31. (a) Show that the pair of equations

$$(u-v)_+ = \sqrt{2}\, e^{(u+v)/2}$$
$$(u+v)_- = \sqrt{2}\, e^{(u-v)/2}$$

provides a Bäcklund transformation linking solutions of $v_{+-} = 0$ (the wave equation in light-cone coordinates) to those of $u_{+-} = e^u$ (the Liouville equation).

(b) Starting from d'Alembert's general solution $v = f(x^+) + g(x^-)$ of the wave equation, use the Bäcklund transformation from part (a) to obtain the corresponding solutions of the Liouville equation for u. [**Hint**: Set $u(x^+, x^-) = 2U(x^+, x^-) + f(x^+) - g(x^-)$. You might simplify the notation by setting $f(x^+) = \log(F'(x^+))$ and $g(x^-) = -\log(G'(x^-))$, where prime means first derivative.]

32. Consider the Bäcklund transformation

$$v_x + \frac{1}{2}uv = 0$$
$$v_t + \frac{1}{2}u_x v - \frac{1}{4}u^2 v = 0\,.$$

(a) Show that these equations taken together imply that v satisfies the linear heat equation $v_t = v_{xx}$, while u satisfies Burgers' equation $u_t + uu_x - u_{xx} = 0$.
[**Hint**: for v, solve the first equation for u and substitute in the second; for u, start by cross-differentiating.]

(b) Find the *general* travelling-wave solution for $v(x,t)$ and, via the Bäcklund transformation, re-obtain the travelling-wave for Burgers' equation found in question 17.

(c) * The linear equation satisfied by $v(x,t)$ allows for the linear superposition of solutions. Use this fact, and your answers to part (b), to construct solutions for v and then u which describe the interaction of *two* travelling waves.

(d) * Sketch your solutions functions of x at fixed times both before and after the interaction, and also draw their trajectories in the (x,t) plane, perhaps starting with the help of a computer. Are the travelling waves of Burgers' equation true solitons, in the sense given in lectures?
[**Hints**: Examine the asymptotics of the solution viewed from frames moving at various velocities V (that is, set $X_V = x - Vt$ and consider $t \to \pm\infty$ keeping X_V finite). This should allow you to isolate various travelling waves in these limits, and to decide whether they preserve their form under interactions. For definiteness, consider the case $c_1 > c_2 > 0$, where c_1 and c_2 are the velocities of the two separate travelling waves before they were superimposed. A further hint: as well as the 'expected' special values for V, namely c_1 and c_2, be careful about what happens when $V = c_1 + c_2$.]

33. (a) Show that the two equations

$$v_x = -u - v^2$$
$$v_t = 2u^2 + 2uv^2 + u_{xx} - 2u_x v$$

are a Bäcklund transformation relating solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

and the wrong sign modified KdV (mKdV) equation

$$v_t - 6v^2 v_x + v_{xxx} = 0\,.$$

(Note the appearance of the Miura transform in the Bäcklund transformation.)

(b) Taking $u = c^2$, where c is a constant, as a seed solution of the KdV equation, find the corresponding solution of the wrong sign mKdV equation.

34. The 2-soliton solution of the sine-Gordon equation with Bäcklund parameters a_1 and a_2 is

$$u(x,t) = 4\arctan\left(\mu\,\frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}}\right)\,, \qquad \theta_i = \varepsilon_i\gamma_i(x - v_i t - \bar{x}_i)$$

where $\mu = (a_2{+}a_1)/(a_2{-}a_1)$, $v_i = (a_i^2{-}1)/(a_i^2{+}1)$, $\gamma_i = 1/\sqrt{1{-}v_i^2}$, $\varepsilon_i = \mathrm{sign}(a_i)$, and $\bar{x}_1$ and $\bar{x}_2$ are constants, as in the lectures. Rewriting u as a function of $X_V \equiv x - Vt$ and t, show that, for $V \neq v_1, v_2$ (and $v_1 \neq v_2$)

$$\lim_{\substack{t\to\infty \\ X_V \text{ finite}}} u = 2n\pi\,,$$

where n is an integer. If $v_2 > v_1 > 0$ and $\varepsilon_i = 1$, how does the parity of n (whether it is even or odd) depend on the value of v relative to v_1 and v_2?
[**Hints**: First show that $|\theta_i| \to +\infty$ as $t \to \pm\infty$; then consider each of the four possible options $(\theta_1, \theta_2) \to (+\infty, +\infty)$, $(-\infty, -\infty)$, $(+\infty, -\infty)$, $(-\infty, +\infty)$. Remember that $\arctan(0) = m\pi$ and $\arctan(\pm\infty) = \pm\pi/2{+}m\pi$, where the ambiguities of $m\pi$, $m \in \mathbb{Z}$, encode the multivalued nature of the $\arctan$ function.]

35. Find the asymptotics of the 2-soliton sine-Gordon solution defined in problem 34, in the case $a_2 > a_1 > 0$, as $t \to \pm\infty$ with $X_{v_2} \equiv x{-}v_2 t$ held finite.

36. Show by direct analysis (as in the lectures) that taking a_1 and a_2 of opposite signs in problem 34 results in a two-kink, or two-antikink, solution to the sine-Gordon equation.

37. (a) The argument of the arctangent in the sine-Gordon 2-soliton solution of problem 34 is a continuous function of x for all $x \in \mathbb{R}$. In particular, it is never infinite. What does this imply about the range of u? [**Hint**: consider the graph of $\tan u/4$.]

(b) By taking the limits of this function as $x \to \pm\infty$ (with $t = \bar{x}_1 = \bar{x}_2 = 0$ for simplicity), show that the topological charge of this two-soliton solution is 0 if $\mathrm{sign}(a_1) = \mathrm{sign}(a_2)$, and ± 2 if $\mathrm{sign}(a_1) = -\mathrm{sign}(a_2)$, in units where the topological charge of a kink is 1.

38. Consider the two-soliton solution of the sine-Gordon equation from problem 34 with complex Bäcklund parameters $a_1 = a_2^* := a \in \mathbf{C}$ and with vanishing integration constants, as is appropriate to find the breather solution. Show that

$$\begin{aligned} \mathrm{Re}(\theta_1) &= +\mathrm{Re}(\theta_2) = \gamma(x - vt)\cos\varphi\ , \\ \mathrm{Im}(\theta_1) &= -\mathrm{Im}(\theta_2) = \gamma(vx - t)\sin\varphi\ , \end{aligned}$$

where $\varphi = \arg(a)$ and

$$\begin{aligned} v &= \frac{|a|^2 - 1}{|a|^2 + 1} \\ \gamma &= \frac{1}{\sqrt{1 - v^2}} = \frac{1 + |a|^2}{2|a|}\ . \end{aligned}$$

39. The *stationary* breather solution of the sine-Gordon equation (that is the breather solution with $v = 0$) has the form

$$\tan\frac{u}{4} = \frac{\cos\varphi}{\sin\varphi} \cdot \frac{\sin(t\sin\varphi)}{\cosh(x\cos\varphi)}\ .$$

Show that in the limit $\varphi \to 0$, in which the kink and antikink that form the breather are very loosely bound, the time period τ of a single oscillation of the breather scales like $\tau \sim |\varphi|^{-1}$, and the spatial size $x_{\max}$ of the breather scales like $x_{\max} \sim -\log\varphi$.
[**Hint**: You could define $x_{\max}$ as the value of x at which $\tan(u/4) = 1$ when the oscillatory factor in the numerator is at its maximum. Focus only on the parametric dependence on φ, ignoring all numerical factors.]

40. We have seen in lectures that the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ for the field $u(x,t)$ that describes the profile of a wave translates into the following equation for the new variable $w(x,t) = \int dx\ u$:

$$w_t + 3w_x^2 + w_{xxx} = 0\ .$$

Let $w = 2\frac{\partial}{\partial x}\log f = 2f_x/f$ where $f(x,t)$ is a nowhere vanishing function of x and t, so that $u = 2\frac{\partial^2}{\partial x^2}\log f$. The aim of this exercise is to rewrite the equation for w as an equation for f.

(a) Express w_t, w_x, w_{xx} and w_{xxx} in terms of f and its derivatives.

(b) Show that the equation for $w_t + 3w_x^2 + w_{xxx} = 0$ can be rewritten as

$$ff_{xt} - f_x f_t + 3f_{xx}^2 - 4f_x f_{xxx} + ff_{xxxx} = 0\ ,$$

which is known as the *quadratic form* of the KdV equation.

41. The Hirota bilinear differential operator $D_t^m D_x^n$ is defined for any pair of natural numbers (m, n) by

$$D_t^m D_x^n (f \cdot g) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n f(x,t) g(x',t') \Bigg|_{\substack{x'=x \\ t'=t}}$$

and maps a pair of functions $(f(x,t), g(x,t))$ into a single function.

(a) Prove that the Hirota operators $B_{m,n} := D_t^m D_x^n$ are bilinear, *i.e.* for all constants a_1, a_2

$$B_{m,n}(a_1 f_1 + a_2 f_2 \cdot g) = a_1 B_{m,n}(f_1 \cdot g) + a_2 B_{m,n}(f_2 \cdot g) \ ,$$
$$B_{m,n}(f \cdot a_1 g_1 + a_2 g_2) = a_1 B_{m,n}(f \cdot g_1) + a_2 B_{m,n}(f \cdot g_2) \ .$$

(b) Prove the symmetry property

$$B_{m,n}(f \cdot g) = (-1)^{m+n} B_{m,n}(g \cdot f) \ .$$

(c) Compute the Hirota derivatives $D_t^2(f \cdot g)$ and $D_x^4(f \cdot g)$, and verify that your expression for the latter is consistent with the result for $D_x^4(f \cdot f)$ given in lectures.

42. Define a "not-Hirota" bilinear differential operator $\tilde{D}_t^m \tilde{D}_x^n$ by

$$\tilde{D}_t^m \tilde{D}_x^n (f \cdot g) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'} \right)^m \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right)^n f(x,t) g(x',t') \Bigg|_{\substack{x'=x \\ t'=t}}$$

(note the plus signs!).

(a) Compute $\tilde{D}_x(f \cdot g)$ and $\tilde{D}_t(f \cdot g)$, verifying that in both cases the answer is given by the corresponding 'ordinary' derivative of the product $f(x,t)g(x,t)$.

(b) How does this result generalise for arbitrary not-Hirota differential operators? Prove your claim.

(c) Compare your answer with the Hirota operators defined above.

43. (a) If $\theta_i = a_i x + b_i t + c_i$, prove that

$$D_t D_x (e^{\theta_1} \cdot e^{\theta_2}) = (b_1 - b_2)(a_1 - a_2) e^{\theta_1 + \theta_2}.$$

(b) Prove the corresponding result for $D_t^m D_x^n (e^{\theta_1} \cdot e^{\theta_2})$, as quoted in lectures.

44. Prove that

$$D_t^m D_x^n (f \cdot 1) = \frac{\partial^m}{\partial t^m} \frac{\partial^n}{\partial x^n} f \ .$$

45. Consider the function f, such that $u = 2\frac{\partial^2}{\partial x^2} \log f$ is the KdV field, which corresponds to a 2-soliton solution:

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 = 1 + \epsilon \left(e^{\theta_1} + e^{\theta_2}\right) + \epsilon^2 \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 e^{\theta_1 + \theta_2} ,$$

where $\theta_i = a_i x - a_i^3 t + c_i$, with a_i and c_i constants. Check that $B(f_1 \cdot f_2) = 0$ and $B(f_2 \cdot f_2) = 0$, where $B = D_x(D_t + D_x^3)$, and show that this implies that the above expansion, which is truncated at order ϵ^2, is a solution of the bilinear form of the KdV equation.

46. Derive the solution of the bilinear form of the KdV equation $D_x(D_t + D_x^3)(f \cdot f) = 0$ which represents the 3-soliton solution, in the form

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3$$

where $f_1 = \sum_{i=1}^{3} e^{\theta_i}$. [This includes proving that the higher order terms in the ϵ expansion can be consistently set to zero, as in problem 45.]

47. Show that the Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0$$

can be written in the bilinear form

$$(D_t^2 - D_x^2 - D_x^4)(f \cdot f) = 0$$

where $u = 2\frac{\partial^2}{\partial x^2} \log f$.

48. Show that the following higher-dimensional version of the KdV equation,

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0$$

for the field $u(x, y, t)$, also known as the Kadomtsev-Petviashvili (KP) equation, can be written in the bilinear form

$$(D_t D_x + D_x^4 + 3\sigma^2 D_y^2)(f \cdot f) = 0$$

where $u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \log f(x, y, t)$.

49. It is given that the system of Hirota equations

$$\begin{cases} (D_x^2 - D_t^2 - 1)(f \cdot g) = 0 \\ (D_x^2 - D_t^2)(f \cdot f) = (D_x^2 - D_t^2)(g \cdot g) \end{cases}$$

yields solutions $u = 4\arctan(g/f)$ of the sine-Gordon equation. Let $\theta_i = a_i x + b_i t + c_i$, where a_i, b_i, c_i are constants.

(a) Take

$$f = 1 , \qquad g = \epsilon e^{\theta_1}$$

and work order by order in powers of ϵ to find the one-soliton solution of the sine-Gordon equation.

(b) Taking e^{θ_i} as in the solution of the previous part, repeat the exercise for

$$f = 1 + \epsilon^2 f_2 , \qquad g = \epsilon(e^{\theta_1} + e^{\theta_2}) ,$$

and check that the Hirota equations are satisfied to all orders in ϵ.