MATHEMATICS WORKBOOK

May 26, 2004

Introduction

The Mathematical Tripos is designed to be accessible to students who are familiar with the content of a typical single mathematics A-level. Half of any A-level syllabus is so-called 'common core', but the material chosen for the other half varies between examining boards and according to the choice of modules.

The purpose of this workbook is to present a set of material which it would be useful to know before starting the Tripos. It is not *necessary* to know all this material, but it would make your first term more comfortable if you did know it. Rather than present a list of topics, the workbook contains questions on each topic that are supposed to be straightforward; by tackling these questions, you will see how much knowledge is expected.

Most of you will be familiar with most of the material covered here. It is still worth sketching out a solution to each question: it will be good revision and anyway you never really know that you have understood a problem until you do it. The answers are given at the end of the workbook. Note that none of the questions requires the use of a calculator.

If you find some of the material is unfamiliar, you should to look it up, if you have time. It is all explained in *Mathematics - The Core Course for A Level* by L. Bostock and S. Chandler published by Nelson Thornes, 1981 (ISBN 0859503062), or in *Further Pure Mathematics*, by Bostock, Chandler and C. Rourke, published by Nelson Thornes, 1982 (ISBN 0859501035). You will also probably find most of it in any standard A-level (or the equivalent) text. If a particular area is really unfamiliar to you, it would be worth doing exercises from a text book to supplement those given here. If you have difficulties with some questions, don't worry; you will have oppportunities to cover the material when you get to Cambridge.

You may also find that you need to get into the swing of mathematics again after your long break (especially if you took a gap year). The best way to do this is to practise problem-solving. One source of problems is past STEP papers (available from OCR Publications, PO Box 5050, Annesley, Nottingham NG15 0DL) which will be especially useful if you did not take STEP. Alternatively, your Director of Studies may supply you with some work of this sort.

Please e-mail comments or corrections to faculty@maths.cam.ac.uk.

May 26, 2004

Algebra

Although computers and even calculators are very good at algebra, all mathematicians agree that it is important to be able to do routine algebra quickly and accurately. You should be able to state elementary series expansions including binomial, sine and cosine, and ln series.

1 Factorization.

Factorize the following polynomials:

(i) $x^2 - 3x + 2\,;$ (ii) $3x^3 - 3x^2 - 6x\,;$

(iii) $x^2 - x - 1\,;$ (iv) $x^3 - 1\,;$

(v) $x^4 - 3x^3 - 3x^2 + 11x - 6.$

Notes In part (iii) you will need the quadratic formula to find the factors; part (iv) has one linear and one quadratic factor; for part (v) you can use the factor theorem.

2 More factorization.

Find the values of x for which $x^3 < 2x^2 + 3x$.

3 Partial fractions.

Express the following in partial fractions:

(i) $\dfrac{2}{(x+1)(x-1)}\,;$ (ii) $\dfrac{1}{x^3+1}\,;$

(iii) $\dfrac{4x+1}{(x+1)^2(x-2)}\,;$ (iv) $\dfrac{x^2-7}{(x-2)(x+1)}.$

Note It is best for these purposes not to use the 'cover-up rule'; there are at least two other ways which involve elementary mathematics, whereas the cover-up rule works for more sophisticated reasons and to most users is simply a recipe (which does not always work).

4 Completing the square.

Find the smallest value (for real x and y) of:

(i) $x^2 - 2x + 6$; (ii) $x^4 + 2x^2 + y^4 - 2y^2 + 3$;

(iii) $\sin^2 x + 4\sin x$.

Note Of course, you could find the smallest value by calculus, but expressing the function as a perfect square plus a remainder term is a surprisingly useful technique – for example, when integrating a function with a quadratic denominator.

5 Exponentials and logs.

(i) The exponential function e^x can be defined by the series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Use this definition to show that $\dfrac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$.

(ii) The natural log function $\ln t$ can be defined (for $t > 0$) as the inverse of the exponential function, so that $\ln e^x = x$. Set $t = e^x$ and use the relationship $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 1\Big/\dfrac{\mathrm{d}t}{\mathrm{d}x}$ to show that

$$\frac{\mathrm{d}\ln t}{\mathrm{d}t} = \frac{1}{t}.$$

(iii) Assuming that the exponential function has the property $e^s e^t = e^{s+t}$, prove that $\ln(xy) = \ln x + \ln y$.

(iv) The definition of a^x for any a is $e^{x \ln a}$. Prove that $a^x a^y = a^{x+y}$ and $a^x b^x = (ab)^x$.

Note If you have not thought of defining a^x in this way, it is worth considering how else you could give it a meaning when x is not an integer.

6 Binomial expansions.

(i) Find the coefficient of x^k (for $0 \leqslant k \leqslant 10$) in the binomial expansion of $(2+3x)^{10}$.

(ii) Use the binomial theorem to find the expansion in powers of x up to x^4 of $(1+x+x^2)^6$, by writing it in the form $\big(1+(x+x^2)\big)^6$.

(iii) Use binomial expansions to find the expansion in powers of x up to x^4 of $(1-x^3)^6(1-x)^{-6}$.

(iv) Find the first four terms in the binomial expansion of $(2+x)^{\frac{1}{2}}$.

7 Series expansions of elementary functions.

Using only the series expansions $\sin x = x - x^3/3! + x^5/5! + \cdots$, $\cos x = 1 - x^2/2! + x^4/4! + \cdots$, $\mathrm{e}^x = 1 + x + x^2/2! + x^3/3! + \cdots$ and $\ln(1+x) = x - x^2/2 + x^3/3 + \cdots$, find the series expansions of the following functions:

(i) $\tan x$ (up to the x^5 term); (ii) $\sin x \cos x$ (up to the x^5 term);

(iii) $\dfrac{e^x + e^{-x}}{2}$ (up to the x^5 term); (iv) $\ln(e^x)$ (up to the x^3 term);

(v) $\dfrac{1-\cos^2 x}{x^2}$ (up to the x^2 term).

Notes Do part (ii) without using a trig. formula, and compare your answer with the expansion for $\sin(2x)$. The function in part (iii) is $\cosh x$, of which more later. The interesting thing about the function in part (v) is that the series shows it is 'well-behaved' at $x = 0$, despite appearances.

8 Proof by induction.

Prove by induction that the following results are valid.

(i) $$a + ar + ar^2 + \cdots + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right).$$

(ii) $$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$

Note Methods of proof will be discussed in some detail in our first term course *Numbers and Sets*. Most students will have met mathematical induction at school; if you haven't, you will probably want to try out this straightforward but important method of proof.

9 Arithmetic and geometric progressions.

(i) Find the sum of all the odd integers from 11 to 99.

(ii) Evaluate $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$.

(iii) Find $\sin\theta + 2\sin^3\theta + 4\sin^5\theta + \cdots$. (What ranges of values of θ are allowed?).

(iv) Estimate roughly the approximate number of times a piece of paper has to be torn in half, placing the results of each tearing in a stack and then doing the next tearing, for the stack of paper to reach the moon.

Note The dots in (ii) and (iv) indicate that the series has an infinite number of terms. For part (iii), recall that the expansion $a(1+r+r^2+\cdots)$ only converges if $-1 < r < 1$. You may find the approximation $10^3 = 2^{10}$ useful for part (iv). The distance from the Earth to the Moon is about 4×10^5 km.

Trigonometry

It is not necessary to learn all the various trigonometrical formulae; but you should certainly know what is available. The double-angle formulae (such as $\tan 2x = 2\tan x/(1-\tan^2 x)$) are worth learning, as are the basic formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$, the Pythagoras-type identities (such as $\sec^2 x = 1 + \tan^2 x$) and a few special values (such as $\sin\pi/4 = 1/\sqrt{2}$) that can be deduced from right-angled triangles with sides $(1, 1, \sqrt{2})$ or $(1, \sqrt{3}, 2)$.

10 Basic identities.

Starting from the identity $\sin(A+B) = \sin A \cos B + \cos A \sin B$, use the basic properties of the trig. functions (such as $\sin(-A) = -\sin A$) to prove the following:

(i) $\sin(A-B) = \sin A \cos B - \cos A \sin B$;

(ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$;

(iii) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$;

(iv) $\sin C + \sin D = 2 \sin \frac{1}{2}(C+D) \cos \frac{1}{2}(C-D)$;

(v) $\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\dfrac{a+b}{1-ab}\right)$.

Notes For part (ii), recall that $\cos A = \sin(\pi/2 - A)$. You can use part (iii) to help with part (v).

11 Trig. equations.

(i) Solve the following equations:

(a) $\sin(x + \frac{\pi}{6}) + \sin(x - \frac{\pi}{6}) = \dfrac{\sqrt{3}}{2}$; (b) $\cos x + \cos 2x + \cos 3x = 0$.

(ii) Write down the value of $\cot(\pi/6)$ and use a double angle formula to show that $\cot(\pi/12)$ satisfies the equation $c^2 - 2\sqrt{3}c - 1 = 0$. Deduce that $\cot(\pi/12) = 2 + \sqrt{3}$.

12 Trig. identities using Pythagoras.

Prove the following identities:

(i) $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta\,;$ (ii) $(\cot\theta + \operatorname{cosec}\theta)^2 = \dfrac{1+\cos\theta}{1-\cos\theta}\,;$

(iii) $\cos\theta = \dfrac{1-t^2}{1+t^2}\,;$ $\sin\theta = \dfrac{2t}{1+t^2}\,;$ $\tan\theta = \dfrac{2t}{1-t^2}\,,$ where $t = \tan\frac{1}{2}\theta$.

Complex numbers

No prior knowledge of complex numbers is assumed in our first year course because some A-level syllabuses do not cover complex numbers. Nevertheless, the material is lectured quite fast, so if you have not met complex numbers before, working through the following examples with the help of a textbook would make the first few weeks of the course more comfortable. A complex number z can be written as $x + iy$, where x is the real part, y is the imaginary part and $i^2 = -1$. The modulus of z (written $|z|$ or r) is $\sqrt{(x^2+y^2)}$ and the argument (written $\arg z$ or θ) is defined by $x = r\cos\theta$, $y = r\sin\theta$ and $-\pi < \theta \leqslant \pi$. The complex conjugate of z (written z^*) is $x - iy$. The inverse, z^{-1}, of z is the complex number that satisfies $z^{-1}z = 1$ (for $z \neq 0$).

13 Algebra of complex numbers.

Use the definitions above with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ to show:

(i) $z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)\,;$ (ii) $|z|^2 = zz^*\,;$

(iii) $z^{-1} = \dfrac{z^*}{|z|^2}\,;$ (iv) $|z_1z_2| = |z_1||z_2|\,;$

(v) $\arg(z_1z_2) = \arg z_1 + \arg z_2$ (assume that $0 < \arg z_1 \leqslant \pi/4$ and $0 < \arg z_2 \leqslant \pi/4$).

Give a sketch of the x–y plane (called also the complex plane or the Argand diagram) showing the points representing the complex numbers $z_1 = 1 - i$, $z_2 = -\sqrt{3} + i$. Verify results (iii), (iv) and (v) for these numbers.

14 De Moivre's theorem.

(i) Show by means of series expansions that

$$\cos\theta + i\sin\theta = e^{i\theta}$$

and deduce that $\cos\theta - i\sin\theta = e^{-i\theta}$ and that $z = re^{i\theta}$.
Deduce also that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

(ii) Use the above result to show that $re^{i\theta} = 1$ (for real θ and $r > 0$) if and only if $r = 1$ and $\theta = 2n\pi$ for some integer n.

(iii) Use part (ii) to find the three distinct roots of the equation $z^3 = 1$. Draw them on the complex plane and convert them from modulus-argument form to real–imaginary form.

15 Geometry of the complex plane.

(i) Show by transforming to cartesian coordinates that the equation $|z-c| = r$, where c is a complex number, describes a circle.

(ii) Show that the equation $\arg z = \alpha$, where α is a constant, describes a line.

(iii) Show by means of a diagram that $|z_1+z_2| \leqslant |z_1|+|z_2|$ for any two complex numbers. Deduce that $|z_1 - z_3| \leqslant |z_1| + |z_3|$ and $|z_4 - z_2| \geqslant |z_4| - |z_2|$ for any complex numbers z_1, z_2, z_3 and z_4. Under what circumstances is $|z_1 + z_2| = |z_1| + |z_2|$?

Hyperbolic functions

Prior knowledge of hyperbolic functions is not assumed for our mathematics course, but it is worth getting to know the definitions and basic properties, which are given below. Hyperbolic functions are very similar to trigonometric functions, and many of their properties are direct analogues of the properties of trigonometric functions. The definitions are

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

and $\operatorname{cosech} x = (\sinh x)^{-1}$, $\operatorname{sech} x = (\cosh x)^{-1}$, $\coth x = (\tanh x)^{-1}$.

16 Basic properties.

Give a rough sketch of the graphs of the six hyperbolic functions. Show from the above definitions that

(i) $\cosh^2 x - \sinh^2 x = 1\,;$

(ii) $\dfrac{\mathrm{d}(\sinh x)}{\mathrm{d}x} = \cosh x\,; \qquad \dfrac{\mathrm{d}(\cosh x)}{\mathrm{d}x} = \sinh x\,;$

(iii) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y\,;$

(iv) $\cosh(ix) = \cos x, \quad \sinh(ix) = i \sin x, \quad \cos(ix) = \cosh(x), \quad \sin(ix) = i \sinh x.$

Notes You can use (iv) to prove (i) and (iii) using the corresponding trig. identities. In fact, (iv) is behind the rule which says that any formula involving trig. functions becomes the corresponding formula involving hyperbolic functions if you change the sign of every product of two odd functions (such as $\sin x$ or $\tan x$).

17 Further properties.

Use the definitions, and the results of the previous question, to show that

(i) $\operatorname{sech}^2 x = 1 - \tanh^2 x$ (ii) $\dfrac{\mathrm{d}(\tanh x)}{\mathrm{d}x} = \operatorname{sech}^2 x$

(iii) $\dfrac{\mathrm{d}^2(\sinh x)}{\mathrm{d}x^2} = \sinh x$ (iv) $\sinh^{-1} x = \ln\left(x + \sqrt{(x^2+1)}\right).$

Note For part (iv), note that $\sinh^{-1} x$ is the inverse function, not the reciprocal.

Conic sections

You should be familiar with basic coordinate geometry (lines, circles, tangents and normals to curves, etc). Conic sections (parabola, ellipse and hyperbola) are introduced in the first term course Algebra and Geometry, but you would find it very useful to do a little preliminary work on coordinate and parametric equations for conic sections if you have not seen them before. They are not only important in pure mathematics: they also arise, for example, as celestial orbits in the second term Dynamics course.

18 Basic definitions.

(i) Parabola. The point (x, y) has the property that its distance from the point $(a, 0)$ is equal to its distance from the line $x = -a$. Sketch the locus of the point using this information. Show that $y^2 = 4ax$.

(ii) Ellipse and hyperbola. The point (x, y) has the property that its distance from the point $(ae, 0)$ is e times its distance from the line $x = ae^{-1}$. Sketch the locus of the point for $0 < e < 1$ and for $e > 1$. Show that $x^2/a^2 + y^2/b^2 = 1$ if $0 < e < 1$ and that $x^2/a^2 - y^2/b^2 = 1$ if $e > 1$, where $b = a\sqrt{|1 - e^2|}$.

(iii) Rectangular hyperbola. Show that the hyperbola $x^2/a^2 - y^2/b^2 = 1$ becomes the rectangular hyperbola $XY = 1$ in the new coordinates given by $x = a(X + Y)/2$, $y = b(X - Y)/2$.

19 Parametric equations.

Show that the curves with parametric form $(x, y) = (at^2, 2at)$, $(a\cos\theta, b\sin\theta)$ and $(a\cosh\theta, b\sinh\theta)$ are conic sections.

20 Polar coordinates.

By converting to cartesian coordinates, show that:

(i) $r\cos(\theta - \alpha) = c$ describes a straight line;

(ii) $r = \ell\cos\theta$ describes a circle;

(iii) $r^2\cos(\theta + \alpha)\cos(\theta - \alpha) = c^2$ describes a hyperbola;

(iv) $r^{-1} = k\cos\theta + m$ can describe any conic section, the type depending on the values of k and m.

Notes Here, c, k, ℓ, m and α are constants. Special cases, such as $\alpha = 0$ in part (iii), can be ignored. In each case, you should give a careful description (x and y intercepts, e, etc) of the curves.

Matrices and vectors

Matrices and vectors arise in all branches of mathematics and form an indispensable part of a mathematician's toolkit. You don't need to know much about them at the moment: they are dealt with in detail in the first term course Algebra and Geometry. However, it is as well to get to know the rules of matrix multiplication and the definition of a determinant. We consider here only 2×2 matrices. It is also worth understanding the basic geometric uses of vectors. The rule for multiplying the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}.$$

21 Matrix multiplication.

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(i) Show that $\mathbf{AB} \neq \mathbf{BA}$. (This shows that matrix multiplication is not commutative.)

(ii) Show that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$. (This illustrates that matrix multiplication is associative).

(iii) Without using any formulae, find a matrix $\mathbf{D} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\mathbf{AD} = \mathbf{I}$.

(iv) Show that $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

(v) Show that $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$.

Notes The symbol $^{\mathrm{T}}$ denotes the transpose of the matrix, i.e. the matrix obtained by exchanging rows and columns: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. The determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$.

22 Position vectors.

Show that the points with position vectors

$$\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\\0\end{pmatrix}, \quad \begin{pmatrix}4\\3\\-2\end{pmatrix},$$

lie on a straight line and give the equation of the line in the two forms

$$\text{(i)} \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b} \qquad \text{(ii)} \quad \frac{x - x_0}{c} = \frac{y - y_0}{d} = \frac{z - z_0}{e}.$$

23 Scalar products.

The three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are defined by

$$\mathbf{A} = (1, 3, 4), \qquad \mathbf{B} = (2, 1, 3), \qquad \mathbf{C} = (3, 3, 2).$$

(i) Order the vectors by magnitude.

(ii) Use the scalar product to find the angles between the pairs of vectors (a) $\mathbf{A}$ and $\mathbf{B}$, (b) $\mathbf{B}$ and $\mathbf{C}$, leaving your answer in the form of an inverse cosine.

(iii) Find the lengths of the projections of the vectors (a) $\mathbf{A}$ onto $\mathbf{B}$, (b) $\mathbf{B}$ onto $\mathbf{A}$.

(iv) Find an equation of the plane, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, through the points with position vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Show that the normal to this plane is $(1, 0, 1)$ and find an equation of the plane in the form $\mathbf{r}.\mathbf{n} = p$, where $\mathbf{n}$ is a unit vector.

Note The projection of $\mathbf{A}$ onto $\mathbf{B}$ is the component of $\mathbf{A}$ in the direction of the vector $\mathbf{B}$.

Differentiation

Differentiation of standard functions, products, quotients, implicit function expressions, and functions of a function (using the chain rule) should be routine.

24 Direct differentiation.

Differentiate $y(x)$ with respect to x in the following cases:

(i) $y = \ln\left(x + \surd(1+x^2)\right)$; (ii) $y = a^x$;

(iii) $y = x^x$; (iv) $y = \sin^{-1} \dfrac{x}{\sqrt{1+x^2}}$.

Notes Simplify your answer to part (i). For (ii), see the definition in question 5(iv). Can you see why the answer to (iv) is surprisingly simple?

25 Parametric differentiation.

Show that if $x = a\cos\theta$, $y = b\sin\theta$ then $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ for $y > 0$.

26 Stationary points.

Find the stationary points of the function

$$\mathrm{f}(x) = \frac{x}{x^2 + a^2},$$

where $a > 0$, classifying them as either maximum or minimum. Sketch the curve (without using a calculator).

Integration

You need to be able to recognise standard integrals (without having to leaf through a formula book) and evaluate them (referring to your formula book, if necessary). You need to be familiar with the techniques of integration by parts and by substitution.

27 Indefinite integrals.

Calculate the following integrals:

(i) $\displaystyle\int a^x \, dx\,;$ (ii) $\displaystyle\int \frac{1}{x^2 - 2x + 6} \, dx\,;$

(iii) $\displaystyle\int e^{ax} \cos(bx) \, dx\,;$ (iv) $\displaystyle\int e^{ax} e^{ibx} \, dx\,;$

(v) $\displaystyle\int \frac{1}{1 - x^3} \, dx\,;$ (vi) $\displaystyle\int \operatorname{cosec} x \, dx\,;$

(vii) $\displaystyle\int \sec x \, dx\,;$ (viii) $\displaystyle\int \frac{1}{\sqrt{(c^2 + m^2 y^2)}} \, dy\,;$

(ix) $\displaystyle\int \tan^{-1} x \, dx\,;$ (x) $\displaystyle\int x^3 e^{x^2} \, dx\,.$

Notes For part (i), see question 5(iv). For part (ii), see question 4(i). Note that you can obtain (iii) from (iv) by taking the real part. Use partial fractions for part (v). For (vi) and (vii), use the substitution $t = \tan(x/2)$ rather than the trick of multiplying top and bottom by e.g. $\sec x + \tan x$. Try also deriving (vii) from (vi) by means of the substitution $y = \pi/2 - x$. For (viii), either substitute $my = c \sinh \theta$ or, if you are not keen on hyperbolic functions, see question 24(i). Use integration by parts for (ix) and (x).

Differential equations

There is a course on differential equations in the first term for which very little knowledge is assumed.

28 First order equations.

Find the general solution (i.e. with a constant of integration) of the following equations.

(i) $y\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\,;$

(ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = my\,;$

(iii) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + 1\,;$

(iv) $\dfrac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = 1\,;$

(v) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{c^2 - k^2y^2}\,;$

(vi) $z\dfrac{\mathrm{d}z}{\mathrm{d}y} + k^2y = 0.$

Note For part (iv), find a function ('integrating factor') $w(x)$ such that the equation can be written $\dfrac{\mathrm{d}(wy)}{\mathrm{d}x} = w$.

29 Second order equations.

(i) Show, by substituting $y = e^{mx}$ into the equation, that there are two values of m for which e^{mx} satisfies

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0, \qquad (*)$$

where a, b and c are constants.

(ii) Show (by substitution) that if both y_1 and y_2 satisfy the equation $(*)$, then so also does the function y defined by $y = Ay_1 + By_2$, where A and B are constants.

(iii) Find the two values of m for which e^{mx} satisfies the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

and write down a solution that contains two arbitrary constants.

You have just solved the above equation ($y'' + 3y' + 2y = 0$) by guessing solutions. Instead, let $z = y' + 2y$. Then show that $z' + z = 0$, solve for z and then solve the resulting first-order differential equation of y, thereby proving that the solution you found above is in fact the most general solution.

(iv) Show by substitution that $e^{px}\sin qx$ satisfies the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2p\frac{\mathrm{d}y}{\mathrm{d}x} + (p^2 + q^2)y = 0.$$

Find the general solution by the method outlined in the second paragraph of (iii) above.

30 Simple harmonic motion.

(i) Show that the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + k^2y = 0$$

can be written as $z\dfrac{\mathrm{d}z}{\mathrm{d}y} + k^2y = 0$, where $z = \dfrac{\mathrm{d}y}{\mathrm{d}x}$. Show that $z = \surd(c^2 - k^2y^2)$, where c is a constant of integration, (compare questions 28(v) and 28(vi)) and hence show that

$$y = R\sin k(x - x_0),$$

where R and x_0 are constants. ($R = c/k$ and x_0 is a new constant of integration.)

(ii) Repeat the steps of part (i) to show that the general solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - k^2y = 0$$

is

$$y = R\sinh k(x - x_0).$$

Show also that this can be written in the form

$$y = Ae^{kx} + Be^{-kx}.$$

Answers

1. (i) $(x-2)(x-1)$ (ii) $3x(x-2)(x+1)$

(iii) $(x-\frac{1}{2}(1-\sqrt{5}))(x-\frac{1}{2}(1+\sqrt{5}))$ (iv) $(x-1)(x^2+x+1)$

(v) $(x-3)(x-1)^2(x+2)$.

2. $x<-1$ or $0<x<3$.

3. (i) $\dfrac{1}{x-1}-\dfrac{1}{x+1}$ (ii) $\dfrac{1}{3}\left(\dfrac{1}{x+1}-\dfrac{x-2}{x^2-x+1}\right)$

(iii) $\dfrac{1}{(x+1)^2}-\dfrac{1}{x+1}+\dfrac{1}{x-2}$ (iv) $1-\dfrac{1}{x-2}+\dfrac{2}{x+1}$.

4. (i) 5; (ii) 2; (iii) -3 (smallest when $\sin x=-1$).

6. (i) $\dfrac{3^k 2^{10-k} 10!}{k!(10-k)!}$ (ii) $1+6x+21x^2+50x^3+90x^4$

(iii) Same as (ii) (iv) $\sqrt{2}\left(1+\frac{1}{4}x-\frac{1}{32}x^2+\frac{1}{128}x^3\right)$.

7. (i) $x+\frac{1}{3}x^3+\frac{2}{15}x^5$ (ii) $x-\frac{2}{3}x^3+\frac{2}{15}x^5$

(iii) $1+\frac{1}{2}x^2+\frac{1}{24}x^4$ (iv) x

(v) $1-\frac{1}{3}x^2$.

9. (i) 2475 (ii) 12

(iii) $\dfrac{\sin\theta}{\cos 2\theta}$ $(-\pi/4+n\pi<\theta<\pi/4+n\pi)$ (iv) about 42 times.

11. (ia) $n\pi+(-1)^n\pi/6$ (ib) $n\pi/2+\pi/4,\ 2n\pi\pm 2\pi/3$.

13. $|z_1|=\sqrt{2}$, $\arg z_1=-\pi/4$, $|z_2|=2$, $\arg z_2=5\pi/6$.

14. (iii) 1, $e^{2i\pi/3}$, $e^{-2i\pi/3}$ or 1, $(-1\pm i\sqrt{3})/2$.

21. (iii) $\begin{pmatrix}3 & -2\\ -1 & 1\end{pmatrix}$.

22. (i) $\mathbf{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ (ii) $x-1=y=1-z$

23. (i) $|\mathbf{A}| > |\mathbf{C}| > |\mathbf{B}|$; (ii) $\cos^{-1} 17/(2\sqrt{91})$, $\cos^{-1} 15/(2\sqrt{77})$, (iii) $17/\sqrt{14}$, $17/\sqrt{26}$

(iv) $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{r}.\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = 5/\sqrt{2}.$

24. (i) $\dfrac{1}{\sqrt{x^2+1}}$ (ii) $a^x \ln a$

(iii) $x^x(1+\ln x)$ (iv) $\dfrac{1}{1+x^2}$ $\left(\sin^{-1} \dfrac{x}{\sqrt{x^2+1}} = \tan^{-1} x\right).$

26. Maximum at $\left(a, \dfrac{1}{2a}\right)$, minimum at $\left(-a, -\dfrac{1}{2a}\right)$.

27. (i) $\dfrac{1}{\ln a} a^x + C$ (ii) $\dfrac{1}{\sqrt{5}} \tan^{-1} \dfrac{x-1}{\sqrt{5}} + C$

(iii) $\dfrac{1}{a^2+b^2}(a \cos bx + b \sin bx)e^{ax} + C$ (iv) $\dfrac{1}{a+ib} e^{(a+ib)x} + C$

(v) $\dfrac{1}{6} \ln \dfrac{x^2+x+1}{(x-1)^2} + \dfrac{1}{\sqrt{3}} \tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right) + C$

(vi) $\ln \tan(x/2) + C$ (vii) $\ln\left(\dfrac{1+\tan(x/2)}{1-\tan(x/2)}\right) + C$

(viii) $m^{-1}\left(\ln(my) + \sqrt{m^2y^2+c^2}\right) + C$ or $m^{-1} \sinh^{-1}(my/c) + C$

(ix) $x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$ (x) $\frac{1}{2}(x^2-1)e^{x^2} + C$

28. (i) $y^2 = x^2 + C$ (ii) $y = Ce^{mx}$

(iii) $y = Cx - 1$ (iv) $y = C \sec x + \tan x$

(v) $ky = c \sin k(x - x_0)$ (vi) $z^2 + k^2y^2 = C$

29. (i) Solution of $am^2 + bm + c = 0$ (iii) $m = -1$ and $m = -2$; $y = Ae^{-x} + Be^{-2x}$

(iv) Set $z = y' - (p + iq)y$.