# 1 A quick sketch of Lagrangian mechanics

**Ex 1** The real-valued and continuous function $f(t)$, defined for $t \in [t_1, t_2]$, is such that

$$\int_{t_1}^{t_2} dt\ f(t)\delta u(t) = 0 \tag{1.1}$$

for all real-valued and continuous (infinitesimal) functions $\delta u(t)$ with $\delta u(t_1) = \delta u(t_2) = 0$. Prove that $f(t)$ is identically zero for all $t \in [t_1, t_2]$. This is sometimes called the fundamental lemma of the calculus of variations.
[**Hint**: consider $\delta u(t) = \epsilon(t - t_1)(t_2 - t)f(t)$ for $0 < \epsilon \ll 1$.]

**Solution 1** Note that the suggestion for $\delta u(t)$ does indeed satisfy $\delta u(t_1) = \delta u(t_2) = 0$. Taking this option, we have that

$$0 = \epsilon \int_{t_1}^{t_2} dt\ (t - t_1)(t_2 - t)f(t)^2$$

Now $(t - t_1)(t_2 - t) > 0\ \forall t \in (t_1, t_2)$, while $f(t)^2 \geq 0\ \forall t \in (t_1, t_2)$. So the only way for the integral to vanish is for the integrand to be everywhere zero, which in turn requires $f(t) = 0\ \forall t \in (t_1, t_2)$ (and, since $f$ is continuous, the same must also be true at the end points $t_1$ and $t_2$).

**Ex 2** The position $u = u(t)$ of a point particle is a function of time $t$. The particle is described by the **action**

$$S[u] = \int_{t_1}^{t_2} dt\ L(u, u_t)\ , \tag{1.2}$$

where the **Lagrangian** $L(u, u_t)$ does not depend explicitly on time. Therefore the Lagrangian depends on time only through the time dependence of $u(t)$ and $u_t(t)$.

1. Use the chain rule to compute the total time derivative of the Lagrangian

$$\frac{d}{dt}L(u(t), u_t(t)) = \frac{\partial L}{\partial u}u_t + \frac{\partial L}{\partial u_t}u_{tt}\ . \tag{1.3}$$

2. Show that the Euler-Lagrange equation for $u$ implies the *Beltrami identity*

$$\frac{d}{dt}\left(u_t\frac{\partial L}{\partial u_t} - L\right) = 0\ , \tag{1.4}$$

which states the conservation of the total energy (or **Hamiltonian**)

$$E = u_t\frac{\partial L}{\partial u_t} - L\ . \tag{1.5}$$

3. For a particle described by the Lagrangian

$$L = T - V \ , \tag{1.6}$$

where $T(u_t) = \frac{m}{2}u_t^2$ is the kinetic energy and $V(u)$ is the potential energy, show that the conserved total energy is $E = T + V$.

**Solution 2**

1. $$\frac{d}{dt}L(u(t), u_t(t)) = \frac{\partial L}{\partial u}\frac{du}{dt} + \frac{\partial L}{\partial u_t}\frac{du_t}{dt} = \frac{\partial L}{\partial u}u_t + \frac{\partial L}{\partial u_t}u_{tt}$$

as claimed.

2. $$\begin{aligned}\frac{d}{dt}\left(u_t\frac{\partial L}{\partial u_t} - L\right) &= u_{tt}\frac{\partial L}{\partial u_t} + u_t\frac{d}{dt}\frac{\partial L}{\partial u_t} - \frac{\partial L}{\partial u}u_t - \frac{\partial L}{\partial u_t}u_{tt} \\ &= u_t\frac{d}{dt}\frac{\partial L}{\partial u_t} - \frac{\partial L}{\partial u}u_t \\ &= u_t\left(\frac{d}{dt}\frac{\partial L}{\partial u_t} - \frac{\partial L}{\partial u}\right) \ = \ 0\end{aligned}$$

with the final equality following from the Euler-Lagrange equation.

3. With

$$L = T - V = \frac{m}{2}u_t^2 - V(u)$$

we have

$$E = u_t\frac{\partial L}{\partial u_t} - L = mu_t^2 - \left(\frac{m}{2}u_t^2 - V(u)\right) = \frac{m}{2}u_t^2 + V(u) = T + V \, .$$

**Ex 3**

1. Consider a field theory described by a field $w(x,t)$. If the Lagrangian density $\mathcal{L}$ is a function of $w$, $w_x$, $w_{xx}$, and $w_t$, and if

$$\delta S[w] = \delta \iint dxdt \ \mathcal{L}(w, w_x, w_{xx}, w_t) = 0 \ ,$$

derive the (generalised) Euler-Lagrange equation (or *equation of motion*)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial w_t} + \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial w_x} - \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial w_{xx}} - \frac{\partial \mathcal{L}}{\partial w} = 0 \ . \tag{1.7}$$

[**Hint**: mimic the derivation in the notes, taking into account the extra term involving $(\partial\mathcal{L}/\partial w_{xx})\delta w_{xx}$ that appears in the variation of $S$.]

2. What happens if $\mathcal{L}$ also depends on $w_{xxx}$? And if $\mathcal{L}$ depends on $\partial^n w/\partial x^n$?

3. If $\mathcal{L} = \frac{1}{2}w_x w_t + w_x^3 - \frac{1}{2}w_{xx}^2$, use the result of part 1 to find the equation of motion for $w$, and verify that if $w$ is satisfies this equation, then $u \equiv w_x$ solves the KdV equation.

**Solution 3**

1. Arguing as in the notes,

$$\begin{aligned}\delta S[w] &= S[w+\delta w] - S[w] \\ &= \iint dxdt\, (\mathcal{L}(w{+}\delta w, w_x{+}\delta w_x, w_{xx}{+}\delta w_{xx}, w_t{+}\delta w_t) - \mathcal{L}(w, w_x, w_{xx}, w_t)) \\ &= \iint dxdt \left( \frac{\partial \mathcal{L}}{\partial w}\delta w + \frac{\partial \mathcal{L}}{\partial w_x}\delta w_x + \frac{\partial \mathcal{L}}{\partial w_{xx}}\delta w_{xx} + \frac{\partial \mathcal{L}}{\partial w_t}\delta w_t \right) \\ &= \iint dxdt \left( \frac{\partial \mathcal{L}}{\partial w} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial w_x} + \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial w_{xx}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial w_t} \right) \delta w\,.\end{aligned}$$

To go from the third line to the fourth, we integrated the term proportional to $\delta w_x$ once by parts with respect to $x$, the term proportional to $\delta w_{xx}$ twice by parts with respect to $x$ (so the minus sign turned back into a plus), and the term proportional to $\delta w_t$ once by parts with respect to $t$. Requiring the final line to be zero for all $\delta w$ implies the generalised Euler-Lagrange equation quoted in the question. (As usual, we're assuming suitable conditions on $w$ at infinity so that the boundary terms in the integrations by parts vanish.)

2. If $L$ also depends on $w_{xxx}$, the same argument with now three integrations by parts for the (new) term proportional to $\delta w_{xxx}$ gives

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial w_t} + \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial w_x} - \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial w_{xx}} + \frac{d^3}{dx^3}\frac{\partial \mathcal{L}}{\partial w_{xxx}} - \frac{\partial \mathcal{L}}{\partial w} = 0\ .$$

Likewise, if $\mathcal{L}$ depends on $\partial^n w/\partial x^n$, there will be an extra term

$$(-1)^{n+1}\frac{d^n}{dx^n}\frac{\partial \mathcal{L}}{\partial w_{x\ldots x}}$$

in the Euler-Lagrange equation, where $w_{x\ldots x} \equiv \partial^n w/\partial x^n$.

3. With $\mathcal{L} = \frac{1}{2}w_x w_t + w_x^3 - \frac{1}{2}w_{xx}^2$ we have

$$\frac{\partial \mathcal{L}}{\partial w} = 0\ ,\quad \frac{\partial \mathcal{L}}{\partial w_x} = \tfrac{1}{2}w_t + 3w_x^2\ ,\quad \frac{\partial \mathcal{L}}{\partial w_{xx}} = -w_{xx}\ ,\quad \frac{\partial \mathcal{L}}{\partial w_t} = \tfrac{1}{2}w_x$$

and so the generalised Euler-Lagrange equation is

$$\frac{d}{dt}\tfrac{1}{2}w_x + \frac{d}{dx}(\tfrac{1}{2}w_t + 3w_x^2) + \frac{d^2}{dx^2}w_{xx} = w_{xt} + 6w_x w_{xx} + w_{xxxx} = 0$$

and so $u = w_x$ does indeed satisfy the KdV equation.

**Ex 4** 1. **“$\phi^4$ theory”**

A field $u(x,t)$ has kinetic and potential energies

$$\begin{aligned} T &= \int_{-\infty}^{+\infty} dx \, \frac{1}{2} u_t^2 \\ V &= \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} u_x^2 + \frac{\lambda}{4} (u^2 - a^2)^2 \right] . \end{aligned} \tag{1.8}$$

Write down the Lagrangian density for the system, and use the Euler-Lagrange equations to show that $u(x,t)$ must satisfy the equation of motion

$$u_{tt} - u_{xx} + \lambda u (u^2 - a^2) = 0 \, . \tag{1.9}$$

2. **“$\phi^6$ theory”**

Repeat the exercise for a field with kinetic and potential energies

$$\begin{aligned} T &= \int_{-\infty}^{+\infty} dx \, \frac{1}{2} u_t^2 \\ V &= \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} u_x^2 + \frac{\lambda}{6} u^2 (u^2 - a^2)^2 \right] , \end{aligned} \tag{1.10}$$

and derive the equation of motion

$$u_{tt} - u_{xx} + \lambda u (u^2 - a^2)(u^2 - \tfrac{a^2}{3}) = 0 \, . \tag{1.11}$$

**Solution 4** The Lagrangian densities are

$$\mathcal{L}_{\phi^4} = \frac{1}{2} u_t^2 - \frac{1}{2} u_x^2 - \frac{\lambda}{4} (u^2 - a^2)^2$$

for $\phi^4$, and

$$\mathcal{L}_{\phi^6} = \frac{1}{2} u_t^2 - \frac{1}{2} u_x^2 - \frac{\lambda}{6} u^2 (u^2 - a^2)^2$$

for $\phi^6$. The equations of motion follow in the standard way; the only slightly tricky calculation is

$$\frac{\partial \mathcal{L}_{\phi^6}}{\partial u} = \frac{\lambda}{6} \left( 2u(u^2 - a^2)^2 + 4u^2(u^2 - a^2)u \right) = \frac{\lambda}{3} u(u^2 - a^2) \left( u^2 - a^2 - 4u^2 \right)$$

and this leads to the given equation of motion.

# 2 Adding boundaries

**Ex 5** A field $u(x,t)$ on the half-line $x \in (-\infty, 0]$ has kinetic energy $T$ and potential energy $V$, where

$$T = \int_{-\infty}^{0} dx\, \frac{1}{2} u_t^2 \,, \qquad V = \int_{-\infty}^{0} dx \Big[\frac{1}{2}u_x^2 + \frac{\lambda}{4}(u^2 - a^2)^2\Big] + \mu(u-u_0)^2\Big|_{x=0} \,, \tag{2.1}$$

and $a > 0$, $\lambda > 0$, $\mu$ and $u_0$ are real constants. (This is a version of the $\phi^4$ theory with boundary.)

1. Write down the bulk Lagrangian density for the system, and use the boundary version of the Euler-Lagrange equations to derive the equation of motion and boundary condition for $u(x,t)$.
2. Using the equations derived in part 1, show that the total energy $E_{\text{boundary}} = T + V$ is conserved.

**Solution 5**

1. Bulk Lagrangian density:

$$\mathcal{L} = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{\lambda}{4}(u^2 - a^2)^2$$

Bulk ($x < 0$) equation of motion:

$$0 = -\frac{\partial \mathcal{L}}{\partial u} + \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_t} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} = \lambda(u^2 - a^2)u + u_{tt} - u_{xx}$$

Boundary contributions to the kinetic and potential energy are

$$A(u_t(0,t)) = 0 \quad \text{and} \quad B(u(0,t)) = \mu(u(0,t) - u_0)^2$$

so the boundary contribution to the action is $-M = A - B$ with

$$M(u(0,t), u_t(0,t)) = \mu(u(0,t) - u_0)^2 \,.$$

The boundary contribution to the infinitesimal variation of the action, $\delta S$, is

$$\int_{-\infty}^{\infty} dt \left( \frac{\partial \mathcal{L}}{\partial u_x}\bigg|_{x=0} - \frac{\partial M}{\partial u} + \frac{\partial}{\partial t}\frac{\partial M}{\partial u_t} \right) \delta u(0,t)$$

(if you forget this, you can derive it as in the notes). This must be zero for all $\delta u(0,t)$ and hence the $x = 0$ boundary condition (bc) is

$$0 = \left.\frac{\partial \mathcal{L}}{\partial u_x}\right|_{x=0} - \frac{\partial M}{\partial u} + \frac{\partial}{\partial t}\frac{\partial M}{\partial u_t} = -u_x(0,t) - 2\mu(u(0,t) - u_0)$$

or equivalently $\frac{1}{2\mu}u_x(0,t) + u(0,t) = u_0$, which is a Robin boundary condition.

2. We have

$$E_{\text{boundary}} = T + V = \int_{-\infty}^{0} \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{\lambda}{4}(u^2 - a^2)^2\right] dx + \mu(u - u_0)^2\Big|_{x=0} ,$$

Taking the time derivative,

$$\begin{aligned}
\frac{d}{dt}E_{\text{boundary}} &= \int_{-\infty}^{0} \Big[u_t u_{tt} + u_x u_{xt} + \lambda(u^2 - a^2)^2 u u_t\Big] dx + 2\mu(u - u_0)u_t\Big|_{x=0} \\
&= \int_{-\infty}^{0} \Big[u_x u_{xt} + u_t \left(u_{tt} + \lambda(u^2 - a^2)^2 u\right)\Big] dx + 2\mu(u - u_0)u_t\Big|_{x=0} \\
&= \int_{-\infty}^{0} \Big[u_x u_{xt} + u_t u_{xx}\Big] dx - u_x u_t\Big|_{x=0} \quad \text{(using bulk EoM and bc)} \\
&= \int_{-\infty}^{0} \frac{\partial}{\partial x}(u_x u_t)\, dx - u_x u_t\Big|_{x=0} \\
&= \Big[u_x u_t\Big]_{-\infty}^{0} - u_x u_t\Big|_{x=0} \\
&= u_x u_t\Big|_{x=0} - u_x u_t\Big|_{x=0} \qquad \text{(using } u_x,\ u_t \to 0 \text{ as } x \to -\infty) \\
&= 0\,.
\end{aligned}$$

Hence the total energy is conserved, as claimed.

**Ex 6** A field $u(x,t)$ satisfies the sine-Gordon equation for $x < 0$ with a Dirichlet boundary condition $u(0,t) = u_0$ imposed at $x = 0$. The kinetic energy density $\mathcal{T}$ and the potential energy density $\mathcal{V}$ are

$$\begin{aligned}
\mathcal{T} &= \frac{1}{2}u_t^2 , \\
\mathcal{V} &= \frac{1}{2}u_x^2 + 1 - \cos u = \frac{1}{2}u_x^2 + 2\sin^2\frac{u}{2} .
\end{aligned} \tag{2.2}$$

1. If a field configuration is to have finite total energy, what values can the field $u(x,t)$ take as $x \to -\infty$?

2. For the case $u(-\infty,t)=0$, complete a square in the integral for the total energy to derive a Bogomol'nyi bound on the energy of solutions. If $|u_0|\le\pi$, find the solution which saturates this bound, either by solving the Bogomol'nyi equation directly, or (equivalently) by 'parking' a single kink or antikink at a suitable location to match the boundary condition.

3. $^*$ Adapt the considerations of parts 1 and 2 to a field satisfying the sinh-Gordon equation $u_{tt}-u_{xx}+\sinh u=0$ for $x<0$, again with a Dirichlet boundary condition $u(0,t)=u_0$ imposed at $x=0$. Note that for half-line problems, singular full-line solutions can be physically relevant, as long as the singularity stays on the discarded ($x>0$) half of the full line.

**Solution 6**

1. The total energy in this case is

$$E=\int_{-\infty}^{0}\Big[\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+2\sin^2\frac{u}{2}\Big]\,dx\,.$$

(Note, since we have a Dirichlet boundary condition and the field at $x=0$ is fixed, the extra boundary term seen in the previous question is not needed here.) As in the situation with no boundaries, for the integral of $2\sin^2\frac{u}{2}$ out to $-\infty$ to be finite, we need $2\sin^2\frac{u}{2}\to 0$, which implies $u(x,t)\to 2n\pi$, as $x\to-\infty$, where $n$ is an integer.

2. This goes much as in the no-boundary case:

$$\begin{aligned}
E&=\int_{-\infty}^{0}\Big[\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+2\sin^2\frac{u}{2}\Big]\,dx\\
&\ge\int_{-\infty}^{0}\Big[\frac{1}{2}u_x^2+2\sin^2\frac{u}{2}\Big]\,dx\\
&=\int_{-\infty}^{0}\Big[\frac{1}{2}\Big(u_x\pm 2\sin\frac{u}{2}\Big)^2\mp 2\sin\frac{u}{2}u_x\Big]\,dx\\
&=\int_{-\infty}^{0}\frac{1}{2}\Big(u_x\pm 2\sin\frac{u}{2}\Big)^2dx\pm\Big[4\cos\frac{u}{2}\Big]_{-\infty}^{0}\\
&\ge 4\Big(1-\cos\frac{u_0}{2}\Big)\,.
\end{aligned}$$

(Note, in passing to the last line the lower set of signs was chosen to give the best possible bound, taking into account the fact that $|\cos\frac{u_0}{2}|\le 1$ for all $u_0$.) To saturate the bound, we need $u_t=0$ for all $x<0$ and $t$, so the field is static, and then $u_x=2\sin\frac{u}{2}$ for all $x<0$. This is exactly the equation that was solved in the full-line case, so the general solution can be "borrowed' from that:

$$u(x)=4\arctan e^{x-x_0}\,.$$

On the full line, $x_0$ is a free parameter, but in this case $u$ has to satisfy the boundary condition imposed at $x = 0$, and this fixes $x_0$:

$$u_0 = 4 \arctan e^{-x_0} \quad \Rightarrow \quad x_0 = -\log \tan \frac{u_0}{4} .$$

(A more 'friendly' question might pick some specific value for $u_0$ for which the value of $x_0$ turns out to be a nice number.)

3. This is a 'starred' question so a little harder than would appear in an exam without rather more hints, but the idea is much the same as just described. The first part goes through as before, with hyperbolic functions instead of trigonometric ones. The only tricky point is at the end: for the sinh-Gordon model there are no nontrivial smooth static solutions to the Bogomolnyi equation on the full line, but there are singular ones, and provided the singularity occurs for $x > 0$ such a solution will work fine for the half-line problem, where we only care about negative values of $x$. For enthusiasts interested in exploring further (but *not* for exam preparation!), some of these ideas were used in papers concerned with the interaction of quantum solitons with boundaries: see for example section 8 of `https://arxiv.org/pdf/hep-th/9407148` (but beware that the notation used in that paper is a little different from the one used here).