3H NUMERICAL ANALYSIS 1997/98. PROBLEM SHEET 3.

3.1 Show that the minimax constant approximation to a function $f \in C[a,b]$ is

$$c = \frac{1}{2}\left[\max_{x\in[a,b]} f(x) + \min_{x\in[a,b]} f(x)\right]$$

and that the error of largest magnitude is

$$E = \frac{1}{2}\left[\max_{x\in[a,b]} f(x) - \min_{x\in[a,b]} f(x)\right].$$

3.2 Show that the minimax linear approximation to $\sin(4\pi x) + ax + b$ on $[0,1]$ is $p_1(x) = ax + b$, whatever the values of a and b.

3.3 Show that, in any Chebyshev series $\sum_{k=0}^{\infty} a_k T_k(x)$, each partial sum S_n is the minimax polynomial of degree n for S_{n+1} on $[-1,1]$. Is S_{n-1} the minimax polynomial of degree $n-1$ for S_{n+1}?

3.4 Find the minimax polynomial of degree less than or equal to 5 for

$$f(x) = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 - \tfrac{1}{8}x^6$$

on the interval $[-1,1]$. What is the maximum error on this interval?

3.5 Show that the minimax quadratic for $f(x) = 144/(x+2)$ on $[0,6]$ is $p_2^*(x) = 69 - 20x + 2x^2$, and that the extreme values of the error occur at the points $x = 0, 1, 4, 6$.

3.6 For a function $f \in C^2[a,b]$, with $f''(x) > 0$ on $[a,b]$, prove that the linear minimax approximation on $[a,b]$ is $\alpha + \beta x$ where

$$\beta = \frac{f(b) - f(a)}{b - a}, \qquad \alpha = \tfrac{1}{2}[f(c) + f(b) - \beta(c+b)]$$

for some $c \in (a,b)$, and explain how c is determined.
To use an iterative method for calculating the square root of a number $A \in [\frac{1}{2}, 2]$ it is necessary to provide an estimate of $\sqrt{A}$. This may be done by finding the minimax linear approximation for $f(x) = x^2 - A$ on $[\frac{1}{2}, 2]$ and then finding the point x_0 at which this straight line cuts the x-axis. Express x_0 in terms of A, and find a numerical upper bound on the error $|\sqrt{A} - x_0|$ for $\frac{1}{2} \le A \le 2$.

3.7 By using the first part of Problem 3.6, or otherwise, prove that the minimax linear approximation to $\sqrt{1+x^2}$ on the interval $[0,1]$ is $\frac{1}{2}(2c+1) + (\sqrt{2}-1)x$ where $c = [(\sqrt{2}-1)/2]^{1/2}$. Obtain a numerical value for the maximum absolute error in this approximation.

3.8 Find the best (1) constant and (2) linear approximation, in the minimax sense, to $\arctan x$ for $0 \le x \le 1$.

3.9 Suppose we decide to approximate x on $[-1,1]$ by quadratics of the form αx^2 where α is real—a silly thing to do, but the exercise has some value. Find best approximations of this type with respect to the L_2, L_1 and L_∞ norms. Is there a unique value of α in each case?

3.10 Calculate the minimax quadratic approximation for $f(x) = |x + \frac{1}{2}|$ on $[-1,1]$. It may be helpful first to draw a sketch to see what to expect.

3.11 Let $p_n^*(x)$ be the minimax polynomial of degree n, on the interval $[-1,1]$, for

$$f(x) = x^{n+2} - \sigma x^{n+1}$$

where σ is a non-negative constant, and let $E_{n+2}(x) = f(x) - p_n^*(x)$. By examining the effect, on the form of $f(x)$, of the change of variable

$$u = \frac{x - \sigma/(n+2)}{1 + \sigma/(n+2)},$$

or otherwise, verify that

$$E_{n+2}(x) = \frac{1}{2^{n+1}}\left(1 + \frac{\sigma}{n+2}\right)^{n+2} T_{n+2}(u)$$

when $\sigma \leq (n+2)\tan^2(\pi/(2n+4))$, and explain why this condition must be imposed.

3.12 Apply the discrete version of the one-point exchange algorithm to calculate the minimax linear approximation to the following function values

x	0	1	2	3	4	5	6
$f(x)$	0.3	4.2	0.1	3.4	5.7	4.9	5.7

Let the initial reference be the point set $\{0, 3, 6\}$. Would you have arrived at the result more quickly if you had allowed more than one point of the reference to change at each iteration?

3.13 Use the method of forced oscillation of the error to find a near-minimax quadratic approximation for $\ln x$ on $[1, 2]$. What is the maximum error in this approximation? What can you conclude about the error in the minimax quadratic approximation for $\ln x$ on $[1, 2]$?

3.14 Find the [3/2] Padé approximant for $\tan^{-1} x$. Use this to estimate π.

3.15 Obtain the [4/1], [3/2], and [2/3] Padé approximants for $\ln(1+x)$ and evaluate them at $x = 1$. Compare the leading error terms of those approximants.

3.16 Find a rational approximation, with numerator and denominator both of second degree, for the Bessel function $J_0(2x)$. Estimate the position of the smallest zero of $J_0(2x)$ for positive x.

3.17 Find some diagonal Padé approximants for the divergent series

$$1 - x + 2!x^2 - 3!x^3 + \cdots$$

obtained by formally expanding the denominator of the integrand in $\int_0^\infty e^{-t}/(1+xt)\,dt$. Compare the numerical values of your Padé approximants when $x = 1$ with the exact value of the integral, which is 0.5963 to four decimal places.

3.18 Show that the [1/2] Padé approximant for e^x is

$$R_{12}(x) = \frac{1 + \frac{1}{3}x}{1 - \frac{2}{3}x + \frac{1}{6}x^2}$$

and that its error may be written as

$$E(x) = e^x - R_{12}(x) = \frac{x^4 \sum_{k=0}^{\infty} d_k x^k}{1 - \frac{2}{3}x + \frac{1}{6}x^2}$$

where

$$d_k = \frac{(k+1)(k+2)}{6(k+4)!} \qquad \text{for } k = 0, 1, \ldots.$$

Show that if $|x| < 0.1$ then $|E(x)| < 1.7 \times 10^{-6}$. Rearrange $R_{12}(x)$ so that it may be evaluated in two divisions and no multiplications. Compare this with the accuracy and computational effort involved in using the Maclaurin polynomial of degree 3.