \leq 1$.

1.15 By economisation based on the shifted Chebyshev polynomials, find a quadratic approximation for $\cos x$ with error less than 0.005 on the interval $[0,1]$. Compare with the result of Problem 1.11.

1.16 Find the Bernstein polynomial $B_n(f;x)$ for $f(x) = x^3$ on $[0,1]$.

1.17 For $f(x) = (1+x^2)^{-1}$ calculate the Bernstein polynomials for $n = 2$ and 3 on the interval $[-1,1]$. Compare the graphs of $f(x)$, $B_2(f;x)$, and $B_3(f;x)$ on this interval.

1.18 Find the Bernstein polynomials of degree 1 and 2 for $f(x) = \sin x$ on the interval $[0, \pi/2]$.

1.19 Prove that

$$S_{m+1} = x(1-x)\left[S_m'(x) + mnS_{m-1}(x)\right],$$

where

$$S_m(x) = \sum_{k=0}^{n} p_{nk}(x)(k-nx)^m$$

and the prime denotes differentiation with respect to x. Hence find $S_3(x)$ and $S_4(x)$.