

1 A quick sketch of Lagrangian mechanics

Ex 1 The real-valued and continuous function $f(t)$, defined for $t \in [t_1, t_2]$, is such that

$$\int_{t_1}^{t_2} dt f(t) \delta u(t) = 0 \quad (1.1)$$

for all real-valued and continuous (infinitesimal) functions $\delta u(t)$ with $\delta u(t_1) = \delta u(t_2) = 0$. Prove that $f(t)$ is identically zero for all $t \in [t_1, t_2]$. This is sometimes called the fundamental lemma of the calculus of variations.

[**Hint**: consider $\delta u(t) = \epsilon(t - t_1)(t_2 - t)f(t)$ for $0 < \epsilon \ll 1$.]

Ex 2 The position $u = u(t)$ of a point particle is a function of time t . The particle is described by the **action**

$$S[u] = \int_{t_1}^{t_2} dt L(u, u_t) , \quad (1.2)$$

where the **Lagrangian** $L(u, u_t)$ does not depend explicitly on time. Therefore the Lagrangian depends on time only through the time dependence of $u(t)$ and $u_t(t)$.

1. Use the chain rule to compute the total time derivative of the Lagrangian

$$\frac{d}{dt} L(u(t), u_t(t)) = \frac{\partial L}{\partial u} u_t + \frac{\partial L}{\partial u_t} u_{tt} . \quad (1.3)$$

2. Show that the Euler-Lagrange equation for u implies the *Beltrami identity*

$$\frac{d}{dt} \left(u_t \frac{\partial L}{\partial u_t} - L \right) = 0 , \quad (1.4)$$

which states the conservation of the total energy (or **Hamiltonian**)

$$E = u_t \frac{\partial L}{\partial u_t} - L . \quad (1.5)$$

3. For a particle described by the Lagrangian

$$L = T - V , \quad (1.6)$$

where $T(u_t) = \frac{m}{2} u_t^2$ is the kinetic energy and $V(u)$ is the potential energy, show that the conserved total energy is $E = T + V$.

- Ex 3** 1. Consider a field theory described by a field $w(x, t)$. If the Lagrangian density \mathcal{L} is a function of w , w_x , w_{xx} , and w_t , and if

$$\delta S[w] = \delta \iint dx dt \mathcal{L}(w, w_x, w_{xx}, w_t) = 0 ,$$

derive the (generalised) Euler-Lagrange equation (or *equation of motion*)

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial w_t} + \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial w_x} - \frac{d^2}{dx^2} \frac{\partial \mathcal{L}}{\partial w_{xx}} - \frac{\partial \mathcal{L}}{\partial w} = 0 . \quad (1.7)$$

[**Hint:** mimic the derivation in the notes, taking into account the extra term involving $(\partial \mathcal{L} / \partial w_{xx}) \delta w_{xx}$ that appears in the variation of S .]

2. What happens if \mathcal{L} also depends on w_{xxx} ? And if \mathcal{L} depends on $\partial^n w / \partial x^n$?
 3. If $\mathcal{L} = \frac{1}{2} w_x w_t + w_x^3 - \frac{1}{2} w_{xx}^2$, use the result of part 1 to find the equation of motion for w , and verify that if w satisfies this equation, then $u \equiv w_x$ solves the KdV equation.

Ex 4 1. “ ϕ^4 theory”

A field $u(x, t)$ has kinetic and potential energies

$$\begin{aligned} T &= \int_{-\infty}^{+\infty} dx \frac{1}{2} u_t^2 \\ V &= \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} u_x^2 + \frac{\lambda}{4} (u^2 - a^2)^2 \right] . \end{aligned} \quad (1.8)$$

Write down the Lagrangian density for the system, and use the Euler-Lagrange equations to show that $u(x, t)$ must satisfy the equation of motion

$$u_{tt} - u_{xx} + \lambda u(u^2 - a^2) = 0 . \quad (1.9)$$

2. “ ϕ^6 theory”

Repeat the exercise for a field with kinetic and potential energies

$$\begin{aligned} T &= \int_{-\infty}^{+\infty} dx \frac{1}{2} u_t^2 \\ V &= \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} u_x^2 + \frac{\lambda}{6} u^2 (u^2 - a^2)^2 \right] , \end{aligned} \quad (1.10)$$

and derive the equation of motion

$$u_{tt} - u_{xx} + \lambda u(u^2 - a^2)(u^2 - \frac{a^2}{3}) = 0 . \quad (1.11)$$

2 Adding boundaries

Ex 5 A field $u(x, t)$ on the half-line $x \in (-\infty, 0]$ has kinetic energy T and potential energy V , where

$$\begin{aligned} T &= \int_{-\infty}^0 dx \frac{1}{2} u_t^2, \\ V &= \int_{-\infty}^0 dx \left[\frac{1}{2} u_x^2 + \frac{\lambda}{4} (u^2 - a^2)^2 \right] + \mu (u - u_0)^2 \Big|_{x=0}, \end{aligned} \quad (2.1)$$

and $a > 0$, $\lambda > 0$, μ and u_0 are real constants. (This is a version of the ϕ^4 theory with boundary.)

1. Write down the bulk Lagrangian density for the system, and use the boundary version of the Euler-Lagrange equations to derive the equation of motion and boundary condition for $u(x, t)$.
2. Using the equations derived in part 1, show that the total energy $E_{\text{boundary}} = T + V$ is conserved.

Ex 6 A field $u(x, t)$ satisfies the sine-Gordon equation for $x < 0$ with a Dirichlet boundary condition $u(0, t) = u_0$ imposed at $x = 0$. The kinetic energy density \mathcal{T} and the potential energy density \mathcal{V} are

$$\begin{aligned} \mathcal{T} &= \frac{1}{2} u_t^2, \\ \mathcal{V} &= \frac{1}{2} u_x^2 + 1 - \cos u = \frac{1}{2} u_x^2 + 2 \sin^2 \frac{u}{2}. \end{aligned} \quad (2.2)$$

1. If a field configuration is to have finite total energy, what values can the field $u(x, t)$ take as $x \rightarrow -\infty$?
2. For the case $u(-\infty, t) = 0$, complete a square in the integral for the total energy to derive a Bogomol'nyi bound on the energy of solutions. If $|u_0| \leq \pi$, find the solution which saturates this bound, either by solving the Bogomol'nyi equation directly, or (equivalently) by 'parking' a single kink or antikink at a suitable location to match the boundary condition.
3. * Adapt the considerations of parts 1 and 2 to a field satisfying the sinh-Gordon equation $u_{tt} - u_{xx} + \sinh u = 0$ for $x < 0$, again with a Dirichlet boundary condition $u(0, t) = u_0$ imposed at $x = 0$. Note that for half-line problems, singular full-line solutions can be physically relevant, as long as the singularity stays on the discarded ($x > 0$) half of the full line.