1. **QUESTION**:
Describe the sample space and all 16 events for a trial in which two coins are thrown and each shows either a *head* or a *tail*.

SOLUTION:
The sample space is $\mathcal{S} = \{hh,\ ht,\ th,\ tt\}$. As this has 4 elements there are $2^4 = 16$ subsets, namely ϕ, hh, ht, th, tt, $\{hh,\ ht\}$, $\{hh,\ th\}$, $\{hh,\ tt\}$, $\{ht,\ th\}$, $\{ht,\ tt\}$, $\{th,\ tt\}$, $\{hh,\ ht,\ th\}$, $\{hh,\ ht,\ tt\}$, $\{hh,\ th,\ tt\}$, $\{ht,\ th,\ tt\}$ and finally $\{hh,\ ht,\ th,\ tt\}$.

2. **QUESTION**:
A fair coin is tossed, and a fair die is thrown. Write down sample spaces for

   (a) the toss of the coin;
   (b) the throw of the die;
   (c) the combination of these experiments.

Let A be the event that a head is tossed, and B be the event that an odd number is thrown. Directly from the sample space, calculate $\mathrm{P}(A \cap B)$ and $\mathrm{P}(A \cup B)$.

SOLUTION:

   (a) $\{Head, Tail\}$
   (b) $\{1, 2, 3, 4, 5, 6\}$
   (c) $\{(1 \cap Head), (1 \cap Tail), \ldots, (6 \cap Head), (6 \cap Tail)\}$

Clearly $\mathrm{P}(A) = \frac{1}{2} = \mathrm{P}(B)$. We can assume that the two events are independent, so

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = \frac{1}{4}.$$

Alternatively, we can examine the sample space above and deduce that three of the twelve equally likely events comprise $A \cap B$.
Also, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{3}{4}$, where this probability can also be determined by noticing from the sample space that nine of twelve equally likely events comprise $A \cup B$.

3. **QUESTION**:
A bag contains fifteen balls distinguishable only by their colours; ten are blue and five are red. I reach into the bag with both hands and pull out two balls (one with each hand) and record their colours.

   (a) What is the *random phenomenon*?
   (b) What is the *sample space*?
   (c) Express the *event* that the ball in my left hand is red as a subset of the sample space.

SOLUTION:

   (a) The random phenomenon is (or rather the phenomena are) the colours of the two balls.
   (b) The sample space is the set of all possible colours for the two balls, which is

$$\{(B, B), (B, R), (R, B), (R, R)\}.$$

   (c) The event is the subset $\{(R, B), (R, R)\}$.

4. **QUESTION**:
M&M sweets are of varying colours and the different colours occur in different proportions. The table below gives the probability that a randomly chosen M&M has each colour, but the value for tan candies is missing.

Colour	Brown	Red	Yellow	Green	Orange	Tan
Probability	0.3	0.2	0.2	0.1	0.1	?

(a) What value must the missing probability be?

(b) You draw an M&M at random from a packet. What is the probability of each of the following events?

   i. You get a brown one or a red one.
   ii. You don't get a yellow one.
   iii. You don't get either an orange one or a tan one.
   iv. You get one that is brown or red or yellow or green or orange or tan.

SOLUTION:

(a) The probabilities must sum to 1.0 Therefore, the answer is $1-0.3-0.2-0.2-0.1-0.1 = 1-0.9 = .1$.

(b) Simply add and subtract the appropriate probabilities.

   i. $0.3 + 0.2 = 0.5$ since it can't be brown and red simultaneously (the events are incompatible).
   ii. $1 - \text{P(yellow)} = 1 - 0.2 = 0.8$.
   iii. $1 - \text{P(orange} \quad \text{or} \quad \text{tan)} = 1 - \text{P(orange)} - \text{P(tan)} = 1 - 0.1 - 0.1 = 0.8$ (since orange and tan are incompatible events).
   iv. This must happen; the probability is 1.0

5. **QUESTION**:
You consult Joe the bookie as to the form in the 2.30 at Ayr. He tells you that, of 16 runners, the favourite has probability 0.3 of winning, two other horses each have probability 0.20 of winning, and the remainder each have probability 0.05 of winning, excepting Desert Pansy, which has a worse than no chance of winning. What do you think of Joe's advice?

SOLUTION:
Assume that the sample space consists of a win for each of the 16 different horses. Joe's probabilities for these sum to 1.3 (rather than unity), so Joe is *incoherent*, albeit profitable! Additionally, even "Dobbin" has a non-negative probability of winning.

6. **QUESTION**:
Not all dice are fair. In order to describe an unfair die properly, we must specify the probability for each of the six possible outcomes. The following table gives answers for each of 4 different dice.

	Probabilities			
Outcome	Die 1	Die 2	Die 3	Die 4
1	1/3	1/6	1/7	1/3
2	0	1/6	1/7	1/3
3	1/6	1/6	1/7	-1/6
4	0	1/6	1/7	-1/6
5	1/6	1/6	1/7	1/3
6	1/3	1/7	2/7	1/3

Which of the four dice have validly specified probabilities and which do not? In the case of an invalidly described die, explain why the probabilities are invalid.

SOLUTION:

(a) Die 1 is valid.
(b) Die 2 is invalid; The probabilities do not sum to 1. In fact they sum to 41/42.
(c) Die 3 is valid.
(d) Die 4 is invalid. Two of the probabilities are negative.

7. **QUESTION**:
A six-sided die has four green and two red faces and is balanced so that each face is equally likely to come up. The die will be rolled several times. You must choose one of the following three sequences of colours; you will win £25 if the first rolls of the die give the sequence that you have chosen.

R G R R R
R G R R R G
G R R R R R

Without making any calculations, explain which sequence you choose. (In a psychological experiment, 63% of 260 students who had not studied probability chose the second sequence. This is evidence that our intuitive understanding of probability is not very accurate. This and other similar experiments are reported by A. Tversky and D. Kahneman, *Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment*, Psychological Review **90** (1983), pp. 293–315.)

SOLUTION:
Without making calculations, the sequences are identical except for order for the first five rolls. Consequently, these sequences have the same probability up to and including the first five rolls. The second and third sequences must now be less probable than the first, as an extra roll, with probability less than one, is involved. Hence the first sequence is the most probable.
Calculation requires the notion of independence. Two methods. Firstly, work out the probabilities for the sequences: The probability of a red on an individual roll is $\frac{2}{6} = \frac{1}{3}$ and the probability of a green is $\frac{2}{3}$. Hence, since successive rolls are independent, the probability of the first sequence is

$$\frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{243} = 0.0082.$$

Similarly the probabilities of the other two sequences are

$$\frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{729} = 0.0055,$$

and

$$\frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{729} = 0.0027.$$

The sequence with highest probability is the first one. For a second method, reason as follows. All three sequences begin with five rolls containing one green and four reds. The order in which these green and reds occur is irrelevent, because of independence. So, let H be the event that we obtain one green and four reds in the first five rolls. The three sequences are now H, HG, and HR, with probabilities $\mathrm{P}(H)$, $\mathrm{P}(H)\mathrm{P}(G) = \frac{2}{3}\mathrm{P}(H)$, and $\mathrm{P}(H)\mathrm{P}(R) = \frac{1}{3}\mathrm{P}(H)$. Clearly, the first sequence is more probable than the second, which is more probable than the third.

8. **QUESTION**:
Suppose that for three dice of the standard type all 216 outcomes of a throw are equally likely. Denote the scores obtained by X_1, X_2 and X_3. By counting outcomes in the events find (a) $P(X_1+X_2+X_3 \leq 5)$; (b) $P(\min(X_1, X_2, X_3) \geq i)$ for $i = 1, 2, \ldots\ 6$; (c) $P(X_1 + X_2 < (X_3)^2)$.

SOLUTION:

(a) There are 216 equally likely triples and of these only 10 have a sum ≤ 5 so $P(X_1 + X_2 + X_3 \leq 5) = 10/216$.

(b) The smallest of three numbers is bigger than i only when all three are so $P(\min(X_1, X_2, X_3) \geq i) = P(X_1 \geq i, X_2 \geq i, X_3 \geq i) = (7-i)^3/216$ (picture this group as a cube within the bigger cube of all 216 states).

(c) Of 36 triples with $X_3 = 2$ only 3 have $X_1 + X_2 < 4$ and of 36 triples with $X_3 = 3$, 26 have $X_1 + X_2 < 9$ so that $P((X_3)^2 > X_1 + X_2) = \sum_j P(X_1 + X_2 < j^2, X_3 = j) = 137/216$.

9. **QUESTION**:
You play draughts against an opponent who is your equal. Which of the following is more likely: (a) winning three games out of four or winning five out of eight; (b) winning at least three out of four or at least five out of eight?

SOLUTION:
Let X and Y be the numbers of wins in 4 and 8 games respectively. For 4 games there are $2^4 = 16$ equally likely outcomes e.g. $WLWW$ which has 3 wins so $X = 3$. Using our basic counting principles there will be $\binom{4}{j}$ outcomes containing j wins and so $P(X = 3) = 4 \times 0.5^4 = 0.25$.

Similarly with 8 games there are $2^8 = 256$ equally likely outcomes and this time $P(Y = 5) = 56 \times 0.5^8 = 0.2188$ so the former is larger.

For part (b) remember that $X \geq 3$ means all the outcomes with at least 3 wins out of 4 etc and that we sum probabilities over mutually exclusive outcomes. Doing the calculations, $P(X \geq 3) = 0.25 + 0.0625 = 0.3125$ is less than $P(Y \geq 5) = 0.2188 + 0.1094 + 0.0313 + 0.0039 = 0.3633$ – *we deduce from this that the chance of a drawn series falls as the series gets longer.*

10. **QUESTION**:
Count the number of distinct ways of putting 3 balls into 4 boxes when:

MB all boxes and balls are distinguishable;
BE the boxes are different but the balls are indentical;
FD the balls are identical, the boxes are different but hold at most a single ball.

See if you can do the counting when there are m balls and n boxes.

SOLUTION:
#(MB)$= 4^3 = 64$, #(BE)$= \binom{6}{3} = 20$, #(FD)$= \binom{4}{3} = 4$. The general cases are n^m, $\binom{m+n-1}{m}$ (i.e. arrangements of balls and fences), $\binom{n}{m}$.

11. **QUESTION**:
A lucky dip at a school fête contains 100 packages of which 40 contain tickets for prizes. Let X denote the number of prizes you win when you draw out three of the packages. Find the probability density of X i.e. $P(X = i)$ for each appropriate i.

SOLUTION:
There are $\binom{100}{3}$ choices of three packages (in any ordering). There are $\binom{60}{3}$ choices of three packages without prizes. Hence $P(X = 0) = \binom{60}{3}/\binom{100}{3} \approx 0.2116$. If a single prize is won this can happen in $\binom{40}{1} \cdot \binom{60}{2}$ ways. Hence $P(X = 1) = \binom{40}{1} \cdot \binom{60}{2}/\binom{100}{3} \approx 0.4378$ and similarly $P(X = 2) = \binom{40}{2} \cdot \binom{60}{1}/\binom{100}{3} \approx 0.2894$ and $P(X = 3) = \binom{40}{3}/\binom{100}{3} \approx 0.0611$ (there is some small rounding error in the given values).

12. **QUESTION**:
Two sisters maintain that they can communicate telepathically. To test this assertion, you place the sisters in separate rooms and show sister A a series of cards. Each card is equally likely to depict either a circle or a star or a square. For each card presented to sister A, sister B writes down 'circle', or 'star' or 'square', depending on what she believes sister A to be looking at. If ten cards are shown, what is the probability that sister B correctly matches at least one?

SOLUTION:
We will calculate a probability under the assumption that the sisters are guessing. The probability of at least one correct match must be equal to one minus the probability of no correct matches. Let F_i be the event that the sisters fail to match for the ith card shown. The probability of no correct matches is $\mathrm{P}(F_1 \cap F_2 \cap \ldots \cap F_{10})$, where $\mathrm{P}(F_i) = \frac{2}{3}$ for each i. If we assume that successive attempts at matching cards are independent, we can multiply together the probabilities for these independent events, and so obtain

$$\mathrm{P}(F_1 \cap F_2 \cap \ldots \cap F_{10}) = \mathrm{P}(F_1)\mathrm{P}(F_2)\ldots\mathrm{P}(F_{10}) = (\frac{2}{3})^{10} = 0.0173.$$

Hence the probability of at least one match is $1 - 0.0173 = 0.9827$.

13. **QUESTION**:
An examination consists of multiple-choice questions, each having five possible answers. Suppose you are a student taking the exam. and that you reckon you have probability 0.75 of knowing the answer to any question that may be asked and that, if you do not know, you intend to guess an answer with probability 1/5 of being correct. What is the probability you will give the correct answer to a question?

SOLUTION:
Let A be the event that you give the correct answer. Let B be the event that you knew the answer. We want to find $\mathrm{P}(A)$. But $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^c)$ where $\mathrm{P}(A \cap B) = \mathrm{P}(A|B)\mathrm{P}(B) = 1 \times 0.75 = 0.75$ and $\mathrm{P}(A \cap B^c) = \mathrm{P}(A|B^c)\mathrm{P}(B^c) = \frac{1}{5} \times 0.25 = 0.05$. Hence $\mathrm{P}(A) = 0.75 + 0.05 = 0.8$.

14. **QUESTION**:
Consider the following experiment. You draw a square, of width 1 foot, on the floor. Inside the square, you inscribe a circle of diameter 1 foot. The circle will just fit inside the square.
You then throw a dart at the square in such a way that it is equally likely to fall on any point of the square. What is the probability that the dart falls inside the circle? (Think about area!)
How might this process be used to estimate the value of π?

SOLUTION:
All points in the square are equally likely so that probability is the ratio of the area of the circle to the area of the square. The area of the square is 1 and the area of the circle is $\pi/4$ (since the radius is 1/2). If you don't know π you can estimate it by repeating the experiment a very large number of times. Then π will be approximately the same as the proportion of times the dart fall in the circle multiplied by 4.

15. **QUESTION**:
I have in my pocket ten coins. Nine of them are ordinary coins with equal chances of coming up head and tail when tossed and the tenth has two heads.

(a) If I take one of the coins at random from my pocket, what is the probability that it is the coin with two heads ?

(b) If I toss the coin and it comes up heads, what is the probability that it is the coin with two heads ?

(c) If I toss the coin one further time and it comes up tails, what is the probability that it is one of the nine ordinary coins ?

SOLUTION:
Denote by D the event that the coin is the one with two heads.

(a) $\mathrm{P}(D) = 1/10$.

(b) Denote by H the event that we get a head when we toss the coin. Then we want to find $P(D|H)$. By Bayes theorem, we have

$$P(D|H) = \frac{P(H|D)P(D)}{P(H)}.$$

We have $P(H|D) = 1$ and $P(D) = \frac{1}{10}$. Now, we need to think about H, getting a head, in terms of getting a head with either a double headed or single headed coin. Using the idea of a partition,

$$\begin{aligned} P(H) &= P(H \cap D) + P(H \cap D^c) \\ &= P(H|D)P(D) + P(H|D^c)P(D^c) \\ &= (1)(\frac{1}{10}) + (\frac{1}{2})(\frac{9}{10}) \\ &= \frac{11}{20}. \end{aligned}$$

Finally, here is another way of calculating $P(H)$: think of the bag as containing the possible tosses. As the bag contains 9 fair coins and one double-headed coin, it must contains 11 heads and 9 tails, so that the probability of choosing a head is $11/(11+9) = 11/20$.
To return to the original question, we now obtain the answer

$$P(D|H) = \frac{\frac{1}{10}}{\frac{11}{20}} = \frac{2}{11}.$$

(c) 1. If it comes up tails, it can't be the coin with two heads. Therefore it must be one of the other nine.

16. **QUESTION**:
Let A, B and C be any three events. Draw Venn diagrams to deduce that

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
(b) $(A \cap B)' = A' \cup B'$;
(c) $(A \cup B)' = A' \cap B'$.

SOLUTION:
Just draw the picture!

17. **QUESTION**:
A certain person considers that he can drink and drive: usually he believes he has a negligible chance of being involved in an accident, whereas he believes that if he drinks two pints of beer, his chance of being involved in an accident on the way home is only one in five hundred. Assuming that he drives home from the same pub every night, having drunk two pints of beer, what is the chance that he is involved in at least one accident in one year? Are there any assumptions that you make in answering the question?

SOLUTION:
We must assume that each drive home is *independent* of any other drive home. Write $A_i; i = 1, \ldots, 365$; to be the event that our driver is *not* involved in an accident on day i, with $P(A_i) = 0.998$. We find the probability of at least one accident in a year as unity minus the the probability of no accidents at all, i.e.

$$\begin{aligned} P(\textit{At least one accident}) &= 1 - P(\textit{No accidents}) \\ &= 1 - P(\bigcap_{i=1}^{365} A_i) \\ &= 1 - \prod_{i=1}^{365} P(A_i) \quad (\textit{by independence}) \\ &= 1 - (0.998)^{365} \\ &= 0.5184. \end{aligned}$$

18. **QUESTION**:
Two events A and B are such that $P(A) = 0.5$, $P(B) = 0.3$ and $P(A \cap B) = 0.1$. Calculate

(a) $P(A|B)$;
(b) $P(B|A)$;
(c) $P(A|A \cup B)$;
(d) $P(A|A \cap B)$;
(e) $P(A \cap B|A \cup B)$.

SOLUTION:
(Venn diagrams are helpful in understanding some of the events that arise below.)

(a) $P(A|B) = P(A \cap B)/P(B) = \frac{1}{3}$
(b) $P(B|A) = P(A \cap B)/P(A) = \frac{1}{5}$
(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.7$, and the event $A \cap (A \cup B) = A$, so

$$P(A|A \cup B) = P(A)/P(A \cup B) = \frac{5}{7}.$$

(d) $P(A|A \cap B) = P(A \cap B)/P(A \cap B) = 1$, since $A \cap (A \cap B) = A \cap B$.
(e) $P(A \cap B|A \cup B) = P(A \cap B)/P(A \cup B) = \frac{1}{7}$, since $A \cap B \cap (A \cup B) = A \cap B$.

19. **QUESTION**:
An urn contains r red balls and b blue balls, $r \geq 1$, $b \geq 3$. Three balls are selected, without replacement, from the urn. Using the notion of conditional probability to simplify the problem, find the probability of the sequence Blue, Red, Blue.

SOLUTION:
Let B_i be the event that a blue ball is drawn on the ith draw, and define R_i similarly. We require

$$\begin{aligned} P(B_1R_2B_3) &= P(B_3|R_2B_1)P(R_2|B_1)P(B_1) \\ &= (\frac{b-1}{r+b-2})(\frac{r}{r+b-1})(\frac{b}{r+b}). \end{aligned}$$

20. **QUESTION**:
Three babies are given a weekly health check at a clinic, and then returned randomly to their mothers. What is the probability that at least one baby goes to the right mother?

SOLUTION:
Let E_i be the event that baby i is reunited with its mother. We need $P(E_1 \cup E_2 \cup E_3)$, where we can use the result

$$Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C) + Pr(A \cap B \cap C).$$

for any A, B, C. The individual probabilities are $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$. The pairwise joint probabilities are equal to $\frac{1}{6}$, since $P(E_1E_2) = P(E_2|E_1)P(E_1) = (\frac{1}{2})(\frac{1}{3})$, and the triplet $P(E_1E_2E_3) = \frac{1}{6}$ similarly. Hence our final answer is

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{6} = \frac{2}{3}.$$

21. **QUESTION**:
In a certain town, 30% of the people are Conservatives; 50% Socialists; and 20% Liberals. In this town at the last election, 65% of Conservatives voted, as did 82% of the Socialists and 50% of the Liberals. A person from the town is selected at random, and states that she voted at the last election. What is the probability that she is a Socialist?

SOLUTION:
We organise the problem as follows: let C, S and L be the events that a person is Conservative, Socialist, or Liberal respectively. Let V be the event that a person voted in the last election. We require to find $\mathrm{P}(S|V)$, where the information we are given can be summarised as:

$$\mathrm{P}(C) = 0.3, \;\; \mathrm{P}(S) = 0.5, \;\; \mathrm{P}(L) = 0.2,$$

$$\mathrm{P}(V|C) = 0.65 \;\; \mathrm{P}(V|S) = 0.82, \;\; \mathrm{P}(V|L) = 0.5.$$

Now, by Bayes theorem,

$$\mathrm{P}(S|V) = \frac{\mathrm{P}(V|S)\mathrm{P}(S)}{\mathrm{P}(V)}.$$

Each term is known, excepting $\mathrm{P}(V)$ which we calculate using the idea of a partition. We can calculate $\mathrm{P}(V)$ by associating V with the certain partition $C \cup S \cup L$:

$$\begin{aligned}\mathrm{P}(V) &= \mathrm{P}(V \cap (C \cup S \cup L)) \\ &= \mathrm{P}(VC) + \mathrm{P}(VS) + \mathrm{P}(VL) \\ &= \mathrm{P}(V|C)\mathrm{P}(C) + \mathrm{P}(V|S)\mathrm{P}(S) + \mathrm{P}(V|L)\mathrm{P}(L) \\ &= (0.65)(0.3) + (0.82)(0.5) + (0.5)(0.2) \\ &= 0.705.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{P}(S|V) &= \frac{(0.82)(0.5)}{0.705} \\ &= 0.5816.\end{aligned}$$

22. **QUESTION**:
Three prisoners, A, B, and C, are held in separate cells. Two are to be executed. The warder knows specifically who is to be executed, and who is to be freed, whereas the prisoners know only that two are to be executed. Prisoner A reasons as follows: my probability of being freed is clearly $\frac{1}{3}$ until I receive further information. However, it is clear that at least one of B and C will be executed, so I will ask the warder to name one prisoner *other than myself* who is to be executed. Once I know which of B and C is to be executed, either I will go free or the other, unnamed, prisoner will go free, with equal probability. Hence, by asking the name of another prisoner to be executed, I raise my chances of survival from $\frac{1}{3}$ to $\frac{1}{2}$. Investigate A's reasoning. [Hint: find the conditional probability that A is freed, given that the warder names B to be executed.]

SOLUTION:
A's reasoning is unsound. It does not take into account the latitude that the warder has in naming another prisoner to be executed. To see this, let A_F be the event that A goes free, and let W_B be the event that the warder names B. We need to calculate

$$\mathrm{P}(A_F|W_B) = \frac{\mathrm{P}(W_B|A_F)\mathrm{P}(A_F)}{\mathrm{P}(W_B)}.$$

We have

$$\begin{aligned}\mathrm{P}(W_B) &= \mathrm{P}(W_B|A_F)\mathrm{P}(A_F) + \mathrm{P}(W_B|B_F)\mathrm{P}(B_F) + \mathrm{P}(W_B|C_F)\mathrm{P}(C_F) \\ &= (\frac{1}{2})(\frac{1}{3}) + (0)(\frac{1}{3}) + (1)(\frac{1}{3}) \\ &= \frac{1}{2}\end{aligned}$$

and we find thereby that $\mathrm{P}(A_F|W_B) = \frac{1}{3}$. (This analysis supposes that the warder is equally likely to name either B or C in the situation that both are to be executed.)

23. **QUESTION**:
You're playing duplicate bridge. Your partner has bid two spades, and you have to decide whether to pass or to bid game in spades, namely to bid four spades. You reckon that there is a good chance, 40%, that four spades will make. Otherwise, you think three spades will make about 40% of time, and two spades the rest of the time. Suppose there are no doubles (by the opposition, for penalties). The gains and losses depend on whether you are *vulnerable* or not. The possible outcomes and scores are as follows:

	Not vulnerable Score if you make			Vulnerable Score if you make		
You bid	2 spades	3 spades	4 spades	2 spades	3 spades	4 spades
2 spades	110	140	170	110	140	170
3 spades	-50	140	170	-100	140	170
4 spades	-100	-50	420	-200	-100	620

What should you bid when not vulnerable? What should you bid when vulnerable? Calculate the variation in score for one of the bids.

SOLUTION:
This involves calculating expected values and variances for each bid separately, given the different possible outcomes. Suppose you bid two spades. Let X be the score you obtain. X is a random variable with probability distribution as shown below. The expected value is $\mathrm{E}(X) = \sum_x x\mathrm{P}(X = x)$. The calculations are shown below. For the variance, we also need to calculate $\mathrm{E}(X^2)$.

x	110	140	170	Sum
$\mathrm{P}(X = x)$	0.2	0.4	0.4	1
$x\mathrm{P}(X = x)$	22	56	68	146
$x^2\mathrm{P}(X = x)$	2420	7840	11560	21820

It follows that $\mathrm{E}(X) = 146$ and $\mathrm{E}(X^2) = 21820$ so that

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 21820 - (146)^2 = 504,$$

so that $SD(X) = \sqrt{504} = 22.45$.
The complete set of expected values and standard deviations, for each case, is as follows.

	Not vulnerable		Vulnerable	
You bid	$\mathrm{E}(X)$	SD(X)	$\mathrm{E}(X)$	SD(X)
2 spades	146	22	146	22
3 spades	114	83	104	103
4 spades	128	240	168	371

If you are not vulnerable, you maximise your expected score by bidding two spades. If you are vulnerable, you maximise your expected score by bidding four spades. There is substantial variation amongst the scores, particularly for the higher bids.

24. **QUESTION**:
Tay-Sachs disease is a rare fatal genetic disease occurring chiefly in children, especially of Jewish or Slavic extraction. Suppose that we limit ourselves to families which have (a) exactly three children, and (b) which have both parents carrying the Tay-Sachs disease. For such parents, each child has independent probability $\frac{1}{4}$ of getting the disease.
Write X to be the random variable representing the number of children that will have the disease.

(a) Show **(without using any knowledge you might have about the binomial distribution!)** that the probability distribution for X is as follows:

k	0	1	2	3
$\mathrm{P}(X=k)$	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

(b) Show that $\mathrm{E}(X) = \frac{3}{4}$ and that $\mathrm{Var}(X) = \frac{9}{16}$.

SOLUTION:
To answer this question, it is necessary both to think about *how* certain events can occur, and the *probability* that they occcur. Let H be the event that a child is healthy, and D be the event that a child has the disease. If there are three children, there are 8 possiblities (including different orderings) as shown in the next table, where by the sequence H, H, D we mean that the first two children were born healthy, and the third was born with the disease.

Sequence	Probability	X
H,H,H	$\frac{3}{4}\frac{3}{4}\frac{3}{4} = \frac{27}{64}$	0
H,H,D	$\frac{3}{4}\frac{3}{4}\frac{1}{4} = \frac{9}{64}$	1
H,D,H	$\frac{3}{4}\frac{1}{4}\frac{3}{4} = \frac{9}{64}$	1
D,H,H	$\frac{1}{4}\frac{3}{4}\frac{3}{4} = \frac{9}{64}$	1
H,D,D	$\frac{3}{4}\frac{1}{4}\frac{1}{4} = \frac{3}{64}$	2
D,H,D	$\frac{1}{4}\frac{3}{4}\frac{1}{4} = \frac{3}{64}$	2
D,D,H	$\frac{1}{4}\frac{1}{4}\frac{3}{4} = \frac{3}{64}$	2
D,D,D	$\frac{1}{4}\frac{1}{4}\frac{1}{4} = \frac{1}{64}$	3

The probabilities for each sequence are shown in the second column; successive births are independent so that we can multiply probabilities. Notice that the sum of the probabilities is 1. The random variable X is the number of children having the disease. We see that only one sequence leads to $X = 0$, and this sequence has probability $\frac{27}{64}$. Hence $\mathrm{P}(X = 0) = \frac{27}{64}$. There are three sequences leading to $X = 1$, each with probability $\frac{9}{64}$. Hence $\mathrm{P}(X = 1) = \frac{9}{64} + \frac{9}{64} + \frac{9}{64} = \frac{27}{64}$. The other probabilities are found similarly. It is easy to show that $\mathrm{E}(X) = \frac{3}{4}$ and that $\mathrm{E}(X^2) = \frac{18}{16}$, so that $\mathrm{Var}(X) = \frac{9}{16}$.
Remark. This is an example of a *binomial* distribution with parameters $n = 3$ and $p = \frac{1}{4}$. That is,

$$\mathrm{P}(X = k) = \frac{3!}{k!(3-k)!}(\frac{1}{4})^k(\frac{3}{4})^{3-k}, \quad k = 0, 1, 2, 3.$$

For such distributions it is well known that $\mathrm{E}(X) = np$ and that $\mathrm{Var}(X) = np(1-p)$.

25. **QUESTION**:
A six-sided die has four green and two red faces and is balanced so that each face is equally likely to come up. The die will be rolled several times. Suppose that we score 4 if the die is rolled and comes up green, and 1 if it comes up red. Define the random variable X to be this score. Write down the distribution of probability for X and calculate the expectation and variance for X.

SOLUTION:
The distribution of X is as follows.

k	1	4
P$(X=k)$	$\frac{1}{3}$	$\frac{2}{3}$

$$\begin{aligned} E(X) &= (1)\frac{1}{3}+(4)\frac{2}{3}=3. \\ E(X^2) &= (1^2)\frac{1}{3}+(4^2)\frac{2}{3}=11. \\ Var(X) &= E(X^2)-(E(X))^2=11-9=2. \end{aligned}$$

26. **QUESTION**:
For two standard dice all 36 outcomes of a throw are equally likely. Find $P(X_1+X_2=j)$ for all j and calculate $E(X_1+X_2)$. Confirm that $E(X_1)+E(X_2)=E(X_1+X_2)$.

SOLUTION:
The possible totals are $j=2, 3, \ldots, 12$ and $P(X_1+X_2=j)=(j-1)/36$, $j=2, \ldots, 7$ and $P(X_1+X_2=j)=(13-j)/36$, $j=8, \ldots, 12$. For each of the dice $E(X_i)=21/6=7/2$ while for the total

$$E(X_1+X_2)=\tfrac{1}{36}(2\cdot 1+3\cdot 2+\cdots+7\cdot 6+\cdots+12\cdot 1)=252/36=7$$

27. **QUESTION**:
X takes values 1, 2, 3, 4 each with probability 1/4 and Y takes values 1, 2, 4, 8 with probabilities 1/2, 1/4, 1/8 and 1/8 respectively. Write out a table of probabilities for the 16 paired outcomes which is consistent with the distributions of X and Y. From this find the possible values and matching probabilities for the total $X+Y$ and confirm that $E(X+Y)=E(X)+E(Y)$.

SOLUTION:
There are 16 pairs and infinitely many ways to allocate the probabilities. Selecting one, say $p(1,4)=p(1,8)=1/8$, $p(2,2)=1/4$, $p(3,1)=p(4,1)=1/4$ we see this satisfies $\sum_j p(i,j)=P(X=i)$ and $\sum_i(i,j)=P(Y=j)$. The possible values and probabilities are

t	2	3	4	5	6	7	8	9	10	11	12
p_t	0	0	1/2	3/8	0	0	0	1/8	0	0	0

where for instance $P(X+Y=4)=p(2,2)+p(3,1)=1/2$. From the table, $E(X+Y)=(4\cdot 4+5\cdot 3+9\cdot 1)/8=5$. As $E(X)=5/2$ and $E(Y)=5/2$ the required equality holds.

28. **QUESTION**:
Calculation practice for the binomial distribution. Find $P(X=2)$, $P(X<2)$, $P(X>2)$ when

(a) $n=4$, $p=0.2$; (b) $n=8$, $p=0.1$;
(c) $n=16$, $p=0.05$; (d) $n=64$, $p=0.0125$.

SOLUTION:
(a) $P(X=2)=6\cdot 0.2^2\cdot 0.8^2=0.1536$, $P(X<2)=P(X=0)+P(X=1)=0.4096+0.4096=0.8192$, $P(X>2)=1-P(X\leq 2)=1-0.8192-0.1536=0.0272$.
(b) $P(X=2)=0.1488$, $P(X<2)=0.8131$, $P(X>2)=0.0381$.
(c) $P(X=2)=0.1463$, $P(X<2)=0.8108$, $P(X>2)=0.0429$.
(d) $P(X=2)=0.1444$, $P(X<2)=0.8093$, $P(X>2)=0.0463$.

29. **QUESTION**:
A wholesaler supplies products to 10 retail stores, each of which will independently make an order on a given day with chance 0.35. What is the probability of getting exactly 2 orders? Find the most probable number of orders per day and the probability of this number of orders. Find the expected number of orders per day.

SOLUTION:

Using the independence of orders the chance that only the first two stores place orders is $0.35^2 \cdot 0.65^8$. As there are $10 \times 9/2 = 45$ distinct pairs of stores that could order we have

$$P(X = 2) = 450.35^2 0.65^8 = 0.1757$$

A similar argument works for any number of orders. We say that the number of orders placed has the Bin(10, 0.35) distribution. The formula for x orders is

$$P(X = x) = \binom{10}{x} 0.35^x 0.65^{10-x}$$

The most probable number of orders is 3 (either calculate $P(X = x)$ for a few different x values or look at binomial tables in a textbook) and $P(X = 3) = 120(0.35)^3(0.65)^7 \approx 0.2522$. The expected number of orders is

$$\sum_0^{10} x \cdot P(X = x) = 1 \cdot 0.0725 + 2 \cdot 0.1757 + 3 \cdot 0.2522 + 4 \cdot 0.2377 + \cdots$$

which (barring numericals errors) will give the same answer as the formula $E(X) = np = 10 \times 0.35 = 3.5$.

(The problem of which number is most likely for general n and p was not set but is not all that hard – show that $P(X = x + 1) < P(X = x) \Leftrightarrow x + 1 > (n + 1)p$ and think about what that means)

30. **QUESTION**:
A machine produces items of which 1% at random are defective. How many items can be packed in a box while keeping the chance of one or more defectives in the box to be no more than 0.5? What are the expected value and standard deviation of the number of defectives in a box of that size?

SOLUTION:
Let X be the number of defectives when n items are packed into a box. $P(X = 0) = 0.99^n$ so that $P(X \geq 1) = 1 - 0.99^n$. To ensure $1 - 0.99^n < 0.5$ we must take $n < \log 0.5/\log 0.99 = 68.97$ so $n = 68$. The expected value and standard deviation of X when $n = 68$ are 0.68 and $\sqrt{0.68 \times 0.99} = 0.8205$.

31. **QUESTION**:
Suppose that 0.3% of bolts made by a machine are defective, the defectives occurring at random during production. If the bolts are packaged in boxes of 100, what is the Poisson approximation that a given box will contain x defectives? Suppose you buy 8 boxes of bolts. What is the distribution of the number of boxes with no defective bolts? What is the expected number of boxes with no defective bolts?

SOLUTION:
D is Bin(100, 0.003) which is approximately Poisson with parameter $100 \times 0.003 = 0.3$. Hence $P(D = x) \approx e^{-0.3}(0.3)^x/x!\,,\ x = 0, 1, \ldots$. In particular, $P(D = 0) \approx 0.7408$. Finally, N, the no. of boxes with no defectives is Bin(8, 0.7408) and so $E(N) = 8 \times 0.7408 = 5.926$.

32. **QUESTION**:
Events which occur randomly at rate r are counted over a time period of length s so the event count X is Poisson. Find $P(X = 2)$, $P(X < 2)$ and $P(X > 2)$ when
(a) $r = 0.8$, $s = 1$; (b) $r = 0.1$, $s = 8$; (c) $r = 0.01$, $s = 200$; (d) $r = 0.05$, $s = 200$.

SOLUTION:
(a) and (b) $\lambda = rs = 0.8$ so that $P(X = 2) = e^{-0.8}0.8^2/2 = 0.1438$, $P(X < 2) = 0.4493 + 0.3695 = 0.8088$ and $P(X > 2) = 1 - 0.8088 - 0.1438 = 0.0474$. (c) $\lambda = 2$ so that $P(X = 2) = 0.2707$, $P(X < 2) = 0.4060$, $P(X > 2) = 0.3233$. (d) $\lambda = 10$ so that $P(X = 2) = 0.00227$, $P(X < 2) = 0.00050$, $P(X > 2) = 0.9972$.

33. **QUESTION**:
Given that 0.04% of vehicles break down when driving through a certain tunnel find the probability of (a) no (b) at least two breakdowns in an hour when 2,000 vehicles enter the tunnel.

SOLUTION:
The number of breakdowns X has a binomial distribution which can be approximated by the Pn(λ) distribution with $\lambda = 2000 \times 0.0004 = 0.8$. Hence $P(X = 0) \approx 0.4493$ and $P(X \geq 2) = 1 - P(X \leq 1) \approx 1 - 0.8088 = 0.1912$.

34. **QUESTION**:
Experiments by Rutherford and Geiger in 1910 showed that the number of alpha particles emitted per unit time in a radioactive process is a random variable having a Poisson distribution. Let X denote the count over one second and suppose it has mean 5. What is the probability of observing fewer than two particles during any given second? What is the $P(X \geq 10)$? Let Y denote the count over a separate period of 1.5 seconds. What is $P(Y \geq 10)$? What is $P(X + Y \geq 10)$?

SOLUTION:
$P(X \leq 1) = e^{-5}(1 + 5) = 0.0404$. $P(X \geq 10) = 0.0398$. Y is Pn(7.5) and so $P(Y \geq 10) = 1 - P(Y \leq 9) = 1 - \sum_0^9 P(Y = i) = 0.5113$. $X + Y$ is Pn(12.5) and we find $P(X + Y \geq 10) = 0.7986$.

35. **QUESTION**:
A process for putting chocolate chips into cookies is random and the number of choc chips in a cookie has a Poisson distribution with mean λ. Find an expression for the probability that a cookie contains less than 3 choc chips.

SOLUTION:
The Poisson distribution gives probabilities for each possible number of choc chips but as a cookie can't contain two different numbers simultaneously we *add* the probabilities for the possible values 0, 1 and 2. Hence

$$P(X < 3) = e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)$$

(if you wonder what happened to s, it equals 1 as we're only looking at a single cookie – for 10 cookies we take $s = 10$ etc).

36. **QUESTION**:
Let X have the density $f(x) = 2x$ if $0 \leq x \leq 1$ and $f(x) = 0$ otherwise. Show that X has the mean 2/3 and the variance 1/18. Find the mean and the variance of the random variable $Y = -2X + 3$.

SOLUTION:
To find expected values for continuous random variables we integrate e.g.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x \cdot 2x\, dx = 2/3$$

and similarly

$$\text{Var}(X) = \int_0^1 (x - 2/3)^2 \cdot 2x\, dx = 1/2 - 8/9 + 4/9 = 1/18$$

You could also use the formula $\text{Var}(X) = E(X^2) - E(X)^2$ where

$$E(X^2) = \int_0^1 x^2 \cdot 2x\, dx = 1/2$$

so that $\text{Var}(X) = 1/2 - (2/3)^2 = 1/18$. Use the linearity of expectation for the last bits to get

$$E(Y) = -2 \times 2/3 + 3 = 5/3 \quad \text{and} \quad \text{Var}(Y) = (-2)^2 \times 1/18 = 2/9$$

37. **QUESTION**:
Let the random variable X have the density $f(x) = kx$ if $0 \le x \le 3$. Find k. Find x_1 and x_2 such that $P(X \le x_1) = 0.1$ and $P(X \le x_2) = 0.95$. Find $P(\,|X - 1.8| < 0.6)$.

SOLUTION:
To make $\int_0^3 f(x)dx = 1$ we must take $k = 2/9$. For any c in $(0, 3)$, $P(X \le c) = c^2/9$ so $x_1 = \sqrt{0.9} = 0.9487$ and $x_2 = \sqrt{8.55} = 2.9240$. $P(\,|X - 1.8| < 0.6) = P(1.2 < X < 2.4) = (2.4^2 - 1.2^2)/9 = 0.48$.

38. **QUESTION**:
A small petrol station is supplied with petrol once a week. Assume that its volume X of potential sales (in units of 10,000 litres) has the probability density function $f(x) = 6(x - 2)(3 - x)$ for $2 \le x \le 3$ and $f(x) = 0$ otherwise. Determine the mean and the variance of this distribution. What capacity must the tank have for the probability that the tank will be emptied in a given week to be 5%?

SOLUTION:
Proceed as usual, $E(X) = 5/2$ and $\text{Var}(X) = 1/20$. Let T denote the capacity of the tank. We need to solve $0.05 = \int_T^3 6(x - 2)(3 - x)dx = \int_{T-2}^1 6y(1 - y)\,dy = 1 - 3(T - 2)^2 + 2(T - 2)^3$ and doing this numerically (iterate the equation $T - 2 = \sqrt{0.95}(7 - 2T)^{-1/2}$) we find $T \approx 2.86465$ i.e. the tank should hold approximately 28,650 litres.

39. **QUESTION**:
Find the probability that none of the three bulbs in a set of traffic lights will have to be replaced during the first 1200 hours of operation if the lifetime X of a bulb (in thousands of hours) is a random variable with probability density function $f(x) = 6[0.25 - (x - 1.5)^2]$ when $1 \le x \le 2$ and $f(x) = 0$ otherwise. You should assume that the lifetimes of different bulbs are independent.

SOLUTION:
For a single bulb, $P(X > 1.2) = 6[\frac{3}{2}x^2 - \frac{1}{3}x^3 - 2x]_{1.2}^2 = 0.8960$. Hence $P(\text{no bulbs replaced}) = 0.8960^3 = 0.7193$.

40. **QUESTION**:
Suppose X is $\text{N}(10, 1)$. Find (i) $P[X > 10.5]$, (ii) $P[9.5 < X < 11]$, (iii) x such that $P[X < x] = 0.95$. *You will need to use Standard Normal tables.*

SOLUTION:
Let Z denote a $N(0, 1)$ random variable from now on and let Φ denote its cdf.

(i) $P[X > 10.5] = P[X - 10 > 0.5] = 1 - \Phi(0.5) = 0.3085$; (ii) $P[9.5 < X < 11] = \Phi(1) - \Phi(-0.5) = 0.5328$; (iii) $P[X < x] = P[Z < x - 10] = 0.95$ when $x - 10 = 1.645$ i.e. $x = 11.645$.

41. **QUESTION**:
Suppose X is $\text{N}(-1, 4)$. Find

(a) $P(X < 0)$; (b) $P(X > 1)$; (c) $P(-2 < X < 3)$; (d) $P(|X + 1| < 1)$.

SOLUTION:
As $X = 2Z + 1$ we have (a) $P(X < 0) = P(2Z + 1 < 0) = \Phi(-1/2) = 1 - \Phi(1/2) = 0.3085$; (b) $P(X > 1) = \Phi(0) = 1/2$; (c) $P(-2 < X < 3) = \Phi(1) - \Phi(-3/2) = 0.7745$; (d) $P(|X + 1| < 1) = P(-2 < X < 0) = \Phi(-1/2) - \Phi(-3/2) = 0.2417$.

42. **QUESTION**:
Suppose X is $\text{N}(\mu, \sigma^2)$. For $a = 1$, 2, 3 find $P(\,|X - \mu| < a\sigma)$.

SOLUTION:
$P(\,|X - \mu| < a\sigma) = P(-a < Z < a) = 2\Phi(a) - 1$ so the required values are approximately 0.682, 0.954 and 0.998 respectively.

43. **QUESTION**:
The height of a randomly selected man from a population is normal with $\mu = 178$cm and $\sigma = 8$cm. What proportion of men from this population are over 185cm tall? There are 2.54cm to an inch. What is their height distribution in inches? The heights of the women in this population are normal with $\mu = 165$ cm and $\sigma = 7$cm. What proportion of the women are taller than half of the men?

SOLUTION:
Let M denote the height of a man and W the height of a woman in centimetres. We want to know $P(M > 185) = P(\frac{M-178}{8} > \frac{185-178}{8}) = P(Z > 0.875) \approx 0.19$. Let H denote the height of a man in inches. Then $H = M/2.54$ so that H is $N(70.1\,, (3.15)^2\,)$. Finally $P(M < h) = 0.5$ when $h = 178$ and so $P(W > 178) = P(Z > \frac{13}{7}) = 0.032$ so 3.2% of the women are taller than half of the men.

44. **QUESTION**:
N independent trials are to be conducted, each with "success" probability p. Let $X_i = 1$ if trial i is a success and $X_i = 0$ if it is not. What is the distribution of the random variable $X = X_1 + X_2 + \ldots + X_N$? Express $P[a \leq X \leq b]$ as a sum (where $a \leq b$ and these are integers between 0 and N). Use the central limit theorem to provide an approximation to this probability. Compare your approximation with the limit theorem of De Moivre and Laplace on p1189 of Kreyszig.

SOLUTION:
As X is the total number of successes in N independent trials X is $\text{Bin}(N, p)$. Thus

$$P[a \leq X \leq b] = \sum_{x=a}^{b} \binom{N}{x} p^x (1-p)^{N-x}.$$

As each X_i has $E(X_i) = p$ and $\text{Var}(X_i) = p(1-p)$ *(you should confirm this)* the central limit theorem says that X is approximately $\text{Normal}\big(Np, Np(1-p)\big)$. Let $\bar{a} = (a - Np)/\sqrt{Np(1-p)}$ and $\bar{b} = (b - Np)/\sqrt{Np(1-p)}$. The approximation is $P[a \leq X \leq b] \approx P[\bar{a} < Z < \bar{b}]$ where Z has the standard Normal distribution. The reason for the small correction in the version of this result in Kreyszig is that $P[a \leq X \leq b] = P[a - \delta < X < b + \delta]$ for any $\delta \in (0, 1)$ while the approximation varies with δ – the choice $\delta = 0.5$ is arbitrary but generally sensible.

45. **QUESTION**:
Suppose that of $1,000,000$ live births in Paris over some period, $508,000$ are boys. Suppose X is $\text{Bin}(10^6, 0.5)$ and calculate approximately $P[X \geq 508,000]$. Does it seem reasonable to you that the proportion of males among Parisian babies conceived soon after the above period will be 50%. (Laplace developed his limit theorem in the late 1700's to deal with a question similar to this.)

SOLUTION:
From q17, $\bar{b} = 8,000/500 = 16$ and from the formulæ sheet $1 - \Phi(16) \approx 6.4 \times 10^{-58}$. It seems entirely unreasonable that births at that time and place should be modelled with $p = 0.5$ chance of each sex. Standard practice would be to say that the proportion is within some small distance of $0.508 (= 508,000/10^6\,)$.

46. **QUESTION**:
An airfreight company has various classes of freight. In one of these classes the average weight of packages is 10kg and the variance of the weight distribution is 9kg^2. Assuming that the package weights are independent (it is not the case that a single company is sending a large number of identical packages, for instance), estimate the probability that 100 packages will have total weight more than 1020kg.

SOLUTION:
The central limit theorem says $\sum W_i$ is approximately $N(1,000\,, 30^2\,)$ so that $P(\sum W_i > 1,020) \approx P\big[Z > (1,020 - 1,000)/30 = 0.67\big] = 0.251$.