GALOIS GROUPS OF p-EXTENSIONS OF HIGHER LOCAL FIELDS

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Abstract. Suppose $\mathcal{K}$ is N-dimensional local field of characteristic $p>2$, $\mathcal{G}_{<p}$ is the maximal quotient of $\mathcal{G} == Gal(\mathcal{K}_{sep}/\mathcal{K})$ of period p and nilpotent class $<p$ and $\mathcal{K}_{<p}\subset\mathcal{K}_{sep}$ is such that $Gal(\mathcal{K}_{<p}/\mathcal{K})=\mathcal{G}_{<p}$. We use nilpotent Artin-Schreier theory to identify $\mathcal{G}_{<p}$ with the group $G(\mathcal{L})$ obtained from a profinite Lie $\mathbb{F}_p$-algebra $\mathcal{L}$ via the Campbell-Hausdorff composition law. The canonical $\mathcal{P}$-topology on $\mathcal{K}$ is used to define a dense Lie subalgebra $\mathcal{L}^{\mathcal{P}}$ in $\mathcal{L}$. The algebra $\mathcal{L}^{\mathcal{P}}$ can be provided with a system of $\mathcal{P}$-topological generators and its $\mathcal{P}$-open subalgebras correspond to all N-dimensional extensions of $\mathcal{K}$ in $\mathcal{K}_{<p}$. These results are applied to higher local fields K of characteristic 0 containing non-trivial p-th root of unity. If $\Gamma=Gal(K_{alg}/K)$ we introduce similarly the quotient $\Gamma_{<p}=G(L)$, a dense $\mathbb{F}_p$-Lie algebra $L^{\mathcal{P}}\subset L$, and describe the structure of $L^{\mathcal{P}}$ in terms of generators and relations. The general result is illustrated by explicit presentation of $\Gamma_{<p}$ modulo subgroup of third commutators.

Introduction

Let $p>2$ be a fixed prime number.

0.1. **Higher local fields.** The concept of higher local field K of dimension $N\geqslant 1$ was introduced as an essential ingredient of the theory of higher adeles in the study of arithmetic properties of algebraic varieties. In dimension 0 we just require that K is finite of characteristic p. If $N\geqslant 1$ then K is a complete discrete valuation field with the residue field which is isomorphic to some $(N-1)$-dimensional local field of characteristic p. (In this paper we work with fields, which have most interesting arithmetic properties.)This residue field will be called the first residue field $K^{(1)}$ of K. Similarly, we obtain next residue fields, the last (or N-th) residue field is necessarily finite and will be always denoted by $k\simeq\mathbb{F}_{p^{N_0}}$. For example, 1-dimensional fields appear as either finite extensions of $\mathbb{Q}_p$ or fields of formal Laurent series in one variable with coefficients in a finite field k. Basics of the theory of such fields including highly important concept of special topology ($\mathcal{P}$-topology) together with classification results can be found in [36, 37], cf. also Sect. 2 below.

2010 *Mathematics Subject Classification.* 11S15, 11S20.

Key words and phrases. Local field, Galois group.

One of first considerable achievements of the theory of higher local fields was the construction of higher dimensional generalization of local class field theory, cf. [21, 22, 23, 28, 29, 30] and (for explicit aspects of the theory) [34, 10]. In this setting abelian extensions of N-dimensional fields are described (in a functorial way) in terms of the appropriate Milnor K_N-groups. The group $\Gamma_K = \mathrm{Gal}(K_{sep}/K)$ is soluble and its most interesting part appears as the Galois group $\Gamma_K(p)$ of the maximal p-extension of K. Its structure (as well as of any other p-group) can be described in terms of generators and relations. (Notice that in higher dimensions there are substantial problems with a choice of a set of generators, cf. below.) Any minimal (topological) system of generators of $\Gamma_K(p)$ comes from the lifts of any topological $\mathbb{F}_p$-basis of $\Gamma_K^{ab}/(\Gamma_K^{ab})^p$.

0.2. **Review of 1-dimensional case.** Suppose K is 1-dimensional. In this case (according to class field theory) generators of $\Gamma_K(p)$ come from any $\mathbb{F}_p$-basis of K^*/K^{*p}. This basis can be chosen in a natural way if we fix a choice of uniformizing element of K. For example, suppose $K \simeq \mathbb{F}_p((t))$ and for all $a \in \mathbb{Z}^0(p) := \{a \in \mathbb{Z}_{>0} \mid \gcd(a,p) = 1\} \bigcup \{0\}$, $T_a \in K_{sep}$ are such that $T_a^p - T_a = t^{-a}$. For $b \in \mathbb{Z}^0(p)$, let $\tau_b \in \Gamma_K(p)$ be such that $\tau_b(T_a) - T_a = \delta_{ab}$ (the Kronecker symbol). Then $\{\tau_a \mid a \in \mathbb{Z}^0(p)\}$ is a minimal system of generators in $\Gamma_K(p)$.

The structure of $\Gamma_K(p)$ was described around 1960's as follows:

— if $\mathrm{char}\, K = p$ or $\mathrm{char} K = 0$ and K contains no non-trivial p-th roots of unity the group $\Gamma_K(p)$ is profinite free (I. Shafarevich);

— if K contains a non-trivial p-th root of unity then $\Gamma_K(p)$ has a minimal system of generators containing $[K : \mathbb{Q}_p] + 2$ elements and one (explicitly known) relation (S. Demushkin), cf. [32, 33, 24]. (This result leads to a complete description of Γ_K, cf.[20].)

There is no a straight way to extend the above results to higher local fields for the following reasons.

First, there is no any reasonable choice of generators in $\Gamma_K(p)$. To illustrate this suppose $N = 2$ and $K = \mathbb{F}_p((t_2))((t_1))$ is 2-dimensional local field of iterated Laurent formal series. The extension $K(T)$ such that $T^p - T = t_1^{-1}(1 + t_2 + \dots + t_2^n + \dots)$ is not contained in the composit of all $K(T_n)$, where $T_n^p - T_n = t_1^{-1}t_2^n$, $n \geqslant 0$. As a result, the lifts of elements of the Galois groups of the elementary field extensions $K(T_{a_1a_2})$, where $T_{a_1a_2}^p - T_{a_1a_2} = t_1^{-a_1}t_2^{-a_2}$, generate only very small piece of $\Gamma_K(p)$. This also can be seen at the level of class field theory, where the abelian extensions of N-dimensional local field K are described via the quotients of the K-group $K_N(K)$. This group has no natural system of generators if $N > 1$, but it contains a dense subgroup $K_N^{top}(K)$; this subgroup can be described via (topological) generators and can be taken instead of $K_N(K)$ when studying finite abelian extensions of K.

Another concern is related to the strategy used by Demushkin (in the 1-dimensional case). Let K be a 1-dimensional local field of characteristic 0 containing a non-trivial p-th root of unity. For $s\geqslant 1$, let $C_s^{(p)}$ be the s-th term of the p-central series of $\Gamma_K(p)$. Then we can use the interpretation of the abelian quotient $\Gamma_K(p)/C_2^{(p)}$ in terms of class field theory. Applying formalism of the Galois cohomology we can describe explicitly the action of this quotient on $C_2^{(p)}/C_3^{(p)}$: this involves calculations with Hilbert symbol. As a result we can describe the group theoretic structure of $\Gamma_K(p)/C_3^{(p)}$ in terms of a specially chosen minimal system of generators and one (explicitly given) relation. Luckily, this allows us to recover the structure of $\Gamma_K(p)$ by choosing special lifts of generators which satisfy the simplest possible lift of that relation.

The above strategy was applied in the case of local fields of dimension 2 in [15]. At that time explicit aspects of higher local class field theory, in particular, formulas for the Hilbert symbol, were just developed by the second author of that paper. The authors used the subgroup $K_2^{top}(K)$ of $K_2(K)$ and attempted (following the 1-dimensional strategy) to find the structure of a dense subgroup in $\Gamma_K(p)/C_3^{(p)}$. The paper [15] justifies that in higher dimensions the Demushkin strategy requires enourmous calculations but don't give very much information about $\Gamma_K(p)$. The whole approach should be profoundly revisited at least for the following reason. When we use class field theory and afterwards apply explicit formulas for the Hilbert symbol, we actually move in two opposite directions. For this reason, it makes sense to avoid the use of class field theory and to proceed within the frames of Kummer (or Artin-Schreier) theory from the very beginning. Another important concern is that (abelian) class field theory is not sufficient for understanding the structure of $\Gamma_K(p)$ better than just modulo $C_3^{(p)}$.

In [1, 2] the author initiated the study of $\Gamma_K(p)$ modulo the subgroup of p-th commutators for fields of characteristic p via so-called nilpotent Artin-Schreier theory. Later we applied this theory together with the Fontaine-Wintenberger field-of-norms functor to study the case of 1-dimensional local fields K with non-trivial p-th roots of unity, cf. [11, 12, 13]. As a result, we obtained the description of $\Gamma_{<p}:=\Gamma_K/\Gamma_K^pC_p$ in terms of a specially chosen system of generators which satisfy one relation. (Here $C_p=C_p(\Gamma_K)$ is the closure of the subgroup of p-th commutators in Γ_K.) This could be considered as an alternative approach to Demushkin's result in the 1-dimensional case. Actually, we obtained much more in the above papers: our result gives also an explicit description of the images of all ramification subgroups $\Gamma_K^{(v)}$, $v\geqslant 0$, in $\Gamma_{<p}$, cf. also Sect. 0.4.)

0.3. **Main results.** In this paper we develop a techniques allowing us to study the structure of $\Gamma_{<p}=\Gamma_K/\Gamma_K^pC_p(\Gamma_K)$ in terms of generators

and relations. We consider the cases where either $K = \mathcal{K}$ has characteristic p or K has characteristic 0 and a non-trivial p-th root of unity $\zeta_1 \in K$. In both cases we introduce a dense subgroup $\Gamma_{<p}^{\mathcal{P}}$ of $\Gamma_{<p}$ with new $\mathcal{P}$-topological structure (related to the $\mathcal{P}$-topology on K). The subgroup $\Gamma_{<p}^{\mathcal{P}}$ still allows us to study finite extensions of K in $K_{<p}$ and admits a description in terms of $\mathcal{P}$-topological generators and relations.

Describe the content of the paper in more details. .

a) For an N-dimensional local field $\mathcal{K}$ of characteristic p we apply nilpotent Artin-Schreier theory to fix an identification $\pi : \mathcal{G}_{<p} \simeq G(\mathcal{L})$. Here $\mathcal{G}_{<p} = \mathcal{G}/\mathcal{G}^p C_p(\mathcal{G})$ is the maximal quotient of $\mathcal{G} = \mathrm{Gal}(\mathcal{K}_{sep}/\mathcal{K})$ of period p and nilpotent class $< p$, $\mathcal{L}$ is a profinite Lie $\mathbb{F}_p$-algebra and $G(\mathcal{L})$ is the profinite p-group obtained from $\mathcal{L}$ via the Campbell-Hausdorff composition law. The identification π is defined uniquely up to conjugation after choosing a suitable element $e \in \mathcal{L} \otimes \mathcal{K}$.

b) We use the $\mathcal{P}$-topology on $\mathcal{K}$ to define the Lie subalgebra $\mathcal{L}^{\mathcal{P}}$ in $\mathcal{L}$. This is $\mathcal{P}$-topological algebra provided with a system of $\mathcal{P}$-topological generators. The algebra $\mathcal{L}^{\mathcal{P}}$ is dense in $\mathcal{L}$, i.e. the profinite completion of $\mathcal{L}^{\mathcal{P}}$ coincides with $\mathcal{L}$.

c) With respect to (defined up to conjugation) identifications of nilpotent Artin-Schreier theory $\pi : \mathcal{G}_{<p} \simeq G(\mathcal{L})$ the algebra $\mathcal{L}^{\mathcal{P}}$ gives rise to a class of conjugated subgroups $\mathcal{G}_{<p}^{\mathcal{P}} := \pi^{-1}(\mathcal{L}^{\mathcal{P}})$; the profinite completions of $\mathcal{G}_{<p}^{\mathcal{P}}$ coincide with $\mathcal{G}_{<p}$.

d) The subgroups $\mathcal{G}_{<p}^{\mathcal{P}}$ have $\mathcal{P}$-topological systems of generators and could be used to study N-dimensional local field extensions $\mathcal{K}'$ of $\mathcal{K}$ in $\mathcal{K}_{<p}$. More precisely, $\mathcal{H}$ is an open subgroup in $\mathcal{G}_{<p}$ (with respect to the Krull topology) iff $\mathcal{H}^{\mathcal{P}} := \mathcal{G}_{<p}^{\mathcal{P}} \cap \mathcal{H}$ is a $\mathcal{P}$-open subgroup of finite index in $\mathcal{G}_{<p}^{\mathcal{P}}$. We have also $(\mathcal{G}_{<p} : \mathcal{H}) = (\mathcal{G}_{<p}^{\mathcal{P}} : \mathcal{H}^{\mathcal{P}}) = [\mathcal{K}' : \mathcal{K}]$, where $\mathcal{K}' = \mathcal{K}_{<p}^{\mathcal{H}}$. In particular, $\mathcal{K}'/\mathcal{K}$ is Galois iff $\mathcal{H}^{\mathcal{P}}$ is normal in $\mathcal{G}_{<p}^{\mathcal{P}}$; in this case $\mathrm{Gal}(\mathcal{K}'/\mathcal{K}) = \mathcal{G}_{<p}^{\mathcal{P}}/\mathcal{H}^{\mathcal{P}}$.

e) Suppose $t = (t_1, \dots, t_N)$ is a system of local parameters in $\mathcal{K}$, $\mathrm{m}_{\mathcal{K}}$ is the maximal ideal in the N-valuation ring $\mathcal{O}_{\mathcal{K}}$ of $\mathcal{K}$, $\omega \in \mathrm{m}_{\mathcal{K}}$, and for $1 \leqslant m \leqslant N$, $h_{\omega}^{(m)} \in \mathrm{Aut}\,\mathcal{K}$ are such that $h_{\omega}^{(m)}(t_i) = t_i E(\omega^p)^{\delta_{mi}}$, where $E(X)$ is the Artin-Hasse exponential. Then all lifts of $h_{\omega}^{(m)}$, $1 \leqslant m \leqslant N$, to $\mathcal{K}_{<p}$ form a subgroup $\mathcal{G}_{\omega} \subset \mathrm{Aut}\,\mathcal{K}_{<p}$ containing $\mathcal{G}_{<p}$. Let Γ_{ω} be the maximal quotient of $\mathcal{G}_{\omega}$ of period p and nilpotent class $< p$. If $\bar{\mathcal{G}}$ is the image of $\mathcal{G}_{<p}$ in Γ_{ω} we obtain the following short exact sequence of profinite p-groups

$$1 \longrightarrow \bar{\mathcal{G}} \longrightarrow \Gamma_{\omega} \longrightarrow \langle h_{\omega}^{(1)} \rangle^{\mathbb{Z}/p} \times \cdots \times \langle h_{\omega}^{(N)} \rangle^{\mathbb{Z}/p} \longrightarrow 1\,,$$

the corresponding exact sequence of Lie $\mathbb{F}_p$-algebras (here $\Gamma_\omega = G(L_\omega)$)

$$0 \longrightarrow \bar{\mathcal{L}} \longrightarrow L_\omega \longrightarrow \prod_{1\leqslant m\leqslant N} \mathbb{F}_p h_\omega^{(m)} \longrightarrow 0\,,$$

and define the appropriate dense subalgebra $L_\omega^{\mathcal{P}}$ such that

$$0 \longrightarrow \bar{\mathcal{L}}^{\mathcal{P}} \longrightarrow L_\omega^{\mathcal{P}} \longrightarrow \prod_{1\leqslant m\leqslant N} \mathbb{F}_p l_\omega^{(m)} \longrightarrow 0\,.$$

f) We apply methods from [11, 12] to describe the structure of the Lie algebras $\bar{\mathcal{L}}^{\mathcal{P}} \otimes k$ and $L_\omega^{\mathcal{P}} \otimes k$. In particular, for $1 \leqslant m \leqslant N$, we obtain a recurrent procedure to recover the operators $\mathrm{ad}\,\bar{l}_\omega^{(m)}$, where $\bar{l}_\omega^{(m)}$ are lifts of $l_\omega^{(m)}$ to L_ω, and find explicit formula for $[\bar{l}_\omega^{(m_1)}, \bar{l}_\omega^{(m_2)}] \in \bar{\mathcal{L}}$. These results are illustrated via explicit description of the structure of the Lie algebra $L_\omega^{\mathcal{P}}$ modulo the ideal of third commutators.

g) We apply the results from f) to the explicit description of $\Gamma_{<p} = \Gamma/\Gamma^p C_p(\Gamma)$, where Γ is the Galois group of N-dimensional local field K containing a non-trivial p-th root of unity ζ_1. More precisely, we introduce a canonical class of conjugated dense subgroups $\Gamma_{<p}^{\mathcal{P}}$ in $\Gamma_{<p}$ with $\mathcal{P}$-topological systems of generators. Then we apply Scholl's construction of the field-of-norms functor to identify $\Gamma_{<p}^{\mathcal{P}}$ with $\Gamma_\omega^{\mathcal{P}}$, where $\omega \in \mathrm{m}_K$ is defined in terms related to the p-th root of unity ζ_1. This result is illustrated in the case where $K = \mathbb{Q}_p(\zeta_1)\{\{x\}\}$.

0.4. **Final remarks.** a) In 1-dimensional case the Demushkin relation depends only on the subgroup $\mu(K)$ of roots of unity in K^* and the degree $[K:\mathbb{Q}_p]$. In the general case the structure of $\Gamma_{<p}$ depends only on a special power series constructed from $\zeta_1 \in K$; this series appears in the p-adic Hodge theory as the period of $\mathbb{G}_m$. In particular, the group structure on $\Gamma_{<p}$ is a very weak invariant of the field K.

b) In 1-dimensional case $\Gamma_K(p)$ (as well as Γ_K) has very important additional structure given by the decreasing filtration of ramification subgroups $\Gamma_K(p)^{(v)}$, $v \geqslant 1$. According to [8] the group $\Gamma_K(p)$ together with the additional structure given by the ramification filtration is an absolute invariant of K, cf. also [27, 4] in the context of the whole group Γ_K. The papers [11, 12] contain the description of the group structure of $\Gamma_{<p}$ together with the induced ramification filtration.

c) It would be natural to assume that most interesting structures on $\Gamma_{<p}$ appear as completions of structures defined in terms of the subgroups $\Gamma_{<p}^{\mathcal{P}}$. (In particular, we see that the group structure on $\Gamma_{<p}$ is induced from $\Gamma_{<p}^{\mathcal{P}}$.) As a result, such structures can be studied and described in terms of generators and relations. In particular, it will be natural to expect that the ramification subgroups introduced for higher local fields in [38, 5, 6] satisfy this assumption. In particular, in the case of 2-dimensional higher local field $\mathcal{K}$ of characteristic p the

description of ramification subgroups in $\Gamma_{<p}$ modulo subgroup of third commutators from [6] satisfies this assumption.

d) Recently we found more substantial and natural way to study the ramification filtration of $\Gamma_{<p}$ in the 1-dimensional case, cf. [14]. We expect that the techniques of generators and relations provided by this paper will allow us to develop more substantial and clear approach to the proof of the local analog of the Grothendieck conjecture for all higher local fields.

e) There is still an open question in the description of $\Gamma_{<p}$: we have not yet found explicitly the commutators $[\bar{l}_\omega^{(m_1)}, \bar{l}_\omega^{(m_2)}]$. There is a strong evidence that there are lifts $\bar{l}_\omega^{(m)}$ which commute one with another: we verified this fact modulo $C_4(\Gamma_\omega)$ by direct computation. The existence of commuting elements in sufficiently large Galois groups may have some relation to anabelian geometry, cf. [17].

f) Notice the paper [35] where the case of the Galois group of 2-dimensional fields with the first residue field of CHARACTERISTIC 0 was considered. We are not considering here such fields, but this result is not very far from the Demushkin one. The Galois group here appears as a profinite group with finitely many generators and one relation, i.e. is a group of Poincare type.

0.5. **Notation.** Let G be a topological group. For $s \geqslant 1$, denote by $C_s(G)$ the closure of its subgroup of s-th commutators. Here $C_1(G) = G$ and for $s \geqslant 2$, $C_s(G)$ is the closure of the commutator subgroup $(G, C_{s-1}(G))$. Similarly, if L is a (topological) Lie algebra over some ring R then $C_s(L)$ is the closure of its R-submodule of commutators of order $\geqslant s$. If M and S are R-modules we denote very often by M_S the extension of scalars $M \otimes_R S$.

1. Constructions of nilpotent Artin-Schreier theory

In this section we review basic results of nilpotent Artin-Schreier theory, cf. [1, 2]. This theory allows us to work with p-extensions of fields of characteristic p with Galois groups of nilpotent class $< p$. In these notes we use the simplest case of the theory involving Galois groups of period p. In other words, if $\mathrm{char} K = p$ and $\Gamma = \mathrm{Gal}(K_{sep}/K)$ our approach allows us to work efficiently with subfields of $K_{<p} := K_{sep}^{\Gamma_{<p}}$, where $\Gamma_{<p} = \Gamma/\Gamma^p C_p(\Gamma)$.

1.1. **Groups and Lie algebras of nilpotent class $< p$.** The basic ingredient of the nilpotent Artin-Schreier theory is the equivalence of the category of p-groups of nilpotent class $s_0 < p$ and the category of Lie $\mathbb{Z}_p$-algebras of the same nilpotent class. In the case of objects killed by p this equivalence can be explained as follows.

Let L be a Lie $\mathbb{F}_p$-algebra of nilpotent class $< p$, i.e. $C_p(L) = 0$.

Let $\mathfrak{A}$ be an enveloping algebra of L. Then there is a natural embedding $L \subset \mathfrak{A}$, the elements of L generate the augmentation ideal J of $\mathfrak{A}$ and we have a morphism of algebras $\Delta : \mathfrak{A} \longrightarrow \mathfrak{A} \otimes \mathfrak{A}$ uniquely determined by the conditions $\Delta(l) = l \otimes 1 + 1 \otimes l$ for all $l \in L$. The Poincare-Birkhoff-Witt Theorem then implies:

— $L \cap J^p = 0$;

— $L \bmod J^p = \{a \bmod J^p \mid \Delta(a) \equiv a \otimes 1 + 1 \otimes a \bmod (J \otimes 1 + 1 \otimes J)^p\}$;

— the set $\widetilde{\exp}(L) \bmod J^p$ is identified with the set of all "diagonal elements" $\bmod \deg p$, i.e. with the set of all $a \in 1 + J \bmod J^p$ such that $\Delta(a) \equiv a \otimes a \bmod (J \otimes 1 + 1 \otimes J)^p$ (here $\widetilde{\exp}(x) = \sum_{0 \leqslant i < p} x^i/i!$ is the truncated exponential).

In particular, there is a natural embedding $L \subset \mathfrak{A} \bmod J^p$ and in terms of this embedding the Campbell-Hausdorff formula appears as

$$(l_1, l_2) \mapsto l_1 \circ l_2 = l_1 + l_2 + \frac{1}{2}[l_1, l_2] + \dots, \quad l_1, l_2 \in L,$$

where $\widetilde{\exp}(l_1)\,\widetilde{\exp}(l_2) \equiv \widetilde{\exp}(l_1 \circ l_2) \bmod J^p$. This composition law provides the set L with a group structure and we denote this group by $G(L)$. The group $G(L)$ has period p and nilpotent class $< p$. The correspondence $L \mapsto G(L)$ induces the equivalence of the category of p-groups of period p and nilpotent class $s < p$ and the category of Lie $\mathbb{Z}/p$-algebras of the same nilpotent class s. This equivalence is naturally extended to the similar categories of pro-finite Lie algebras and pro-finite p-groups.

1.2. **Nilpotent Artin-Schreier theory.** Let L be a finite Lie $\mathbb{F}_p$-algebra of nilpotent class $< p$. Consider the extensions of scalars L_K and $L_{sep} := L_{K_{sep}}$. Then the elements of $\Gamma = \mathrm{Gal}(K_{sep}/K)$ and the Frobenius σ act on L_{sep} through the second factor, $L_{sep}|_{\sigma=\mathrm{id}} = L$ and $(L_{sep})^\Gamma = L_K$. If $e \in G(L_K)$ then the set

$$\mathcal{F}(e) = \{f \in G(L_{sep}) \mid \sigma(f) = e \circ f\}$$

is not empty and for any fixed $f \in \mathcal{F}(e)$, the map $\tau \mapsto (-f) \circ \tau(f)$ is a continuous group homomorphism $\pi_f(e) : \Gamma \longrightarrow G(L)$. The correspondence $e \mapsto \pi_f(e)$ has the following properties:

a) if $f' \in \mathcal{F}(e)$ then $f' = f \circ l$, where $l \in G(L)$; in particular, $\pi_f(e)$ and $\pi_{f'}(e)$ are conjugated via l;

b) for any continuous group homomorphism $\pi : \Gamma \longrightarrow G(L)$, there are $e \in G(L_K)$ and $f \in \mathcal{F}(e)$ such that $\pi_f(e) = \pi$;

c) for appropriate elements $e, e' \in G(L_K)$, $f \in \mathcal{F}(e)$ and $f' \in \mathcal{F}(e')$, we have $\pi_f(e) = \pi_{f'}(e')$ iff there is an $x \in G(L_K)$ such that $f' = x \circ f$ and, therefore, $e' = \sigma(x) \circ e \circ (-x)$.

In the case of a profinite Lie algebra $L = \varprojlim_\alpha L_\alpha$, where all L_α are finite Lie $\mathbb{F}_p$-algebras, consider $e = \varprojlim_\alpha e_\alpha \in L_K$, where all $e_\alpha \in L_{\alpha K}$. Then there is $f = \varprojlim_\alpha f_\alpha \in \varprojlim_\alpha \mathcal{F}(e_\alpha) \subset L_{sep}$ (where all $f_\alpha \in \mathcal{F}(e_\alpha)$) and $\pi_f(e) = \varprojlim_\alpha \pi_{f_\alpha}(e_\alpha)$ maps Γ to $G(L) = \varprojlim_\alpha G(L_\alpha)$.

1.3. **The diagonal element and abelian Artin-Schreier theory.** Let $\bar{K} = K/(\sigma - \mathrm{id})K$ and $M = \mathrm{Hom}_{\mathbb{F}_p\text{-lin}}(\bar{K}, \mathbb{F}_p)$. If $\bar{K}$ is provided with discrete topology (as an inductive limit of finite dimensional $\mathbb{F}_p$-subspaces), its dual M has the pro-finite topology and

$$M_{\bar{K}} = \mathrm{Hom}_{\mathbb{F}_p\text{-lin}}(\bar{K}, \bar{K}) \,.$$

Let $\Pi : K \longrightarrow \bar{K}$ be the natural projection and $e \in M_K = \mathrm{Hom}(\bar{K}, K)$ be such that $(\mathrm{id}_M \otimes \Pi)e = \mathrm{id}_{\bar{K}}$. Equivalently, let S be a section of Π and $e := e_S = (\mathrm{id}_M \otimes S)\mathrm{id}_{\bar{K}}$.

In notation of Sect. 1.2 the identification of the abelian Artin-Schreier theory $\pi^{ab} : \Gamma_{<2} := \Gamma/\Gamma^p C_2(\Gamma) \simeq M$ can be obtained as follows:

— choose $f \in M_{sep} := M_{K_{sep}}$ such that $\sigma f - f = e_S$ and for any $\tau \in \Gamma_{<2}$, let $\tau f - f = \pi^{ab}(\tau) \in M_{sep}|_{\sigma = \mathrm{id}} = M$.

Remark. In the above formulas (and in similar situations below if there is no risk of confusion) we use the simpler notation σ and τ instead of $\mathrm{id}_M \otimes \sigma$ and $\mathrm{id}_{M_{sep}} \otimes \tau$.

The map π^{ab} does not depend on a choice of f:

— if $f_1 \in M_{sep}$ is such that $\sigma f_1 - f_1 = e_S$ then $f_1 - f \in M_{sep}|_{\sigma=\mathrm{id}} = M$ and $\tau f_1 - f_1 = \tau f - f$.

The map π^{ab} also does not depend on a choice of S:

— if S' is another section then there is $g \in M_K$ such that $e_{S'} - e_S = \sigma g - g$. Therefore, $f' := f + g$ satisfies the relation $\sigma f' - f' = e_{S'}$ and, $\tau f' - f' = \tau f - f$.

1.4. **Identifications** $\pi_f(e) : \Gamma_{<p} \simeq G(L)$**.**

Let $\widetilde{L}$ be a free Lie $\mathbb{F}_p$-algebra with generating module M and $L = \widetilde{L}/C_p(\widetilde{L})$. (Note that M is a profinite limit of its finite quotients and $\widetilde{L}$ is the corresponding profinite limit of finite Lie algebras.) Consider the natural projection

$$\mathrm{pr} \otimes \Pi : L \otimes_{\mathbb{F}_p} K \longrightarrow L/C_2(L) \otimes_{\mathbb{F}_p} \bar{K} = M_{\bar{K}}$$

and set $\mathcal{E}(L_K) = \{e \in L_K \mid (\mathrm{pr} \otimes \Pi)e = \mathrm{id}_{\bar{K}}\}$. Agree to denote the image of e in M_K by e_S, where S is the appropriate section of Π, cf. Sect. 1.3.

Choose $f \in \mathcal{F}(e)$ and consider the group homomorphism $\pi_f(e) : \Gamma_{<p} \longrightarrow G(L)$ such that for any $\tau \in \Gamma_{<p}$, $\pi_f(e)(\tau) = (-f) \circ \tau(f)$.

Then $\pi_f(e)$ is a group isomorphism (use that Γ is a free pro-p-group and $\pi_f(e)\,\mathrm{mod}\Gamma^pC_2(\Gamma)$ is isomorphism by Sect. 1.3).

If f' is another element from $\mathcal{F}(e)$ then there is $l\in G(L)$ such that $f'=f\circ l$ and $\pi_{f'}(e)(\tau)=(-f')\circ\tau(f')=(-l)\circ\pi_f(e)(\tau)\circ l$ is conjugated to $\pi_f(e)$. Study how $\pi_f(e)$ depends on a choice of $e\in\mathcal{E}(L_K)$.

Proposition 1.1. *If $e,e'\in\mathcal{E}(L_K)$ then there is $x\in L_K$ and a section A of the natural projection* $\mathrm{pr}:L\longrightarrow L/C_2(L)=M$ *such that*

$$e'=\sigma(x)\circ(\mathcal{A}\otimes\mathrm{id}_K)e\circ(-x)\,,$$

where $\mathcal{A}\in\mathrm{Aut}_{Lie}L$ is a unique extension of A.

Proof. Let $\{l_\alpha\mid\alpha\in\mathcal{I}\}$ be an $\mathbb{F}_p$-basis of $\bar{K}$. Let $\hat{l}_\alpha$, $\alpha\in\mathcal{I}$, be the dual (topological) basis for M, i.e. for any $\alpha_1,\alpha_2\in\mathcal{I}$, $\hat{l}_{\alpha_1}(l_{\alpha_2})=\delta_{\alpha_1\alpha_2}$.

Then we have the sections S and S' of Π such that $e_S=\sum\limits_\alpha\hat{l}_\alpha\hat{\otimes}S(l_\alpha)$ and $e_{S'}=\sum\limits_\alpha\hat{l}_\alpha\hat{\otimes}S'(l_\alpha)$.

Apply induction on $r\geqslant 1$ to prove the existence of $x_r\in L_K$ and a section A_r of the projection $L\longrightarrow M$ such that

$$e'\equiv\sigma(x_r)\circ(\mathcal{A}_r\otimes\mathrm{id}_K)e\circ(-x_r)\,\mathrm{mod}\,C_{r+1}(L_K)\,,$$

where $\mathcal{A}_r\in\mathrm{Aut}_{\mathrm{Lie}}L$ is such that $\mathcal{A}_r|_M=A_r$.

If $r=1$ take $A_1=\mathrm{id}_M$ and $x_1=\sum\limits_\alpha\hat{l}_\alpha\otimes x_{1\alpha}$, where all $x_{1\alpha}\in K$ are such that $S'(l_\alpha)-S(l_\alpha)=\sigma(x_{1\alpha})-x_{1\alpha}$.

If $r\geqslant 1$ and the required x_r and A_r exist then there is $l_{r+1}\in C_{r+1}(L_K)$ such that $e'\equiv\sigma x_r\circ(\mathcal{A}_r\otimes\mathrm{id}_K)e\circ(-x_r)\circ l_{r+1}\,\mathrm{mod}\,C_{r+2}(L_K)$.

Using that $K=\mathrm{Im}(S)\oplus(\sigma-\mathrm{id})K$ we can present l_{r+1} as

$$l_{r+1}=l'+\sigma x'-x'$$

where $l'=\sum\limits_\alpha c_\alpha\otimes S(l_\alpha)$, all $c_\alpha\in C_{r+1}(L)$ and $x'\in C_{r+1}(L_K)$. It remains to set $A_{r+1}(\hat{l}_\alpha)=A_r(\hat{l}_\alpha)+c_\alpha$ and $x_{r+1}=x_r+x'$. The proposition is proved. □

Corollary 1.2. *With above notation there is $f'\in\mathcal{F}(e')$ such that for any $\tau\in\Gamma_{<p}$, $\pi_{f'}(e')(\tau)=\mathcal{A}(\pi_f(e)(\tau))$.*

Proof. Let $f'=x\circ(\mathcal{A}\otimes\mathrm{id}_{sep})f$, then $f'\in\mathcal{F}(e')$. Indeed,

$$\sigma(f')=\sigma x\circ(\mathcal{A}\otimes\mathrm{id}_{sep})\sigma(f)=\sigma(x)\circ(\mathcal{A}\otimes\mathrm{id}_K)e_S\circ(\mathcal{A}\otimes\mathrm{id}_{sep})f$$

$$=\sigma(x)\circ(\mathcal{A}\otimes\mathrm{id}_K)e\circ(-x)\circ f'=e'\circ f'\,.$$

Therefore, for any $\tau\in\Gamma_{<p}$, $\pi_{f'}(e')(\tau)$ is equal to

$$(-f')\circ\tau(f')=(\mathcal{A}\otimes\mathrm{id}_{sep})((-f)\circ\tau(f))=\mathcal{A}(\pi_f(e)(\tau))\,.$$

□

By the above corollary, a choice of $e\in\mathcal{E}(L_K)$ determines the class π_e of conjugated identifications $\{\pi_f(e)\mid f\in\mathcal{F}(e)\}$ of $\Gamma_{<p}$ with $G(L)$. When e is replaced by another $e'\in\mathcal{E}(L_K)$ the new class of conjugated identifications $\pi_{e'}$ is obtained from π_e via the composition with some automorphism $\mathcal{A}=\mathcal{A}(e,e')\in\mathrm{Aut}_{\mathrm{Lie}}(L)$ such that $\mathcal{A}\equiv\mathrm{id}_L \bmod C_2(L)$.

1.5. **Compatibility with field extensions.** Suppose K' is a field extension of K in K_{sep}. Consider the above defined objects: M, L, $e\in\mathcal{E}(L_K)$, $f\in\mathcal{F}(e)$ and $\pi=\pi_f(e):\Gamma_{<p}\simeq G(L)$ introduced in the context of the field K. Let $\Gamma'_{<p}$, M', L', $e'\in\mathcal{E}(L'_{K'})$, $f'\in\mathcal{F}(e')$ and $\pi':\Gamma'_{<p}\simeq G(L')$ be the similar objects for the field K'.

The embedding $\mathrm{Gal}(K_{sep}/K')\longrightarrow\mathrm{Gal}(K_{sep}/K)$ induces a natural group homomorphism $\Theta:\Gamma'_{<p}\longrightarrow\Gamma_{<p}$, which can be described in terms of the identifications π and π' as follows.

Consider $e\otimes_K 1\in L_K\otimes_K K'=L_{K'}\supset M_{K'}=\mathrm{Hom}(\bar K,K')$.

Proposition 1.3. *There is a morphism of Lie algebras $\mathcal{A}:L'\longrightarrow L$ and $x\in L_{K'}$ such that*

a) $e\otimes_K 1=\sigma(x')\circ(\mathcal{A}\otimes\mathrm{id}_{\mathcal{K}})e'\circ(-x')$;

b) *for any* $\tau'\in\Gamma'_{<p}$, $\pi(\Theta(\tau'))=\mathcal{A}(\pi'(\tau'))$;

c) *if* $K'\subset K_{<p}$ *then* $\pi(\mathrm{Gal}(K_{<p}/K'))=\mathcal{A}(L')$.

Proof. Let $\{l'_\alpha\mid\alpha\in\mathcal{I}'\}$ be an $\mathbb{F}_p$-basis of $\bar K'=K'/(\sigma-\mathrm{id})K'$. Let $\hat l'_\alpha$, $\alpha\in\mathcal{I}'$, be the dual (topological) basis for M'. Then for a suitable section S' of $\Pi':K'\longrightarrow\bar K'$, we have $e_{S'}=\sum\limits_\alpha\hat l'_\alpha\otimes S'(l'_\alpha)$ and $\{S'(l'_\alpha)|\ \alpha\in\mathcal{I}'\}$ is a basis of $\mathrm{Im}(S')\subset K'$. Proceeding similarly to the proof of Prop.1.1 prove the existence of $x'\in L_{K'}$ and $\widetilde{l}_\alpha\in L$ such that

$$e\otimes_K 1=\sigma(x')\circ\left(\sum_\alpha\widetilde{l}_\alpha\otimes S'(l'_\alpha)\right)\circ(-x')\,.$$

If $A':M'\longrightarrow L$ is a linear map such that for all α, it holds $A'(\hat l'_\alpha)=\widetilde{l}_\alpha$, the above relation appears in the following form

$$e\otimes_K 1=\sigma(x')\circ(\mathcal{A}'\otimes\mathrm{id}_{K'})e'\circ(-x')\,,$$

where $\mathcal{A}'$ is a unique morphism of Lie algebras $L'\longrightarrow L$ such that $\mathcal{A}'|_{M'}=A'$. As a result, the both $(-x')\circ f$ and $(\mathcal{A}'\otimes\mathrm{id}_{sep})f'$ belong to $\mathcal{F}((\mathcal{A}'\otimes\mathrm{id}_{\mathcal{K}'})e')\subset L_{sep}$. So, there is $l\in L$ such that

$$(-x')\circ(f\otimes_K 1)=(\mathcal{A}'\otimes\mathrm{id}_{sep})f'\circ l\,.$$

If $x=x'\circ l$ and $\mathcal{A}=\mathrm{Ad}\,l\cdot\mathcal{A}'\in\mathrm{Hom}_{\mathrm{Lie}}(L',L)$ then the above equality can be rewritten as

$$(-x)\circ(f\otimes_K 1)=(\mathcal{A}\otimes\mathrm{id}_{sep})f'\,.$$

In particular, we have $e\otimes_K 1=\sigma(x)\circ(\mathcal{A}\otimes\mathrm{id}_{K'})e'\circ(-x)$ and for any $\tau'\in\Gamma'_{<p}$, it holds $\pi(\Theta(\tau'))=(-f)\circ\tau'(f)=(\mathcal{A}\otimes\mathrm{id}_{K'})((-f')\circ\tau'(f'))=\mathcal{A}(\pi'(\tau'))$. The proposition is proved. □

1.6. **Lifts of $\phi\in\mathrm{Aut}K$.**

As earlier, $e\in\mathcal{E}(L_K)$, $f\in\mathcal{F}(e)$, $\pi=\pi_f(e):\Gamma_{<p}\simeq G(L)$.

Suppose $\phi\in\mathrm{Aut}K$. We are going to describe (the lifts) $\phi_{<p}\in\mathrm{Aut}K_{<p}$ such that $\phi_{<p}|_K=\phi$.

Let $\phi_*e:=(\mathrm{id}\otimes\phi)e\in L_K$. As earlier, for any given $\phi_{<p}$, establish the existence of $\mathcal{A}=\mathcal{A}(\phi_{<p})\in\mathrm{Aut}_{\mathrm{Lie}}L$ and $C=C(\phi_{<p})\in L_K$ such that

$$\phi_*e=\sigma(C)\circ(\mathcal{A}\otimes\mathrm{id}_K)e\circ(-C)\,. \tag{1.1}$$

Let $\mathfrak{M}(\phi)$ be the set of all pairs $(C,\mathcal{A})$ satisfying (1.1). Let

$$\kappa:\mathfrak{M}(\phi)\longrightarrow\{\phi_{<p}\in\mathrm{Aut}K_{<p}\mid\phi_{<p}|_K=\phi\}$$

be the map defined as follows.

If $(C,\mathcal{A})\in\mathfrak{M}(\phi)$ then $g:=C\circ(\mathcal{A}\otimes\mathrm{id})f\in\mathcal{F}(\phi_*e)$. If $\phi'_{<p}$ is a lift of ϕ then $(\mathrm{id}\otimes\phi'_{<p})f\in\mathcal{F}(\phi_*e)$. Then for some $l\in L$,

$$g=(\mathrm{id}\otimes\phi'_{<p})(f\circ l)=(\mathrm{id}\otimes\phi'_{<p})(\mathrm{id}\otimes\pi^{-1}l)f=(\mathrm{id}\otimes(\phi'_{<p}\cdot\pi^{-1}l))f$$

and the composition $\phi_{<p}:=\phi'_{<p}\cdot\pi^{-1}l$ is a lift of ϕ. It is easy to see that the lift $\phi_{<p}$ does not depend on the above choice of $\phi'_{<p}$. As a result, we can set $\kappa(C,\mathcal{A})=\phi_{<p}$.

Proposition 1.4.

a) *If $\kappa(C,\mathcal{A})=\phi_{<p}$ then for any $\tau\in\Gamma_{<p}$, $\pi(\mathrm{Ad}(\phi_{<p})\tau)=\mathcal{A}(\pi(\tau))$.*

b) *The map κ is a bijection.*

Proof. With above notation $\pi_g(\phi_*e)(\tau)=(-g)\circ\tau(g)=\mathcal{A}(\pi(\tau))$. On the other hand, $g=(\mathrm{id}_L\otimes\phi_{<p})f$ implies that

$$\pi_g(\phi_*e)(\tau)=\phi_{<p}\left((-f)\circ\phi_{<p}^{-1}\tau\phi_{<p}(f)\right)=\pi(\mathrm{Ad}(\phi_{<p})\tau)\,.$$

The part b) is implied by the following three facts:

b_1) *the map κ is injective.*

Indeed, let $\kappa(C_1,\mathcal{A}_1)=\kappa(C_2,\mathcal{A}_2)$. Then

$$C_1\circ(\mathcal{A}_1\otimes\mathrm{id})f=C_2\circ(\mathcal{A}_2\otimes\mathrm{id})f\,.$$

This implies that $(\mathcal{A}_1^{-1}\otimes\mathrm{id})((-C_2)\circ C_1)\circ f=(\mathcal{A}_1^{-1}\mathcal{A}_2\otimes\mathrm{id})f$ and, therefore, for any $\tau\in\Gamma_{<p}$, it holds $\pi(\tau)=(\mathcal{A}_1^{-1}\mathcal{A}_2)(\pi\tau)$ (use that $(\mathcal{A}_1^{-1}\otimes\mathrm{id})((-C_2)\circ C_1)\in L_K$). As a result, $\mathcal{A}_1^{-1}\mathcal{A}_2=\mathrm{id}_L$ and $(C_1,\mathcal{A}_1)=(C_2,\mathcal{A}_2)$.

b_2) *$\{\phi_{<p}\mid\phi_{<p}|_K=\phi\}$ is a principal homogeneous space over $\Gamma_{<p}$ with respect to the action $\phi_{<p}\mapsto\phi_{<p}\cdot\tau$, $\tau\in\Gamma_{<p}$;*

b_3) *the appropriate action of* $\tau \in \Gamma_{<p}$ *on the pair* $(C, \mathcal{A})$ *appears in the form* $(C, \mathcal{A}) \mapsto (C', \mathcal{A}')$, *where for* $l_\tau := \pi(e)\tau$, $C' = C \circ l_\tau$ *and for any* $l \in L$, $\mathcal{A}'(l) = (-l_\tau) \circ \mathcal{A}(l) \circ l_\tau = (\mathrm{Ad} l_\tau \cdot \mathcal{A})(l)$.

The proof of b_2) and b_3) is straightforward. For more details cf. [13]. $\square$

The above formalism allows us to use the identification $\pi = \pi_f(e)$ to work with the group of all lifts $\phi_{<p} \in \mathrm{Aut}\, K_{<p}$ of automorphisms $\phi \in \mathrm{Aut}\, K$. Note that:

– if $(C', \mathcal{A}')$ and $C'', \mathcal{A}'')$ correspond to the lifts $\phi'_{<p}$ and, $\phi''_{<p}$ of, resp., ϕ' and ϕ'', then the couple $(C' \circ (\mathcal{A}' \otimes \mathrm{id}_K) C'', \mathcal{A}'\mathcal{A}'')$ corresponds to the lift $\phi'_{<p}\phi''_{<p}$ of $\phi'\phi''$;

– the elements $\pi(\tau) = l_\tau \in G(\mathcal{L})$ appear as a special case of a lift of id_K and correspond to the pairs $(l_\tau, \mathrm{Ad}\, l_\tau)$, where $\mathrm{Ad}\, l_\tau : l \mapsto (-l_\tau) \circ l \circ l_\tau$.

2. Higher local fields and $\mathcal{P}$-topology

2.1. **Higher local fields.** Let K be an N-dimensional local field, i.e.
– if $N = 0$ then K is finite;
– if $N \geqslant 1$ then K is a complete discrete valuation field such that its residue field is $(N-1)$-dimensional.

If $N \geqslant 1$ then the residue field of K is the FIRST RESIDUE FIELD of K. It will be usually denoted by $K^{(1)}$. The corresponding valuation ring $O_K^{(1)}$ is the FIRST VALUATION RING. We agree to set by induction for all $1 < m \leqslant N$, $K^{(m)} = K^{(m-1)(1)}$ — this is the m-TH RESIDUE FIELD of K. Note that $K^{(N)}$ is 0-dimensional and, therefore, finite.

Define the N-VALUATION RING $\mathcal{O}_K$ of K by induction on N as follows. If $N = 0$ set $\mathcal{O}_K = K$. If $N \geqslant 1$ and $\mathrm{pr} : O_K^{(1)} \longrightarrow K^{(1)}$ is the natural projection then set $\mathcal{O}_K = \mathrm{pr}^{-1}\mathcal{O}_{K^{(1)}}$.

If $N \geqslant 1$ then $\pi := \{\pi_1, \dots, \pi_N\}$ is a SYSTEM OF LOCAL PARAMETERS in K if:
– π_1 is (the first) uniformizer in K;
– $\pi_2, \dots, \pi_N \in O_K^{(1)}$ and their projections $\bar\pi_2, \dots, \bar\pi_N$ to $K^{(1)}$ form a system of local parameters for $K^{(1)}$.

If $[E : K] < \infty$ then the structure of N-dimensional field on K is uniquely extended to E and vice versa.

If $\mathrm{char}\, K = p \neq 0$ and $\pi = \{\pi_1, \dots, \pi_N\}$ is a system of local parameters in K then K appears as a field of iterated formal Laurent series $K = k((\pi_N)) \dots ((\pi_1))$, where $k = K^{(N)} \simeq \mathbb{F}_q$ with $q = p^{N_0}$. This is a part of the classification result, [36]. In this case $K^{(1)}$ is identified with the subfield $k((\pi_N)) \dots ((\pi_2))$ of K. More formally, there is a system of local parameters $\bar\pi := \{\bar\pi_2, \dots, \bar\pi_N\}$ in $K^{(1)}$ such that their lifts to

K form a subset $\{\pi_2, \dots, \pi_N\}$ of π. We use the notation $\iota_{\bar{\pi}}$ for the corresponding embedding of $K^{(1)}$ into K.

If char$K = 0$ we always assume that char$K^{(1)} = p > 0$ – such fields have most interesting arithmetical structure. The appropriate classification result for such fields K can be presented as follows.

Let $K^{(1)} = k((\bar{\pi}_N)) \dots ((\bar{\pi}_2))$ where $\bar{\pi} := \{\bar{\pi}_2, \dots, \bar{\pi}_N\}$ are local parameters for $K^{(1)}$. The elements $\bar{\pi}_2, \dots, \bar{\pi}_N$ form a p-basis in $K^{(1)}$, [16]. We can use this p-basis to construct an (absolutely unramified) lift $K^{(1)}_{\bar{\pi}}$ of $K^{(1)}$ to characteristic 0, [16]. Recall that $K^{(1)}_{\bar{\pi}}$ is the fraction field of the ring $\varprojlim_{m\in\mathbb{N}} O_m(K^{(1)})$, where

$$O_m(K^{(1)}) = W_m(\sigma^{m-1}K^{(1)})[\pi_2, \dots, \pi_N] \subset W_m(K^{(1)})$$

are the lifts of $K^{(1)}$ modulo p^m. (Here $\pi_2, \dots, \pi_N$ are the Teichmuller representatives of $\bar{\pi}_2, \dots, \bar{\pi}_N$.) The field $K^{(1)}_{\bar{\pi}}$ has a natural structure of N-dimensional local field of characteristic 0 with local parameters $\{p, \pi_2, \dots, \pi_N\}$. Now the classification result from [36] states

K is a finite field extension of $K^{(1)}_{\bar{\pi}}$.

In particular, we obtain an analogue $\iota_{\bar{\pi}} : K^{(1)}_{\bar{\pi}} \longrightarrow K$ of the above defined embedding $K^{(1)} \longrightarrow K$ in the characteristic p case.

Note also that,

— there is $\pi_1 \in K$ such that $\{\pi_1, \pi_2, \dots, \pi_N\}$ is a system of local parameters in K;

— the field K contains a (unramified 1-dimensional) local field $F_{ur} = \operatorname{Frac} W(k)$;

— the classification result from [36] states also the existence of a finite totally ramified extension F' of F_{ur} such that $K \subset F'K^{(1)}_{\bar{\pi}}$.

2.2. Definition and basic properties of $\mathcal{P}$-topology.

The topology on N-dimensional local field K (we refer to it as the $\mathcal{P}$-topology) can be introduced as follows, [29, 36, 26].

If $N = 0$ then the $\mathcal{P}$-topology on K is discrete.

Suppose $N \geqslant 1$ and $\pi = \{\pi_1, \dots, \pi_N\}$ is a system of local parameters in K. Then the $\mathcal{P}$-topology on K is introduced by induction on N via the following properties:

(1) any $\xi \in K$ can be uniquely presented as $\mathcal{P}$-convergent series

$$\xi = \sum_a [\alpha_{\bar{a}}]\pi_1^{a_1} \dots \pi_N^{a_N}$$

where the indices $a = (a_1, \dots, a_N) \in \mathbb{Z}^N$, all $\alpha_a \in k$, $[\alpha_a] = \alpha_a$ if char$K = p$ and $[\alpha_{\bar{a}}]$ are the Teichmuller representatives of α_a in $W(k) \subset K^{(1)}_{\bar{\pi}}$ if char$K = 0$;

(2) the $\mathcal{P}$-convergence property of ξ means the existence of integers $A_i(a_1, \dots a_{i-1}) \in \mathbb{Z}$ with $1 \leqslant i \leqslant N$, such that: *if* $\alpha_a \neq 0$ *then*

$$a_1 \geqslant A_1, a_2 \geqslant A_2(a_1), \dots, a_N \geqslant A_N(a_1, \dots, a_{N-1});$$

(3) if $\bar{\pi} := \{\bar{\pi}_2, \dots, \bar{\pi}_n\}$ is a system of local parameters for $K^{(1)}$ used to define the $\mathcal{P}$-topology on $K^{(1)}$ then $\iota_{\bar{\pi}}$ induces the map

$$s_\pi : \sum_{\bar{a}=(a_2,\dots,a_N)} [\alpha_a]\bar{\pi}_2^{a_2}\dots\bar{\pi}_N^{a_N} \mapsto \sum_{\bar{a}=(0,a_2,\dots,a_N)} [\alpha_a]\pi_2^{a_2}\dots\pi_N^{a_N}$$

which is a $\mathcal{P}$-continuous (set-theoretic) section $s : K^{(1)} \longrightarrow O_K^{(1)}$ of the natural projection $O_K^{(1)} \longrightarrow K^{(1)}$.

The above properties imply that:

— a) a base of $\mathcal{P}$-open subsets $\mathcal{U}_\pi(K)$ in K consists of the subsets $\sum_{b\in\mathbb{Z}} \pi_1^b s_\pi(U_b)$, where all $U_b \in \mathcal{U}_{\bar{\pi}}(K^{(1)})$ and for $b \gg 0$, $U_b = K^{(1)}$;

— b) a base $\mathcal{C}_\pi(K)$ of sequentially compact (closed) subsets in K consists of $\sum_{b\in\mathbb{Z}} \pi_1^b s_\pi(C_b)$ such that all $C_b \in \mathcal{C}_{\bar{\pi}}(K^{(1)})$ and for $b \ll 0$, $C_b = 0$;

— c) if $\pi' = \{\pi_1', \dots, \pi_N'\}$ is another system of local parameters for K then the appropriate analogs of the above properties a)-c) also hold (i.e. the concept of $\mathcal{P}$-topology does not depend on the original choice of local parameters in K);

— d) if $[E : K] = n$ and the identification of K-vector spaces $E = K^n$ is induced by a choice of some K-basis in E then $\{U^n \mid U \in \mathcal{U}_\pi(K)\}$ is a base of $\mathcal{P}$-open subsets in L; similarly, $\{C^n \mid C \in \mathcal{C}_\pi(K)\}$ is a base of sequentially $\mathcal{P}$-compact subsets in E;

— e) if $C_1, C_2 \subset \mathcal{C}_\pi(K)$ then C_1C_2 is also sequentially compact (i.e. there is $C \in \mathcal{C}_\pi(K)$ such that $C_1C_2 \subset C$);

— f) K is a $\mathcal{P}$-topological additive group but not a $\mathcal{P}$-topological field; however, $K = \varinjlim_{C\in\mathcal{C}_\pi(K)} C$ and the multiplication $C \times K \longrightarrow K$ is $\mathcal{P}$-continuous (i.e. for any $U \in \mathcal{U}_\pi(K)$ there is an $U' \in \mathcal{U}_\pi(K)$ such that $CU' \subset U$).

Note that the subset of K consisting of the series ξ from above item (1) satisfying the condition:

if $\alpha_a \neq 0$ *then* $a_1 \geqslant A_1, a_2 \geqslant A_2(a_1), \dots, a_N \geqslant A_N(a_1, \dots, a_{N-1})$ *(with a fixed choice of* $A_i(a_1, \dots, a_{i-1})$*,* $1 \leqslant i \leqslant N$*)*

is sequentially compact. The family of all such subsets (with a fixed choice of a system of local parameters $\pi = \{\pi_1, \dots, \pi_N\}$) forms the base $\mathcal{C}_\pi(K)$.

We can similarly describe the base $\mathcal{U}_\pi(K)$:

$U \in \mathcal{U}_\pi(K)$ iff there are $B_1, B_2(a_1), \dots, B_N(a_1, \dots, a_{N-1}) \in \mathbb{Z}$ (depending on U) such that $\xi \in U$ are characterized by the condition:

if $a_1 < B_1$, $a_2 < B_2(a_1)$, $\dots$, $a_N < B_N(a_1, \dots, a_{N-1})$ then $\alpha_a = 0$.

2.3. **$\mathcal{P}$-topology in characteristic p.**
Assume that $K = \mathcal{K}$ has characteristic p and has a system of local parameters $t = \{t_1, \dots, t_N\}$. We will use the simpler notation $\mathcal{U}(\mathcal{K})$ and $\mathcal{C}(\mathcal{K})$ instead of $\mathcal{U}_t(\mathcal{K})$ and $\mathcal{C}_t(\mathcal{K})$ when working with this fixed system of local parameters t. Note that all $C \in \mathcal{C}(\mathcal{K})$ and $U \in \mathcal{U}(\mathcal{K})$ are k-linear vector spaces (where $k = \mathcal{K}^{(N)}$) and their elements appear as (some) formal k-linear combinations of the monomials $t^a := t_1^{a_1} \dots t_N^{a_N}$, where all $a = (a_1, \dots, a_N) \in \mathbb{Z}^N$.

Let $\mathcal{I} = \mathcal{I}(\mathcal{K})$ and $\mathcal{J} = \mathcal{J}(\mathcal{K})$ be the sets of indices such that $\mathcal{C}(\mathcal{K}) = \{C_\alpha \mid \alpha \in \mathcal{I}\}$ and $\mathcal{U}(\mathcal{K}) = \{U_\beta \mid \beta \in \mathcal{J}\}$.

It is easy to see by induction on N that for any $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$,

$$\dim_k C_\alpha / C_\alpha \cap U_\beta < \infty \,. \tag{2.1}$$

Note that $C_\alpha / C_\alpha \cap U_\beta$ are provided with the k-bases consisting of the monomials $t^a \bmod C_\alpha \cap U_\beta$ such that $t^a \in C_\alpha \setminus U_\beta$. These bases are compatible with respect to different choices of α and β. Therefore, $\{t^a \mid t^a \in C_\alpha\}$ is a $\mathcal{P}$-topological k-basis in C_α, a base of $\mathcal{P}$-open neibourhoods in C_α consists of k-vector subspaces containing almost all elements of this basis and $C_\alpha = \varprojlim_\beta C_\alpha / C_\alpha \cap U_\beta$.

Let $\mathcal{N}_{\alpha\beta} = (C_\alpha / C_\alpha \cap U_\beta)^D$ be the dual k-vector space for $C_\alpha / C_\alpha \cap U_\beta$. Then $\dim_k \mathcal{N}_{\alpha\beta} < \infty$, $\mathcal{N}_{\alpha\beta}^D = C_\alpha / C_\alpha \cap U_\beta$ and the spaces $\mathcal{N}_{\alpha\beta}$ are provided with compatible (dual) k-bases

$$\{T_a \mid t^{-a} \in C_\alpha \setminus U_\beta\} \,,$$

where for any t^{-b} with $b \in \mathbb{Z}^N$, $T_a(t^{-b}) = \delta_{ab}$.
Note that $\mathcal{N}_\alpha^{\mathcal{P}} := \mathrm{Hom}_{\mathcal{P}\text{-cont}}(C_\alpha, k) = \varinjlim_\beta \mathrm{Hom}(\mathcal{N}_{\alpha\beta}^D, k) = \varinjlim_\beta \mathcal{N}_{\alpha\beta}$ has a k-basis $\{T_a \mid t^{-a} \in C_\alpha\}$. Therefore, C_α is the set of all formal k-linear combinations of the appropriate monomials t^a and we have a natural identification $C_\alpha = \mathcal{N}_\alpha^{\mathcal{P}\,D}$.

Consider the $\mathbb{F}_p$-vector spaces $\mathcal{E}_\alpha^{\mathcal{P}} := \mathrm{Hom}_{\mathcal{P}\text{-cont}}(C_\alpha, \mathbb{F}_p)$. Then

$$\mathcal{E}_\alpha^{\mathcal{P}} \otimes k = \mathrm{Hom}_{\mathbb{F}_p\text{-lin}, \mathcal{P}\text{-cont}}(C_\alpha, k) = \oplus_{n \in \mathbb{Z}/N_0} \mathcal{N}_\alpha^{\mathcal{P}} \otimes_k k^{(n)} \,,$$

where $k^{(n)}$ is the (twisted) k-module $k \otimes_{\sigma^n} k$. In this identification the Frobenius σ acts through the second factor on the left-hand side and shifts $\mathbb{Z}/N_0$-summands by $+1$ on the right-hand side.
In particular, the extensions of scalars $\mathcal{E}_\alpha^{\mathcal{P}} \otimes k$ have the k-bases

$$\{T_{an} = T_a \otimes 1^{(n)} \mid t^{-a} \in C_\alpha, n \in \mathbb{Z}/N_0\}$$

which are compatible on $\alpha \in \mathcal{I}$. So, $\{T_{an} \mid a \in \mathbb{Z}^N, n \in \mathbb{Z}/N_0\}$ is a topological basis for $\mathcal{E}^{\mathcal{P}} \otimes k$, where $\mathcal{E}^{\mathcal{P}} := \mathrm{Hom}_{\mathcal{P}\text{-cont}}(\mathcal{K}, \mathbb{F}_p) = \varprojlim_{\alpha} \mathcal{E}^{\mathcal{P}}_{\alpha}$.

Proposition 2.1. *Let* $\mathcal{E} := \mathrm{Hom}(\mathcal{K}, \mathbb{F}_p)$. *Then* $\mathcal{E} = \mathcal{E}^{\mathcal{P}\,DD}$ *(the double dual* $\mathbb{F}_p$*-vector space for* $\mathcal{E}^{\mathcal{P}}$*).*

Proof. We verify this on the level of extensions of scalars as follows:

$$\mathrm{Hom}(\mathcal{K}, \mathbb{F}_p) \otimes k = \varprojlim_{\alpha} \mathrm{Hom}(C_\alpha, \mathbb{F}_p) \otimes k =$$

$$\varprojlim_{\alpha} \bigoplus_{n \in \mathbb{Z}/N_0} \mathrm{Hom}_k(C_\alpha, k) \otimes_k k^{(n)} = \varprojlim_{\alpha} \bigoplus_{n \in \mathbb{Z}/N_0} \mathcal{N}_\alpha^{DD} \otimes_k k^{(n)}$$

$$= \varprojlim_{\alpha} \left(\bigoplus_{n \in \mathbb{Z}/N_0} \mathcal{N}_\alpha \otimes_k k^{(n)} \right)^{DD} = \varprojlim_{\alpha} (\mathcal{E}^{\mathcal{P}}_\alpha \otimes k)^{DD} = \mathcal{E}^{\mathcal{P}\,DD} \otimes k\,.$$

□

Corollary 2.2. *The vector space* $\mathrm{Hom}(\mathcal{K}, \mathbb{F}_p)$ *is the profinite completion of its subspace* $\mathrm{Hom}_{\mathcal{P}\text{-cont}}(\mathcal{K}, \mathbb{F}_p)$.

Proof. Use that if L is a $\mathbb{F}_p$-linear space then L^{DD} is canonically isomorphic to the profinite completion of L. We sketch briefly the proof of this fact extracted from [19].

Suppose $L^D = \varinjlim Y_\alpha$, where all Y_α are finite dimensional vector subspaces in L^D. Then $L^{DD} = \varprojlim Y_\alpha^D$. Note that $Y_\alpha \mapsto \mathrm{Ann}(Y_\alpha) \subset L$ is a bijection between the set of finite dimensional subspaces in L^D and the set of finite codimensional subspaces in L, and $Y_\alpha^D \simeq L/\mathrm{Ann}(Y_\alpha)$. Therefore, $L^{DD} \simeq \varprojlim L/Z_\alpha$ where $Z_\alpha = \mathrm{Ann} Y_\alpha$ runs over the set of all finite codimensional subspaces in L. □

3. The group $\mathcal{G}^{\mathcal{P}}_{<p}$

3.1. **Frobenius and $\mathcal{P}$-topology.**

Let $\mathcal{K}$ be an N-dimensional local field of characteristic p. The quotient $\bar{\mathcal{K}} = \mathcal{K}/(\sigma - \mathrm{id})\mathcal{K}$ can be provided with the induced $\mathcal{P}$-topological structure such that the projection $\Pi : \mathcal{K} \longrightarrow \bar{\mathcal{K}}$ is open. Choose a system of local parameters $t = \{t_1, \dots, t_N\}$ in $\mathcal{K}$ and let $\mathcal{C}(\mathcal{K}) = \{C_\alpha \mid \alpha \in \mathcal{I}\}$ and $\mathcal{U}(\mathcal{K}) = \{U_\beta \mid \beta \in \mathcal{J}\}$ be the corresponding bases of sequentially compact and open subsets in $\mathcal{K}$ from Sect. 2.3. Then the images $\bar{C}_\alpha = \Pi(C_\alpha)$ and $\bar{U}_\beta = \Pi(U_\beta)$ form the corresponding bases for $\bar{\mathcal{K}}$.

Choose $\alpha_0 \in k$ with the absolute trace $\mathrm{Tr}_{k/\mathbb{F}_p} \alpha_0 = 1$.

Define $\mathbb{F}_p$-linear operators $\mathcal{S}, \mathcal{R} : \mathcal{K} \longrightarrow \mathcal{K}$ as follows.

Suppose $\alpha \in k^*$.

If $a \in \mathbb{Z}^N$, $a > \bar{0} = (0, \dots, 0) \in \mathbb{Z}^N$ then set $\mathcal{S}(t^a\alpha) = 0$ and $\mathcal{R}(t^a\alpha) = -\sum_{i \geqslant 0} \sigma^i(t^a\alpha)$.

For $a = \bar{0}$, set $\mathcal{S}(\alpha) = \alpha_0 \mathrm{Tr}_{k/\mathbb{F}_p} \alpha$ and $\mathcal{R}(\alpha) = \sum_{0 \leqslant j < i < N_0} (\sigma^j \alpha_0) \sigma^i \alpha$.

If $a = -a_1p^m < \bar{0}$ with $a_1 \in \mathbb{Z}_N^+(p)$ set $\mathcal{S}(t^a\alpha) = t^{-a_1}\sigma^{-m}\alpha$ and $\mathcal{R}(t^a\alpha) = \sum_{1\leqslant i\leqslant m}\sigma^{-i}(t^a\alpha)$.

For $b = \sum\limits_{a\in\mathbb{Z}^N}\alpha_a t^a \in \mathcal{K}$, set $\mathcal{S}(b) = \sum\limits_{a\in\mathbb{Z}^N}\mathcal{S}(\alpha_a t^a)$, $\mathcal{R}(b) = \sum\limits_{a\in\mathbb{Z}^N}\mathcal{R}(\alpha_a t^a)$.

The proof of the following proposition is straightforward. It uses just that $\sigma : \mathcal{K} \longrightarrow \mathcal{K}$ is $\mathcal{P}$-continuous and $\mathcal{K}$ is a $\mathcal{P}$-topological group with respect to addition.

Proposition 3.1. a) *$\mathcal{R}$ and $\mathcal{S}$ are $\mathcal{P}$-continuous.*
b) *For any $b \in \mathcal{K}$, $b = \mathcal{S}(b) + (\sigma - \mathrm{id})\mathcal{R}(b)$.*

Notice that $\mathcal{S}^2 = \mathcal{S}$, $\mathcal{R}^2 = \mathcal{R}$ and $\mathcal{R}\mathcal{S} = \mathcal{S}\mathcal{R} = 0$. In particular, Prop. 3.1 implies that the elements $b \in \mathcal{K}$ can be uniquely presented modulo $(\sigma - \mathrm{id})\mathcal{K}$ in the following form

$$\sum_{a\in\mathbb{Z}_N^+(p)}\gamma_a t^{-a} + \gamma_{\bar{0}}\alpha_0 \tag{3.1}$$

where all $\gamma_a \in k$ and $\gamma_{\bar{0}} \in \mathbb{F}_p$. We have also the following proposition.

Proposition 3.2. a) *The morphism $\Pi(b) \mapsto \mathcal{S}(b)$, where $b \in \mathcal{K}$, defines a $\mathcal{P}$-continuous section $S_{t,\alpha_0} : \bar{\mathcal{K}} \longrightarrow \mathcal{K}$ of Π such that $S_{t,\alpha_0}\Pi(\alpha t^{-a}) = \alpha t^{-a}$ if $a \in \mathbb{Z}_N^+(p)$, $\alpha \in k$, and $S_{t,\alpha_0}(k/(\sigma - \mathrm{id})k) = \mathbb{F}_p\alpha_0 \subset k$.*

b) *For a $\mathcal{P}$-continuous section S of Π there is a $\mathcal{P}$-continuous map $R_S : \mathcal{K} \longrightarrow \mathcal{K}$ such that for any $b \in \mathcal{K}$, $b = S(\Pi(b)) + (\sigma - \mathrm{id})R_S(b)$.*

Proof. Item a) follows from Prop. 3.1. For item b), just notice that

$$b = \mathcal{S}(b) + (\sigma - \mathrm{id})\mathcal{R}(b) = S\Pi(b) + (\mathcal{S} - S\Pi)b + (\sigma - \mathrm{id})\mathcal{R}(b) =$$

$$S\Pi(b) + (\sigma - \mathrm{id})(\mathcal{R}(\mathcal{S} - S\Pi) + \mathcal{R})(b) = S\Pi(b) + (\sigma - \mathrm{id})\mathcal{R}(b - S\Pi b)$$

and $R_S := \mathcal{R}\,(\mathrm{id} - S\Pi)$ is $\mathcal{P}$-continuous. □

3.2. **$\mathcal{P}$-topological module $\bar{\mathcal{K}}^{D\mathcal{P}}$.** Proceed similarly to Sect. 2.3 by setting for all $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$:

1) $\bar{\mathcal{K}}^D_{\alpha\beta} := \mathrm{Hom}(\bar{C}_\alpha/\bar{C}_\alpha \cap \bar{U}_\beta, \mathbb{F}_p) = \mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{C}_\alpha/\bar{C}_\alpha \cap \bar{U}_\beta, \mathbb{F}_p)\,;$

2) $\bar{\mathcal{K}}^{D\mathcal{P}}_\alpha := \varinjlim\limits_\beta \bar{\mathcal{K}}^D_{\alpha\beta} = \mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{C}_\alpha, \mathbb{F}_p)\,;$

3) $\bar{\mathcal{K}}^{D\mathcal{P}} := \varprojlim\limits_\alpha \bar{\mathcal{K}}^{D\mathcal{P}}_\alpha = \mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{\mathcal{K}}, \mathbb{F}_p)\,.$

Remark. 1) For any α and β, $\dim_k \bar{C}_\alpha/(\bar{C}_\alpha \cap U_\beta) < \infty$.

2) $\bar{\mathcal{K}}^D_\alpha = \mathrm{Hom}(\bar{C}_\alpha, \mathbb{F}_p) = \left(\bar{\mathcal{K}}^{D\mathcal{P}}_\alpha\right)^{DD}$, in particular, $\bar{\mathcal{K}}^D_\alpha$ is the profinite completion of $\bar{\mathcal{K}}^{D\mathcal{P}}_\alpha$.

3) It follows from 2) that $\bar{\mathcal{K}}^D = \varprojlim\limits_\alpha \left(\bar{\mathcal{K}}^{D\mathcal{P}}_\alpha\right)^{DD} = \left(\bar{\mathcal{K}}^{D\mathcal{P}}\right)^{DD}$ is the profinite completion of $\bar{\mathcal{K}}^{D\mathcal{P}}$.

Define the $\mathcal{P}$-topology on $\bar{\mathcal{K}}^{D\mathcal{P}}$ as the projective limit of discrete topologies on all $\bar{\mathcal{K}}_\alpha^{D\mathcal{P}}$.

Set for any $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{J}$,

$U_\alpha^D := \{u^D \in \bar{\mathcal{K}}^{D\mathcal{P}} \mid u^D(\bar{C}_\alpha) = 0\} = \mathrm{Ann}(\bar{C}_\alpha)$.

$C_\beta^D = \{c^D \in \bar{\mathcal{K}}^{D\mathcal{P}} \mid c^D(\bar{U}_\beta) = 0\} = \mathrm{Ann}(\bar{U}_\beta)$.

Then in $\bar{\mathcal{K}}^{D\mathcal{P}}$:

a) $\mathcal{U}(\bar{\mathcal{K}}^{D\mathcal{P}}) := \{U_\alpha^D \mid \alpha \in \mathcal{I}\}$ is a base of open neighborhoods;

b) $\mathcal{C}(\bar{\mathcal{K}}^{D\mathcal{P}}) = \{C_\beta^D \mid \beta \in \mathcal{J}\}$ is a base of sequentially compact subsets;

c) $\bar{\mathcal{K}}^{D\mathcal{P}} = \varinjlim_\beta C_\beta^D$ and for any $\beta \in \mathcal{J}$, $C_\beta^D = \varprojlim_\alpha C_\beta^D / C_\beta^D \cap U_\alpha^D$;

d) for any $\alpha \in \mathcal{I}$, $\bar{\mathcal{K}}_\alpha^{D\mathcal{P}} = \bar{\mathcal{K}}^{D\mathcal{P}} / U_\alpha^D$.

These properties are implied easily via the following observations.

Let $\{\alpha_i \mid 1 \leqslant i \leqslant N\}$ be a basis of k over $\mathbb{F}_p$. Consider the set

$$\{\alpha_i t^{-a} \mid 1 \leqslant i \leqslant N, a \in \mathbb{Z}_N^+(p)\} \cup \{\alpha_{\bar{0}}\}\,. \tag{3.2}$$

Then:

— for any $\alpha \in \mathcal{I}$, there is a subset of (3.2) which forms a $\mathcal{P}$-topological basis of $\bar{C}_\alpha$;

— for any $\beta \in \mathcal{J}$, there is a subset of (3.2) which forms a $\mathcal{P}$-topological basis of $\bar{U}_\beta$.

Let

$$\{D_a^{(i)} \mid a \in \mathbb{Z}_N^+(p), 1 \leqslant i \leqslant N\} \cup \{D_{\bar{0}}\} \tag{3.3}$$

be the dual system of elements of $\bar{\mathcal{K}}^{D\mathcal{P}}$ for system (3.2).

Then for any $\alpha \in \mathcal{I}$, there is a subset of (3.3) which forms a $\mathcal{P}$-topological $\mathbb{F}_p$-basis of U_α^D. Similarly, for any $\beta \in \mathcal{J}$, there is a subset of (3.3) which forms a $\mathcal{P}$-topological $\mathbb{F}_p$-basis of C_β^D.

As a result, the pairing $\bar{\mathcal{K}} \times \bar{\mathcal{K}}^{D\mathcal{P}} \longrightarrow \mathbb{F}_p$ is a perfect pairing of $\mathcal{P}$-topological modules. This pairing identifies $\bar{\mathcal{K}}^{D\mathcal{P}}$ with $\mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{\mathcal{K}}, \mathbb{F}_p)$.

Consider the presentations of elements from $\mathcal{S}(\bar{\mathcal{K}}) \subset \mathcal{K}$ in the form (3.1). For $a \in \mathbb{Z}_N^+(p)$ and $n \in \mathbb{Z}/N_0$, let $D_{an} \in \bar{\mathcal{K}}^{D\mathcal{P}} \otimes_{\mathbb{F}_p} k$ be such that $D_{an}(\Pi(\gamma_a t^{-a})) = \sigma^n \gamma_a$ and $D_{an}(\Pi(\alpha_{\bar{0}})) = 0$. If $D_{\bar{0}} \in \bar{\mathcal{K}}^{D\mathcal{P}}$ is the element appeared in (3.3), then $D_{\bar{0}}(\gamma_a t^{-a}) = 0$ and $D_{\bar{0}}(\alpha_{\bar{0}}) = 1$.

The elements of the set

$$\mathcal{D} := \{D_{an} \mid a \in \mathbb{Z}_N^+(p), n \in \mathbb{Z}/N_0\} \cup \{D_{\bar{0}}\} \tag{3.4}$$

form a $\mathcal{P}$-topological basis for $\bar{\mathcal{K}}_k^{D\mathcal{P}}$. In particular:

(1) the elements of $\bar{\mathcal{K}}_k^{D\mathcal{P}}$ appear uniquely as $\mathcal{P}$-convergent series

$$\sum_{\substack{a\in\mathbb{Z}_N^+(p)\\ n\in\mathbb{Z}/N_0}} \gamma_{an}D_{an}+\gamma_{\bar{0}}D_{\bar{0}}\,,$$

where all γ_{an} and $\gamma_{\bar{0}}$ run over k;

(2) the appropriate subsets of (3.4) provide us with compatible k-bases for $\bar{\mathcal{K}}_{\alpha\beta}^D$ and $\bar{\mathcal{K}}_\alpha^{D\mathcal{P}}$;

(3) the elements of $\bar{\mathcal{K}}^{D\mathcal{P}}$ can be presented uniquely as $\mathcal{P}$-convergent series

$$\sum_{\substack{a\in\mathbb{Z}_N^+(p)\\ n\in\mathbb{Z}/N_0}} \sigma^n(\gamma_a)D_{an}+\gamma_{\bar{0}}D_{\bar{0}}\,,$$

where $\gamma_{\bar{0}}\in\mathbb{F}_p$ and for $a\neq\bar{0}$, $\gamma_a\in k$.

Remark. The condition of $\mathcal{P}$-convergence in (1) means that for any $\alpha\in\mathcal{I}$, $\{\gamma_{an}\neq 0\mid t^{-a}\in\bar{C}_\alpha\}$ is finite. Similar condition holds in (3) (where γ_{an} should be replaced by $\sigma^n\gamma_a$).

Let $\otimes^{\mathcal{P}}$ be the $\mathcal{P}$-topological tensor product. Consider

$$\bar{\mathcal{K}}_{\mathcal{K}}^{D\mathcal{P}}:=\mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{\mathcal{K}},\mathcal{K})=\bar{\mathcal{K}}^{D\mathcal{P}}\otimes_{\mathbb{F}_p}^{\mathcal{P}}\mathcal{K}=\bar{\mathcal{K}}_k^{D\mathcal{P}}\otimes_k^{\mathcal{P}}\mathcal{K}\,.$$

The following property is straightforward.

Proposition 3.3. *The elements of $\bar{\mathcal{K}}_{\mathcal{K}}^{D\mathcal{P}}$ can be presented uniquely as $\mathcal{P}$-convergent sums $\sum\limits_{a\in Z^N} m_a t^{-a}$ with coefficients $m_a\in\bar{\mathcal{K}}_k^{D\mathcal{P}}$.*

Remark. The condition of $\mathcal{P}$-convergency in Prop. 3.3 means that for any $\alpha\in\mathcal{I}$, $\beta\in\mathcal{J}$, $\{m_a\neq 0\mid t^{-a}\notin U_\beta, m_a\notin U_\alpha^D\}$ is finite.

3.3. **Lie algebras $\mathcal{L}$ and $\mathcal{L}^{\mathcal{P}}$.**

Let $\widetilde{\mathcal{L}}$ be a free profinite Lie algebra over $\mathbb{F}_p$ with the (profinite) module of free generators $\bar{\mathcal{K}}^D=\mathrm{Hom}(\bar{\mathcal{K}},\mathbb{F}_p)$. Let $\mathcal{L}=\widetilde{\mathcal{L}}/C_p(\widetilde{\mathcal{L}})$. Then $\mathcal{L}$ is a projective limit of finite Lie $\mathbb{F}_p$-algebras generated by the finite quotients of $\bar{\mathcal{K}}^D$.

Let $\widetilde{\mathcal{L}}^{\mathcal{P}}$, resp., $\mathcal{L}^{\mathcal{P}}$, be the Lie subalgebra in $\widetilde{\mathcal{L}}$, resp. in $\mathcal{L}$, generated by the elements of $\bar{\mathcal{K}}^{D\mathcal{P}}=\mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{\mathcal{K}},\mathbb{F}_p)\subset\bar{\mathcal{K}}^D$. Then $C_p(\widetilde{\mathcal{L}}^{\mathcal{P}})=\widetilde{\mathcal{L}}^{\mathcal{P}}\cap C_p(\widetilde{\mathcal{L}})$ (use that the profinite completion of $\bar{\mathcal{K}}^{D\mathcal{P}}$ is $\bar{\mathcal{K}}^D$) and $\mathcal{L}^{\mathcal{P}}=\widetilde{\mathcal{L}}^{\mathcal{P}}/C_p(\widetilde{\mathcal{L}}^{\mathcal{P}})$.

Note that $\mathcal{L}^{\mathcal{P}}$ inherits the $\mathcal{P}$-topological structure from $\bar{\mathcal{K}}^{D\mathcal{P}}$ (use the topology of tensor product on $\sum\limits_{1\leqslant i<p}\left(\bar{\mathcal{K}}^{D\mathcal{P}}\right)^{\otimes i}\supset\mathcal{L}^{\mathcal{P}}$), and the profinite completion of $\mathcal{L}^{\mathcal{P}}$ coincides with $\mathcal{L}$.

Introduce the Lie algebras $\mathcal{L}_\alpha$ with generators $\bar{\mathcal{K}}_\alpha^D=\mathrm{Hom}(\bar{C}_\alpha,\mathbb{F}_p)$ and $\mathcal{L}_\alpha^{\mathcal{P}}$ with generators $\bar{\mathcal{K}}_\alpha^{D\mathcal{P}}=\mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{C}_\alpha,\mathbb{F}_p)$. Then $\mathcal{L}_\alpha^{\mathcal{P}}$ has discrete topology, its profinite completion coincides with $\mathcal{L}_\alpha$, $\mathcal{L}=\varprojlim\limits_\alpha\mathcal{L}_\alpha$

and $\mathcal{L}^{\mathcal{P}} = \varprojlim_{\alpha} \mathcal{L}^{\mathcal{P}}_{\alpha}$. If $\mathcal{L}_{\alpha\beta}$ is the subalgebra in $\mathcal{L}_{\alpha}$ generated by $\bar{\mathcal{K}}^{D}_{\alpha\beta}$ then $\mathcal{L}_{\alpha\beta}$ is finite and $\mathcal{L}^{\mathcal{P}}_{\alpha} = \varinjlim_{\beta} \mathcal{L}_{\alpha\beta}$.

The elements of the Lie algebra $\mathcal{L}^{\mathcal{P}}_{k} := \mathcal{L}^{\mathcal{P}} \otimes_{\mathbb{F}_p} k$ appear as convergent k-linear combinations of the Lie monomials of the form

$$\sum_{D_1,\dots,D_r} \gamma_{D_1,\dots D_r}[\dots[D_1,\dots],D_r]\,.$$

where all $D_1,\dots,D_r$ belong to (3.4). The condition of convergency means that for any $\alpha \in \mathcal{I}$, all but finitely many of these monomials have the zero image in $\mathcal{L}^{\mathcal{P}}_{\alpha}$.

We can describe similarly the enveloping algebra of $\mathcal{L}^{\mathcal{P}}$. Namely, let $\mathfrak{A}^{\mathcal{P}}$ and $\mathfrak{A}^{\mathcal{P}}_{\alpha}$ be enveloping algebras for $\mathcal{L}^{\mathcal{P}}$ and, resp., $\mathcal{L}^{\mathcal{P}}_{\alpha}$ taken modulo p-th powers of the corresponding augmentation ideals. Then $\mathfrak{A}^{\mathcal{P}} = \varprojlim_{\alpha} \mathfrak{A}^{\mathcal{P}}_{\alpha}$ and $\mathfrak{A}^{\mathcal{P}}_{\alpha,k}$ consists of all polynomials of total degree $< p$ in the subset of variables D_{an} from (3.4) satisfying the condition $t^{-a} \notin C_{\alpha}$. In other words, the elements of $\mathfrak{A}^{\mathcal{P}}$ are characterized in the algebra $\mathfrak{A}$ as $\mathcal{P}$-continuous polynomials on $\bar{\mathcal{K}}$ with values in $\mathbb{F}_p$ of total degree $< p$. Of course, $\mathfrak{A}$ and $\mathcal{L}$ can be recovered as the profinite completion of $\mathfrak{A}^{\mathcal{P}}$ and, resp., $\mathcal{L}^{\mathcal{P}}$.

3.4. **Class of conjugated subgroups** $\mathrm{cl}^{\mathcal{P}}(\mathcal{G}_{<p})$.

Let $\mathcal{G} = \mathrm{Gal}(\mathcal{K}_{sep}/\mathcal{K})$ be the absolute Galois group of the field $\mathcal{K}$.

If $\mathcal{G}_{<2} := \mathcal{G}/\mathcal{G}^p C_2(\mathcal{G})$ is the maximal abelian quotient of period p of $\mathcal{G}$ then the classical Artin-Schreier duality $\bar{\mathcal{K}} \times \mathcal{G}_{<2} \longrightarrow \mathbb{F}_p$ allows us to identify $\mathcal{G}_{<2}$ with $\bar{\mathcal{K}}^D = \left(\bar{\mathcal{K}}^{D\mathcal{P}}\right)^{DD}$ and to introduce a dense subgroup $\mathcal{G}^{\mathcal{P}}_{<2} := \bar{\mathcal{K}}^{D\mathcal{P}}$ in $\mathcal{G}_{<2}$. Note that with respect to this identification, the elements $D^{(i)}_a \in \bar{\mathcal{K}}^{D\mathcal{P}}$ from (3.3) appear as elements of $\mathcal{G}_{<2}$ such that if $T_{bj} \in \mathcal{K}_{sep}$ are such that $T^p_{bj} - T_{bj} = \alpha_j t^{-b}$ (in notation from Sect. 2.3) then $D^{(i)}_a(T_{bj}) = T_{bj} + \delta_{ab}\delta_{ij}$. Similarly, the elements $D_{an} \in \bar{\mathcal{K}}^{D\mathcal{P}}_k$ from (3.4) act as follows: if $b \in \mathbb{N}$, $\gcd(b,p) = 1$ and $T_b \in \mathcal{K}_{sep}$ is such that $T^q_b - T_b = t^{-b}$ then for $0 \leqslant m < N_0$, $D_{an}(T^{p^m}_b) = T^{p^m}_b + \delta_{ab}\delta_{nm}$.

The identification of local class field theory $\mathcal{G}_{<2} \simeq K_N(\mathcal{K})/p$ induces the identification $\mathcal{G}^{\mathcal{P}}_{<2} \simeq K^{top}_N(\mathcal{K})/p$, where K^{top}_N is the topological version of the functor K_N, cf. e.g. [28, 29]. The subgroup $\mathcal{G}^{\mathcal{P}}_{<2}$ is considerably smaller than $\mathcal{G}_{<2}$ but its profinite completion recovers the whole $\mathcal{G}_{<2}$. In particular, $\mathcal{G}^{\mathcal{P}}_{<2}$ can be used instead of $\mathcal{G}_{<2}$ when studying finite (abelian) extensions of $\mathcal{K}$ inside $\mathcal{K}_{<2} = \mathcal{K}^{C_2(\mathcal{G}_{<p})}_{<p}$. Our target is to introduce an analog of $\mathcal{G}^{\mathcal{P}}_{<2}$ in the case of p-extensions of nilpotent class $< p$.

From now on we will consider only $e \in \mathcal{E}(\mathcal{L}^{\mathcal{P}}_{\mathcal{K}}) := \mathcal{L}^{\mathcal{P}}_{\mathcal{K}} \cap \mathcal{E}(\mathcal{L}_{\mathcal{K}})$. Under this assumption if S is a section of the projection $\Pi : \mathcal{K} \longrightarrow \bar{\mathcal{K}}$ such that $e \bmod C_2(\mathcal{L}^{\mathcal{P}}_{\mathcal{K}}) = e_S$ then S is $\mathcal{P}$-continuous.

As earlier, choose $f \in \mathcal{F}(e)$ and set $\pi = \pi_f(e) : \mathcal{G}_{<p} \simeq G(\mathcal{L})$.

Definition. $\mathrm{cl}^{\mathcal{P}}(\mathcal{G}_{<p})$ is the class of conjugated subgroups of $\mathcal{G}_{<p}$ containing $\pi_f(e)^{-1}G(\mathcal{L}^{\mathcal{P}})$.

Theorem 3.4. *The class $\mathrm{cl}^{\mathcal{P}}(\mathcal{G}_{<p})$ does not depend on the choices of $e \in \mathcal{E}(\mathcal{L}^{\mathcal{P}}_{\mathcal{K}})$ and $f \in \mathcal{F}(e)$.*

Proof. Suppose $e' \in \mathcal{E}(\mathcal{L}^{\mathcal{P}}_{\mathcal{K}})$, $f' \in \mathcal{F}(e')$ and set $\pi' = \pi_{f'}(e')$. We must prove that $\pi^{-1}G(\mathcal{L}^{\mathcal{P}})$ and $(\pi')^{-1}G(\mathcal{L}^{\mathcal{P}})$ are conjugated in $\mathcal{G}_{<p}$.

Lemma 3.5. *There is an $x \in \mathcal{L}^{\mathcal{P}}_{\mathcal{K}}$ and a $\mathcal{P}$-continuous section $A : \bar{\mathcal{K}}^{D\mathcal{P}} \longrightarrow \mathcal{L}^{\mathcal{P}}$ of the natural projection $\mathcal{L}^{\mathcal{P}} \longrightarrow \mathcal{L}^{\mathcal{P}}/C_2(\mathcal{L}^{\mathcal{P}}) = \bar{\mathcal{K}}^{D\mathcal{P}}$ such that*

$$e' = \sigma(x) \circ (\mathcal{A} \otimes^{\mathcal{P}}_{\mathbb{F}_p} \mathrm{id}_{\mathcal{K}})e \circ (-x)\,,$$

where $\mathcal{A} \in \mathrm{Aut}_{Lie}\mathcal{L}^{\mathcal{P}}$ is such that $\mathcal{A}|_{\bar{\mathcal{K}}^{D\mathcal{P}}} = A$.

Proof of lemma. The proof appears as a $\mathcal{P}$-topological version of the proof of Prop. 1.1, where we use the ($\mathcal{P}$-continuous) operators from Prop. 3.1 and 3.2.

Let $\{l_\alpha \mid \alpha \in \mathcal{I}\}$ be a $\mathcal{P}$-topological $\mathbb{F}_p$-basis of $\bar{\mathcal{K}}$. Let $\hat{l}_\alpha$, $\alpha \in \mathcal{I}$, be the dual ($\mathcal{P}$-topological) $\mathbb{F}_p$-basis for $\bar{\mathcal{K}}^{D\mathcal{P}}$, i.e. for any $\alpha_1, \alpha_2 \in \mathcal{I}$, $\hat{l}_{\alpha_1}(l_{\alpha_2}) = \delta_{\alpha_1\alpha_2}$. Then for the corresponding sections S and S', we have the $\mathcal{P}$-convergent series $e_S = \sum\limits_\alpha \hat{l}_\alpha \otimes^{\mathcal{P}} S(l_\alpha)$ and $e'_{S'} = \sum\limits_\alpha \hat{l}_\alpha \otimes^{\mathcal{P}} S'(l_\alpha)$.

Apply induction on $r \geqslant 1$ to prove the existence of $x_r \in \mathcal{L}^{\mathcal{P}}_{\mathcal{K}}$ and a section A_r of the projection $\mathcal{L}^{\mathcal{P}} \longrightarrow \bar{\mathcal{K}}^{D\mathcal{P}}$ such that

$$e' \equiv \sigma(x_r) \circ (\mathcal{A}_r \otimes \mathrm{id}_{\mathcal{K}})e \circ (-x_r) \,\mathrm{mod}\, C_{r+1}(\mathcal{L}^{\mathcal{P}}_{\mathcal{K}})\,,$$

with the appropriate $\mathcal{P}$-continuous automorphism $\mathcal{A}_r$ of $\mathcal{L}^{\mathcal{P}}$.

If $r = 1$ take $A_1 = \mathrm{id}_M$ and $x_1 = \sum\limits_\alpha \hat{l}_\alpha \otimes^{\mathcal{P}} x_{1\alpha}$, where all $x_{1\alpha} = \mathcal{R}(S'(l_\alpha) - S(l_\alpha)) \in \mathcal{K}$ and $\mathcal{R}$ is the operator from Prop. 3.1. Note that $x_1 \in \mathcal{L}^{\mathcal{P}}_{\mathcal{K}}$ because $\mathcal{R}$ is $\mathcal{P}$-continuous.

If $r \geqslant 1$ and such x_r and A_r exist then there is $l_{r+1} \in C_{r+1}(\mathcal{L}^{\mathcal{P}}_{\mathcal{K}})$ such that $e' \equiv \sigma x_r \circ (\mathcal{A}_r \otimes^{\mathcal{P}} \mathrm{id}_{\mathcal{K}})e \circ (-x_r) \circ l_{r+1} \,\mathrm{mod}\, C_{r+2}(\mathcal{L}^{\mathcal{P}}_{\mathcal{K}})$.

Let $l_{r+1} = \sum_\alpha c_\alpha \otimes^{\mathcal{P}} b_\alpha$ with all $c_\alpha \in C_{r+1}(\mathcal{L}^{\mathcal{P}})$ and $b_\alpha \in \mathcal{K}$. Then

$$l_{r+1} = l' + \sigma(x') - x'\,,$$

where $l' = \sum\limits_\alpha c_\alpha \otimes^{\mathcal{P}} (S\Pi)(b_\alpha)$ and $x' = \sum\limits_\alpha c_\alpha \otimes^{\mathcal{P}} R_S(b_\alpha)$ belong to $C_{r+1}(\mathcal{L}^{\mathcal{P}}_{\mathcal{K}}) \subset \mathcal{L}^{\mathcal{P}}_{\mathcal{K}}$. It remains to set $A_{r+1}(\hat{l}_\alpha) = A_r(\hat{l}_\alpha) + c_\alpha$ and $x_{r+1} = x_r + x'$. The lemma is proved. ☐

Remark. The main reason why the proof of Prop. 1.1 works in the $\mathcal{P}$-topological context is that $\mathcal{L}^{\mathcal{P}}_{\mathcal{K}} = \mathcal{L}^{\mathcal{P}} \otimes^{\mathcal{P}} \mathcal{K}$ is stable with respect to the action of $\mathcal{P}$-continuous operators on the factor $\mathcal{K}$.

Continue the proof of the theorem.

Denote by the same symbol $\mathcal{A}$ the (unique) extension of $\mathcal{A}$ to $\mathcal{L}$ (use that $\mathcal{L}$ is a profinite completion of $\mathcal{L}^{\mathcal{P}}$). Then $f''=x\circ(\mathcal{A}\otimes\mathrm{id}_{sep})f\in\mathcal{F}(e')$ and for any $\tau\in\mathcal{G}_{<p}$, $\pi_{f''}(e')(\tau)=(\mathcal{A}\cdot\pi)\tau$.

Then $(\mathcal{A}\cdot\pi)(\mathcal{G}^{\mathcal{P}}_{<p})=\mathcal{A}(\mathcal{L}^{\mathcal{P}})=\mathcal{L}^{\mathcal{P}}$. This implies that $\pi^{-1}G(\mathcal{L}^{\mathcal{P}})=\pi_{f''}(e')^{-1}G(\mathcal{L}^{\mathcal{P}})$. But $f',f''\in\mathcal{F}(e')$ implies that $(\pi')^{-1}G(\mathcal{L}^{\mathcal{P}})$ and $\pi_{f''}(e')^{-1}G(\mathcal{L}^{\mathcal{P}})$ are conjugated in $\mathcal{G}_{<p}$. □

3.5. **Galois $\mathcal{P}$-correspondence.** As earlier, consider $\bar{\mathcal{K}}^D$, $\bar{\mathcal{K}}^{D\mathcal{P}}$, $e\in\mathcal{E}(\mathcal{L}^{\mathcal{P}}_{\mathcal{K}})$, $f\in\mathcal{F}(e)$ and $\pi:=\pi_f(e):\mathcal{G}_{<p}\simeq G(\mathcal{L})$. Suppose $\mathcal{K}'$ is a finite field extension of $\mathcal{K}$ in $\mathcal{K}_{sep}$. Let $\mathcal{G}'_{<p}$, $\bar{\mathcal{K}}'^D$, $\bar{\mathcal{K}}'^{D\mathcal{P}}$, $e'\in\mathcal{E}(\mathcal{L}'^{\mathcal{P}}_{\mathcal{K}'})$, $f'\in\mathcal{F}(e')$ and $\pi'=\pi_{f'}(e'):\mathcal{G}'_{<p}\simeq G(\mathcal{L}')$ be the similar objects for the field $\mathcal{K}'$.

The natural morphism of profinite groups $\Theta:\mathcal{G}'_{<p}\longrightarrow\mathcal{G}_{<p}$ can be described in the terms of identifications π and π' by Prop. 1.3. It admits the following $\mathcal{P}$-version.

Proposition 3.6. *Suppose $\mathcal{G}'^{\mathcal{P}}_{<p}\in\mathrm{cl}^{\mathcal{P}}\mathcal{G}'_{<p}$. Then:*

a) *there is $\mathcal{G}^{\mathcal{P}}_{<p}\in\mathrm{cl}^{\mathcal{P}}\mathcal{G}_{<p}$ such that $\Theta(\mathcal{G}'^{\mathcal{P}}_{<p})\subset\mathcal{G}^{\mathcal{P}}_{<p}$;*

b) *$(\mathcal{G}^{\mathcal{P}}_{<p}:\Theta(\mathcal{G}'^{\mathcal{P}}_{<p}))=(\mathcal{G}_{<p}:\Theta(\mathcal{G}'_{<p}))$, i.e. $\Theta(\mathcal{G}'^{\mathcal{P}}_{<p})=\Theta(\mathcal{G}'_{<p})\cap\mathcal{G}^{\mathcal{P}}_{<p}$.*

Proof. a) We can assume that $f'\in\mathcal{F}(e')$ is such that $\mathcal{G}'^{\mathcal{P}}_{<p}=\pi'^{-1}G(\mathcal{L}'^{\mathcal{P}})$. Then we can apply the $\mathcal{P}$-topological version of the proof of Prop. 1.3 to establish the existence of $\mathcal{P}$-continuous $\mathcal{A}\in\mathrm{Hom}_{\mathrm{Lie}}(\mathcal{L}'^{\mathcal{P}},\mathcal{L}^{\mathcal{P}})$ and $x'\in\mathcal{L}^{\mathcal{P}}_{\mathcal{K}'}$ such that

$$e\otimes^{\mathcal{P}}_{\mathcal{K}}1_{\mathcal{K}'}=\sigma(x')\circ(\mathcal{A}\otimes\mathrm{id}_{\mathcal{K}})e'\circ(-x')\,. \tag{3.5}$$

From (3.5) it follows that both $(-x')\circ f$ and $(\mathcal{A}\otimes\mathrm{id}_{\mathcal{K}'})f'$ belong to $\mathcal{F}((\mathcal{A}\otimes\mathrm{id})e')\subset\mathcal{L}_{sep}$. Therefore, there is $l\in\mathcal{L}$ such that

$$(-x')\circ f=(\mathcal{A}\otimes\mathrm{id}_{sep})f'\circ l\,.$$

As a result, for any $\tau'\in\mathcal{G}'_{<p}$, $\pi(\Theta(\tau'))=(-l)\circ\mathcal{A}(\pi'(\tau'))\circ l$, and

$$\pi(\Theta(\mathcal{G}'^{\mathcal{P}}_{<p}))=(-l)\circ\mathcal{A}(\mathcal{L}'^{\mathcal{P}})\circ(-l)\subset(-l)\circ\mathcal{L}^{\mathcal{P}}\circ l\,.$$

Equivalently, for $g=\pi^{-1}(l)\in\mathcal{G}_{<p}$, we have

$$\Theta(\mathcal{G}'^{\mathcal{P}}_{<p})\subset(-g)\circ\pi^{-1}(\mathcal{L}^{\mathcal{P}})\circ g\in\mathrm{cl}^{\mathcal{P}}\mathcal{G}_{<p}\,.$$

So, we can take $\mathcal{G}^{\mathcal{P}}_{<p}=\mathrm{Ad}(g)(\pi^{-1}\mathcal{L}^{\mathcal{P}})$.

b) Assume that in the notation from a), $l=0$. This guarantees $\Theta(\mathcal{G}'^{\mathcal{P}}_{<p})\subset\mathcal{G}^{\mathcal{P}}_{<p}$ and $\pi=\mathcal{A}\cdot\pi'$, where $\pi=\pi_f(e):\mathcal{G}_{<p}\simeq G(\mathcal{L})$, $\pi'=\pi_{f'}(e'):\mathcal{G}'_{<p}\simeq G(\mathcal{L}')$ and $\mathcal{A}:\mathcal{L}'\longrightarrow\mathcal{L}$ is induced by Θ.

Let $p^n=[\mathcal{K}':\mathcal{K}]=(\mathcal{G}_{<p}:\Theta(\mathcal{G}'_{<p}))$.

- *The case $[\mathcal{K}':\mathcal{K}]=p$.*

Here $\mathcal{K}'/\mathcal{K}$ is Galois of degree p, $(\mathcal{L}:\mathcal{A}(\mathcal{L}'))=p$, $\mathcal{A}(\mathcal{L}')$ is an ideal in $\mathcal{L}$, and $\mathcal{A}(\mathcal{L}')=C_2(\mathcal{L})+L$, where $L\subset\bar{\mathcal{K}}^D$ is of index p.

Let $\mathcal{A}(\mathcal{L}'^{\mathcal{P}}) = C_2(\mathcal{L}^{\mathcal{P}}) + L^0 \subset \mathcal{A}(\mathcal{L}')$, where in notation from Sect. 3.2,

$$L^0 \subset \bar{\mathcal{K}}^{\mathcal{P}D} = \varprojlim_{\alpha} \mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{C}_\alpha, \mathbb{F}_p)\,.$$

Let $\mathcal{K}' = \mathcal{K}(T')$, where $T'^p - T' = c \in \mathcal{K}$. Then $\mathcal{K}' = \mathcal{K}^H_{<p}$, with $H = \Theta(\mathcal{G}'_{<p})$ and $\pi(H) = G(C_2(\mathcal{L}) + L)$. Therefore, $L \subset \bar{\mathcal{K}}^D$ is characterized by the trivial action on T' or, equivalently, $L = \mathrm{Ann}(\bar{c})$, where $\bar{c} \in \bar{\mathcal{K}}$ is the image of c under the natural projection $\Pi : \mathcal{K} \longrightarrow \bar{\mathcal{K}}$.

We can assume that for some index α_0, $\Pi(c) \in \bar{C}_{\alpha_0}$, because $\bar{\mathcal{K}}$ is the union of all $\bar{C}_\alpha = \Pi(C_\alpha)$. As a result:

– the $\mathcal{P}$-subgroup $H^{\mathcal{P}}$ appears in the form $\pi^{-1}G(C_2(\mathcal{L}^{\mathcal{P}}) + L^0)$;

– L^0 is the preimage of a subspace $L^0_{\alpha_0} \subset \mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{C}_{\alpha_0}, \mathbb{F}_p)$;

– $L^0_{\alpha_0}$ consists of all FINITE $\mathbb{F}_p$-linear combinations of the elements $D^{(i)}_a \in \bar{C}_{\alpha_0}$ from (3.3) which annihilate $\bar{c}$. This means that $L^0_{\alpha_0}$ is of index p in $\mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{C}_{\alpha_0}, \mathbb{F}_p)$, L^0 is of index p in $\bar{\mathcal{K}}^{\mathcal{P}D}$, $\mathcal{A}(\mathcal{L}'^{\mathcal{P}})$ is of index p in $\mathcal{L}^{\mathcal{P}}$ and b) is proved in the case $n = 1$.

Inductive step.

Suppose $n \geqslant 2$ and b) is proved for field extensions of degree p^{n-1}.
Consider the tower $\mathcal{K} \subset \mathcal{K}_1 \subset \mathcal{K}'$, $[\mathcal{K}_1 : \mathcal{K}] = p$, $[\mathcal{K}' : \mathcal{K}] = p^{n-1}$.
Using similar notation for $\mathcal{K}'$ and $\mathcal{K}_1$ we have:

(1) the fields tower $\mathcal{K} \subset \mathcal{K}_1 \subset \mathcal{K}' \subset \mathcal{K}_{<p} \subset \mathcal{K}_{1,<p} \subset \mathcal{K}'_{<p}$,

(2) the compatible identifications:
 – $\mathcal{G}_{<p} \simeq G(\mathcal{L}) \subset \mathcal{G}_{1,<p} \simeq G(\mathcal{L}_1) \subset \mathcal{G}'_{<p} \simeq G(\mathcal{L}')$,
 – $\mathcal{G}^{\mathcal{P}}_{<p} \simeq G(\mathcal{L}^{\mathcal{P}}) \subset \mathcal{G}^{\mathcal{P}}_{1,<p} \simeq G(\mathcal{L}_1) \subset \mathcal{G}'^{\mathcal{P}}_{<p} \simeq G(\mathcal{L}'^{\mathcal{P}})$

(3) the natural group homomorphisms:
 – $\Theta_1 : \mathcal{G}_{1,<p} \longrightarrow \Theta_1(\mathcal{G}_{1,<p}) \subset \mathcal{G}_{<p}$,
 – $\Theta' : \mathcal{G}'_{<p} \longrightarrow \Theta'(\mathcal{G}'_{<p}) \subset \mathcal{G}_{1,<p}$
 – $\Theta_1 : \Theta'(\mathcal{G}_{<p}) \longrightarrow \Theta(\mathcal{G}'_{<p}) \subset \Theta_1(\mathcal{G}_{1,<p}) \subset \mathcal{G}_{<p}$,

(4) the restrictions of the above Θ, Θ', Θ_1 to the corresponding $\mathcal{P}$-subgroups satisfy analogs of relations from c).

Note that $\mathrm{Ker}\,\Theta_1 = \mathrm{Gal}(\mathcal{K}_{1,<p}/\mathcal{K}_{<p}) = J$ is the profinite closure of $\mathcal{J}^{\mathcal{P}} := \mathcal{G}^{\mathcal{P}}_{1,<p} \cap J = \mathrm{Ker}\,\Theta_1|_{\mathcal{G}^{\mathcal{P}}_{1,<p}}$. Therefore, $\Theta_1(\mathcal{G}^{\mathcal{P}}_{1,<p}) = \mathcal{G}_{1,<p}/\mathrm{Ker}\,J^{\mathcal{P}}$.

Similarly, Θ_1 induces a group epimorphic map $\Theta'(\mathcal{G}_{<p}) \longrightarrow \Theta(\mathcal{G}'_{<p})$ with the kernel J, the corresponding epimorphism $\Theta'(\mathcal{G}^{\mathcal{P}}_{<p}) \longrightarrow \Theta(\mathcal{G}'^{\mathcal{P}}_{<p})$ has the kernel $J^{\mathcal{P}}$ and $\Theta(\mathcal{G}'^{\mathcal{P}}_{<p}) = \Theta'(\mathcal{G}^{\mathcal{P}}_{<p})/J^{\mathcal{P}}$.

Therefore, $(\mathcal{G}^{\mathcal{P}}_{1,<p} : \Theta'(\mathcal{G}^{\mathcal{P}}_{<p})) = (\Theta_1(\mathcal{G}^{\mathcal{P}}_{1,<p}) : \Theta(\mathcal{G}'^{\mathcal{P}}_{<p}))$. By the inductive assumption, this index equals p^{n-1}. Finally, using the case $n = 1$ we obtain $(\mathcal{G}^{\mathcal{P}}_{<p} : \Theta(\mathcal{G}'^{\mathcal{P}}_{<p})) = p^n$. □

Definition. If $\mathcal{H} = \Theta(\mathcal{G}'_{<p})$ then we set $\mathcal{H}^{\mathcal{P}} = \mathcal{H} \cap \mathcal{G}^{\mathcal{P}}_{<p}$.

Clearly, the conjugacy class of $\mathcal{H}^{\mathcal{P}}$ in its profinite completion $\mathcal{H}$ is well defined.

Corollary 3.7. a) *Any extension $\mathcal{K}'$ of $\mathcal{K}$ in $\mathcal{K}_{<p}$ (in the category of N-dimensional local fields) appears in the form $\mathcal{K}_{<p}^{\mathcal{H}}$, where $\mathcal{H}$ is the profinite completion of a $\mathcal{P}$-closed subgroup $\mathcal{H}^{\mathcal{P}}$ of some $\mathcal{G}_{<p}^{\mathcal{P}}\in \mathrm{cl}^{\mathcal{P}}\mathcal{G}_{<p}$.*

b) *In the above notation, $\mathcal{K}'$ is Galois over $\mathcal{K}$ iff the subgroup $\mathcal{H}^{\mathcal{P}}$ of $\mathcal{G}_{<p}^{\mathcal{P}}$ is normal, and* $\mathrm{Gal}(\mathcal{K}'/\mathcal{K})=\mathcal{G}_{<p}^{\mathcal{P}}/\mathcal{H}^{\mathcal{P}}$.

It remains to characterize the subgroups $\mathcal{H}^{\mathcal{P}}\subset\mathcal{G}_{<p}^{\mathcal{P}}$ such that $\mathcal{K}^{\mathcal{H}}$ is an extension of $\mathcal{K}$ in $\mathcal{K}_{<p}$.

Proposition 3.8. *Let $H\subset\mathcal{G}_{<p}^{\mathcal{P}}$ be a subgroup. Then $H=\mathcal{H}^{\mathcal{P}}$, where $\mathcal{K}^{\mathcal{H}}=\mathcal{K}'$ is N-dimensional field extension of $\mathcal{K}$ iff*

a) *$(\mathcal{G}_{<p}^{\mathcal{P}}:H)<\infty$;*

b) *H is $\mathcal{P}$-open in $\mathcal{G}_{<p}^{\mathcal{P}}$.*

Proof. If $\mathcal{K}'$ is field extension of $\mathcal{K}$ in the category of N-dimensional fields then $[\mathcal{K}':\mathcal{K}]=(\mathcal{G}_{<p}:\mathcal{H})=(\mathcal{G}_{<p}^{\mathcal{P}}:\mathcal{H}^{\mathcal{P}})<\infty$ and by Prop. 3.6 $\mathcal{H}^{\mathcal{P}}$ is $\mathcal{P}$-closed. It is also $\mathcal{P}$-open as a closed subgroup of finite index in $\mathcal{G}_{<p}^{\mathcal{P}}$.

To proceed in the opposite direction note that $H=G(L)$, where L is a Lie subalgebra in $\mathcal{L}^{\mathcal{P}}$ and the index $(\mathcal{L}^{\mathcal{P}}:L)$ is a power of p. Choose an increasing sequence of Lie algebras $L=L_0\subset L_1\cdots\subset L_n=\mathcal{L}^{\mathcal{P}}$ where each L_{i-1} is ideal in L_i and $(L_i:L_{i-1})=p$. As a result, we can proceed by induction and it will be sufficient to consider the case $n=1$.

Then $L\supset C_2(\mathcal{L}^{\mathcal{P}})$ and $L=C_2(\mathcal{L}^{\mathcal{P}})+L^0$, where

$$L^0\subset \mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{\mathcal{K}},\mathbb{F}_p)=\varprojlim_{\alpha}\mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{C}_\alpha,\mathbb{F}_p)\,,$$

cf. notation from Sect. 3.2. Since L^0 is $\mathcal{P}$-open there is an index α_0 such that L^0 is the preimage of a subgroup $L^0_{\alpha_0}$ of index p in $\mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{C}_{\alpha_0},\mathbb{F}_p)$. Therefore, there is $\bar{c}\in\bar{C}_{\alpha_0}$ such that $L^0_{\alpha_0}=\mathrm{Ann}\,\bar{c}$ in $\mathrm{Hom}_{\mathcal{P}\text{-cont}}(\bar{C}_{\alpha_0},\mathbb{F}_p)$.

Let $\mathcal{K}'=\mathcal{K}(T')$, where $T'^p-T'=\Pi^{-1}\bar{c}$. Then $\mathcal{K}'=\mathcal{K}_{<p}^{\mathcal{H}}$, where the subgroup $\mathcal{H}$ of $\mathcal{G}_{<p}$ is such that $\mathcal{H}=G(C_2(\mathcal{L})+L)$ and $L\subset\bar{\mathcal{K}}^D$ is characterized by the trivial action on T'. Therefore, the corresponding $\mathcal{P}$-subgroup $\mathcal{H}^{\mathcal{P}}=G(C_2(\mathcal{L}^{\mathcal{P}})+L^{\mathcal{P}})$, where $L^{\mathcal{P}}$ consists of finite $\mathbb{F}_p$-linear combinations of the elements $D_a^{(i)}$ from (3.3) which annihilate c. Therefore, $L^{\mathcal{P}}=L^0$ and $\mathcal{H}^{\mathcal{P}}=H$. □

3.6. **More general $\mathcal{P}$-groups.** Suppose $G'\subset\mathrm{Aut}\,\mathcal{K}'$. For example, $\mathcal{K}'/\mathcal{K}$ is Galois and $G'=\mathrm{Gal}(\mathcal{K}'/\mathcal{K})$. Consider the group $\Gamma'\subset\mathrm{Aut}\,\mathcal{K}'_{<p}$ of all lifts of the elements of G' to $\mathcal{K}'_{<p}$. These lifts can be treated in terms of couples $(C'_g,\mathcal{A}'_g)$, where $g\in G'$, $C'_g\in\mathcal{L}'_{\mathcal{K}'}$ and $\mathcal{A}'\in\mathrm{Aut}_{\mathrm{Lie}}\mathcal{L}'$, cf. Sect. 1.6. This description uses the identification $\pi'=\pi_{f'}(e'):\mathcal{G}'_{<p}\simeq G(\mathcal{L}')$. After applying π'^{-1} we obtain the exact sequence

$$1\longrightarrow\mathcal{G}'_{<p}\longrightarrow\Gamma'\longrightarrow G'\longrightarrow 1\,.$$

Consider a subgroup $\Gamma'^{\mathcal{P}}$ of Γ' coming from $C'_g \in \mathcal{L}'^{\mathcal{P}}_{\mathcal{K}} = \mathcal{L}'^{\mathcal{P}} \otimes^{\mathcal{P}} \mathcal{K}'$ and $\mathcal{P}$-continuous $\mathcal{A}'_g$. (For example, use the $\mathcal{P}$-continuous operators $\mathcal{R}$ and $\mathcal{S}$ from Sect. 3.1 to recover the corresponding pairs $(C'_g, \mathcal{A}'_g)$, cf. e.g. Sect. 4.2 below.) We obtain the following exact sequence

$$1 \longrightarrow \mathcal{G}'^{\mathcal{P}}_{<p} \longrightarrow \Gamma'^{\mathcal{P}} \longrightarrow G' \longrightarrow 1\,.$$

This construction of the subgroup $\Gamma'^{\mathcal{P}}$ of Γ' does not depend on a choice of "$\mathcal{P}$-continuous" lifts of elements of $g \in G'$ (all such lifts differ by elements of $\mathcal{G}'^{\mathcal{P}}_{<p}$).

The above construction in the case $G' = \mathrm{Gal}(\mathcal{K}'/\mathcal{K})$ allows us to recover (uniquely up to isomorphism) the group $\mathcal{G}^{\mathcal{P}}_{<p}$ from $\mathcal{H}^{\mathcal{P}} = \Theta(\mathcal{G}'^{\mathcal{P}}_{<p})$. Even more, if $\mathcal{K} \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_{<p}$, $\mathcal{H}_i = \mathrm{Gal}(\mathcal{K}_{<p}/\mathcal{K}_i)$, $\mathcal{H}^{\mathcal{P}}_i = \mathcal{H}_i \cap \mathcal{G}^{\mathcal{P}}_{<p}$ with $i = 1, 2$, and $\mathcal{K}_2/K_1$ is Galois then $\mathcal{H}^{\mathcal{P}}_2 = \mathcal{H}_2 \cap \mathcal{G}^{\mathcal{P}}_{<p}$ is uniquely (up to isomorphism) recovered from $\mathcal{H}^{\mathcal{P}}_1$.

4. The groups $\mathcal{G}^{\mathcal{P}}_{\omega}$ and $\Gamma^{\mathcal{P}}_{\omega}$

As earlier, $\mathcal{K}$ is N-dimensional local field of characteristic p with fixed system of local parameters $t = \{t_1, \dots, t_N\}$ and the last residue field $k \simeq \mathbb{F}_{p^{N_0}}$. Fix $\alpha_0 \in k$ such that $\mathrm{Tr}_{k/\mathbb{F}_p}(\alpha_0) = 1$. Let $S = S_{t,\alpha_0}$ be the section from Prop. 3.2.

Take $e := e_S = \sum\limits_{a \in \mathbb{Z}^+_N(p)} t^{-a} D_{a\,0} + \alpha_{\bar{0}} D_{\bar{0}}$, choose $f \in \mathcal{F}(e)$ and consider $\pi = \pi_f(e) : \mathcal{G}_{<p} \simeq G(\mathcal{L})$.

Fix $\bar{c}^0 = (c^0_1, \dots, c^0_N) \in p\mathbb{Z}^N$ such that $c^0_1 > 0$, set $t^{\bar{c}^0} := t_1^{c^0_1} \dots t_N^{c^0_N}$. Choose $\omega = \sum\limits_{\iota \geqslant \bar{0}} \beta_\iota t^{(\bar{c}^0/p)+\iota} \in t^{\bar{c}_0/p} \mathcal{O}^*_{\mathcal{K}}$, where all $\beta_\iota = \beta_\iota(\omega) \in k$, $\beta_{\bar{0}} \neq 0$. Let $E(X) = \exp\left(\sum\limits_{i \geqslant 0} X^{p^i}/p^i\right)$ be the Artin-Hasse exponential.

4.1. **Automorphisms $h^{(m)}_{\omega}$.** For $1 \leqslant m \leqslant N$, let $h^{(m)}_{\omega}$ be the $\mathcal{P}$-continuous automorphism of $\mathcal{K}$ such that $h^{(m)}_{\omega}|_k = \mathrm{id}$, $h^{(m)}_{\omega}(t_m) = t_m E(\omega^p)$ and for all $j \neq m$, $h^{(m)}_{\omega}(t_j) = t_j$. Let $\mathrm{m}_{\mathcal{K}}$ be the maximal ideal in $\mathcal{O}_{\mathcal{K}}$. Clearly, $\mathrm{m}_{\mathcal{K}}$ consists of all $\mathcal{P}$-convergent k-linear combinations of t^a, where $a \in \mathbb{Z}^N_{>\bar{0}}$.

For $n \in \mathbb{Z}$, let $h^{(m)n}_{\omega}$ be the n-th iteration of $h^{(m)}_{\omega}$ and, similarly, denote by $h^{(m_1)}_{\omega} h^{(m_2)}_{\omega}$ the composition of $h^{(m_1)}_{\omega}$ and $h^{(m_2)}_{\omega}$.

Proposition 4.1.

a) *For any $n \geqslant 0$, $h^{(m)n}_{\omega}(t_m) \equiv t_m E(n\,\omega^p) \bmod t^{p\bar{c}^0} \mathrm{m}_{\mathcal{K}}$;*

b) $h^{(m_1)}_{\omega} h^{(m_2)}_{\omega} \equiv h^{(m_2)}_{\omega} h^{(m_1)}_{\omega} \bmod t^{p\bar{c}^0} \mathrm{m}_{\mathcal{K}}$.

Proof. Note that $h_\omega^{(m)}(t_m) \equiv t_m \operatorname{mod} t^{\bar c^0}\mathrm{m}_{\mathcal K}$ and this implies for any $\iota \geqslant \bar 0$, that $h_\omega^{(m)}(t^{\bar c^0+p\iota}) \equiv t^{\bar c^0+p\iota} \operatorname{mod} t^{p\bar c^0}\mathrm{m}_{\mathcal K}$. As a result,

$$h_\omega^{(m)}(\omega^p) \equiv \omega^p \operatorname{mod} t^{p\bar c^0}\mathrm{m}_{\mathcal K}\,. \tag{4.1}$$

Apply induction on $n \geqslant 0$ to prove part a) of the proposition.

If it is proved for some $n \geqslant 0$ then

$$h_\omega^{(m)n+1}(t_m) \equiv h_\omega^{(m)}(t_m E(n\omega^p)) \equiv t_m E(\omega^p)E(n\omega^p) \equiv t_m E((n+1)\omega^p)$$

modulo $t^{p\bar c^0}\mathrm{m}_{\mathcal K}$ (use that $E(X+Y) \equiv E(X)E(Y) \operatorname{mod} \deg p$).

Similarly, relation (4.1) implies part b). □

Remark. The above proposition can be stated also for the truncated exponential $\widetilde{\exp}(X) = 1 + X + \dots + X^{p-1}/(p-1)!$ instead of $E(X)$.

4.2. **The groups $\mathcal G_\omega$ and $\mathcal G_\omega^{\mathcal P}$.** Let $\hat h_\omega^{(m)} \in \operatorname{Aut}\mathcal K_{<p}$ be such that $\hat h_\omega^{(m)}|_{\mathcal K} = h_\omega^{(m)}$. Denote by $\mathcal G_\omega$ the subgroup in $\operatorname{Aut}\mathcal K_{<p}$ generated by the elements of $\mathcal G_{<p}$ and the lifts $\hat h_\omega^{(m)}$ with $1 \leqslant m \leqslant N$. The elements $\hat h_\omega$ of $\mathcal G_\omega$ are characterised by the property $\hat h_\omega|_{\mathcal K} \in \langle h_\omega^{(1)}, \dots, h_\omega^{(N)}\rangle \subset \operatorname{Aut}\mathcal K$. They can be uniquely specified by the couples $(C, \mathcal A) \in \mathcal L_{\mathcal K} \times \operatorname{Aut}\mathcal L$ such that

$$\hat h_\omega(f) = C \circ (\mathcal A \otimes \mathrm{id})f$$

or, equivalently, such that (where $h_\omega = \hat h_\omega|_{\mathcal K}$)

$$(\mathrm{id}_{\mathcal L} \otimes h_\omega)e = \sigma C \circ (\mathcal A \otimes \mathrm{id})e \circ (-C)\,. \tag{4.2}$$

If $\hat h'_\omega$ corresponds to $(C', \mathcal A')$ then the composition $\hat h'_\omega \cdot \hat h_\omega$ corresponds to the couple $(h'_\omega(C) \circ \mathcal A(C'), \mathcal A\mathcal A')$, where $h'_\omega = \hat h'_\omega|_{\mathcal K}$. With this notation the subgroup $\mathcal G_{<p} \subset \mathcal G_\omega$ is identified with the subgroup of couples $(l, \operatorname{Ad} l)$, where $l \in \mathcal L$. Indeed, under the identification $\pi = \pi_f(e) : \mathcal G_{<p} \simeq G(\mathcal L)$ from Prop. 1.4, if $\tau \in \mathcal G_{<p}$ then $\tau(f) = f \circ l = l \circ (\operatorname{Ad} l \otimes \mathrm{id})f$.

Suppose $1 \leqslant m \leqslant N$ and $\hat h_\omega^{(m)}$ is specified via the couple $(C^{(m)}, \mathcal A^{(m)})$. Relation (4.2) can be treated via the following recurrent procedure.

Suppose $s \geqslant 1$ and the couple $(C_s^{(m)}, \mathcal A_s^{(m)})$ satisfies relation (4.2) modulo s-th commutators $C_s(\mathcal L_{\mathcal K})$. Use the operators $\mathcal R$ and $\mathcal S$ from Sect. 3.1 to obtain $C'_s \in C_{s+1}(\mathcal L_{\mathcal K})$ and $\mathcal A'_s \in \operatorname{Hom}_{\mathrm{Lie}}(\mathcal L, C_{s+1}\mathcal L)$ such that

$$\sigma C'_s - C'_s + (\mathcal A'_s \otimes \mathrm{id}_{\mathcal K})e \equiv$$
$$(\mathrm{id}_{\mathcal L} \otimes h_\omega^{(m)})e - \sigma C_s^{(m)} \circ (\mathcal A_s^{(m)} \otimes \mathrm{id}_{\mathcal K})e \circ (-C_s^{(m)}) \operatorname{mod} C_{s+1}(\mathcal L_{\mathcal K})\,.$$

Then the couple $(C_s + C'_s, \mathcal A_s + \mathcal A'_s)$ satisfies (4.2) modulo $C_{s+2}(\mathcal L_{\mathcal K})$.

Denote by $\hat h_\omega^{0(m)}$ the lift of $h_\omega^{(m)}$ which corresponds to the couple $(C^{0(m)}, \mathcal A^{0(m)}) := (C_p^{(m)}, \mathcal A_p^{(m)})$. Note that $C^{0(m)} \in \mathcal L_{\mathcal K}^{\mathcal P}$ and $\mathcal A^{0(m)}|_{\mathcal L^{\mathcal P}}$ is a $\mathcal P$-continuous automorphism of the Lie algebra $\mathcal L^{\mathcal P}$.

Using that $(\mathrm{id}_{\mathcal L} \otimes h_\omega^{(m)})e \in \mathcal L_{\mathcal K}^{\mathcal P}$ and $\mathcal L \cap \mathcal L_{\mathcal K}^{\mathcal P} = \mathcal L^{\mathcal P}$ we obtain the following property:

Proposition 4.2. *A lift $\hat{h}_\omega^{(m)}$ corresponds to a couple $(C^{(m)}, \mathcal{A}^{(m)})$ with $C^{(m)} \in \mathcal{L}_\mathcal{K}^\mathcal{P}$ and $\mathcal{A}^{(m)} \in \mathrm{Aut}_{\mathcal{P}\text{-}cont}(\mathcal{L}^\mathcal{P})$, if and only if there is $l \in \mathcal{L}^\mathcal{P}$ such that $C^{(m)} = C^{0(m)} \circ l$ and $\mathcal{A}^{(m)} = \mathrm{Ad}\, l \cdot \mathcal{A}^{0(m)}$.*

Definition. $\mathcal{G}_\omega^\mathcal{P} \subset \mathcal{G}_\omega$ is a subgroup generated by $\mathcal{G}_{<p}^\mathcal{P} = \pi^{-1}\mathcal{L}^\mathcal{P}$ and the lifts $\hat{h}_\omega^{0(m)}$, $1 \leqslant m \leqslant N$.

Remark. (i) The elements of the group $\mathcal{G}_\omega^\mathcal{P}$ are specified via the couples $(C, \mathcal{A}) \in \mathcal{L}_\mathcal{K}^\mathcal{P} \times \mathrm{Aut}_{\mathcal{P}\text{-cont}}\mathcal{L}^\mathcal{P}$ (which satisfy relation (4.2)).

(ii) The profinite completion of $\mathcal{G}_\omega^\mathcal{P}$ coincides with $\mathcal{G}_\omega$.

Obviously, we have the following natural short exact sequences:

$$1 \longrightarrow \mathcal{G}_{<p} \longrightarrow \mathcal{G}_\omega \longrightarrow \langle h_\omega^{(1)}, \dots, h_\omega^{(N)} \rangle \longrightarrow 1\,, \tag{4.3}$$

$$1 \longrightarrow \mathcal{G}_{<p}^\mathcal{P} \longrightarrow \mathcal{G}_\omega^\mathcal{P} \longrightarrow \langle h_\omega^{(1)}, \dots, h_\omega^{(N)} \rangle \longrightarrow 1\,, \tag{4.4}$$

The structure of (4.3) can be uniquely recovered from (4.4) by going to profinite completions.

4.3. **The commutator subgroups $C_s(\mathcal{G}_\omega^\mathcal{P})$.**

Define the weight function in $\mathcal{L}_k^\mathcal{P}$ by setting for $s \in \mathbb{N}$ and $(s-1)\bar{c}^0 \leqslant a < s\bar{c}^0$,

$$\mathrm{wt}(D_{an}) = s\,.$$

Introduce the ideal $\mathcal{L}_{\bar{c}_0}^\mathcal{P}(s)$ of $\mathcal{L}^\mathcal{P}$ such that $\mathcal{L}_{\bar{c}^0}^\mathcal{P}(s)_k$ is generated by all $[\dots[D_{a_1n_1}, D_{a_2n_2}], \dots, D_{a_rn_r}]$ with $\sum_i \mathrm{wt}(D_{a_in_i}) \geqslant s$. Clearly, for any s_1, s_2, it holds $[\mathcal{L}_{\bar{c}^0}^\mathcal{P}(s_1), \mathcal{L}_{\bar{c}^0}^\mathcal{P}(s_2)] \subset \mathcal{L}_{\bar{c}^0}^\mathcal{P}(s_1+s_2)$.

Consider the lifts $h_\omega^{0(m)} \in \mathcal{G}_\omega^\mathcal{P}$ from Sect. 4.2. Denote by $\mathrm{Ad}^{(m)}$ the automorphism of $G(\mathcal{L}^\mathcal{P})$ obtained from conjugation by $\hat{h}_\omega^{0(m)}$ on $\mathcal{G}_{<p}^\mathcal{P}$ with respect to the identification $\pi (= \pi_f(e)) : \mathcal{G}_{<p}^\mathcal{P} \simeq G(\mathcal{L}^\mathcal{P})$.

Let for $a \in \mathbb{Z}_N^+(p)$, $\mathrm{Ad}_k^{(m)}(D_{a\,0}) = D_{a\,0}^{(m)}$ and $\mathrm{Ad}_k^{(m)}(D_{\bar{0}}) = D_{\bar{0}}^{(m)}$.

Lemma 4.3. *For any $1 \leqslant m, m' \leqslant N$,*

a) *$D_{\bar{0}}^{(m)} \equiv D_{\bar{0}} \bmod \mathcal{L}_{\bar{c}^0}^\mathcal{P}(3) + \mathcal{L}_{\bar{c}^0}^\mathcal{P}(2) \cap C_2(\mathcal{L}^\mathcal{P})$;*

b) *if $a = (\bar{a}^{(1)}, \dots, \bar{a}^{(N)}) \in \mathbb{Z}_N^+(p)$ and $\mathrm{wt}(D_{a\,n}) = s$ then*

$$D_{a\,0}^{(m)} \equiv D_{a\,0} - \sum_{\imath \geqslant \bar{0}} A_\imath a^{(m)} D_{a+\bar{c}^0+p\imath,0}$$

modulo $\mathcal{L}_{\bar{c}_0}^\mathcal{P}(s+2)_k + \mathcal{L}_{\bar{c}_0}^\mathcal{P}(s+1)_k \cap C_2(\mathcal{L}_k^\mathcal{P})$, where the elements $A_\iota \in k$ are such that $E(\omega^p) = 1 + \sum_{\iota \geqslant \bar{0}} A_\iota t^{\bar{c}^0 + p\iota}$;

c) *the commutator $(\hat{h}_\omega^{0(m)}, \hat{h}_\omega^{0(m_1)}) \in \pi^{-1} G(\mathcal{L}_{\bar{c}_0}^\mathcal{P}(2))$.*

We shall prove this lemma after finishing the proof of Prop. 4.4 below.

Note that lemma 4.3 implies $\pi C_2(\mathcal{G}_\omega^{\mathcal{P}})\subset G(\mathcal{L}_{\bar{c}_0}^{\mathcal{P}}(2))$.

Set $\mathcal{L}_\omega^{\mathcal{P}}(1)=\mathcal{L}^{\mathcal{P}}$.

For $s\geqslant 2$, let $\mathcal{L}_\omega^{\mathcal{P}}(s)\subset\mathcal{L}^{\mathcal{P}}$ be such that $\pi C_s(\mathcal{G}_\omega^{\mathcal{P}})=G(\mathcal{L}_\omega^{\mathcal{P}}(s))$.

Proposition 4.4. *For $1\leqslant s\leqslant p$, $\mathcal{L}_\omega^{\mathcal{P}}(s)=\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s)$.*

Proof. Use induction on $s\geqslant 1$.

Clearly, $\mathcal{L}_\omega^{\mathcal{P}}(1)=\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(1)$.

Suppose $s_0\geqslant 1$ and for $1\leqslant s\leqslant s_0$, $\mathcal{L}_\omega^{\mathcal{P}}(s)=\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s)$.

Let $\mathcal{L}_{\mathrm{lin}}^{\mathcal{P}}=\left(\sum\limits_{a,n}kD_{a\,n}\right)\cap\mathcal{L}^{\mathcal{P}}$ be "the subspace of linear terms" in $\mathcal{L}^{\mathcal{P}}$.

We have the following properties for all $s\leqslant s_0$:

– $\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s+1)=\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s+1)\cap\mathcal{L}_{\mathrm{lin}}^{\mathcal{P}}+\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s+1)\cap C_2(\mathcal{L}^{\mathcal{P}})$;

– $\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s+1)\cap C_2(\mathcal{L}^{\mathcal{P}})=\sum\limits_{s_1+s_2=s+1}\left[\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s_1),\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s_2)\right]$;

– $\mathcal{L}_\omega^{\mathcal{P}}(s+1)$ is the ideal in $\mathcal{L}^{\mathcal{P}}$ generated by $[\mathcal{L}_\omega^{\mathcal{P}}(s),\mathcal{L}^{\mathcal{P}}]$ and the elements $\mathrm{Ad}^{(m)}(l)\circ(-l)$, where $l\in\mathcal{L}_\omega^{\mathcal{P}}(s)$ and $1\leqslant m\leqslant N$. (If $s_0=1$ we do need part c) of Lemma 4.3.)

Now statements a) and b) of Lemma 4.3 imply:

(c1) *if $l\in\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s)$ then $\mathrm{Ad}^{(m)}(l)\circ(-l)\in\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s+1)$;*

(c2) *if $l\in\mathcal{L}_{\mathrm{lin}}^{\mathcal{P}}\cap\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s+1)$ then there are m and $l'\in\mathcal{L}_{\mathrm{lin}}^{\mathcal{P}}\cap\mathcal{L}(s)$ such that*

$$\mathrm{Ad}^{(m)}(l')\circ(-l')\equiv l\,\mathrm{mod}\,\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s+1)\cap C_2(\mathcal{L}^{\mathcal{P}})$$

(*use that $A_{\bar{0}}\neq 0$ and for any $a=(a^{(1)},\dots,a^{(N)})\in\mathbb{Z}_N^+(p)$, there is m such that $a^{(m)}\not\equiv 0\,\mathrm{mod}\,p$*).

Then $[\mathcal{L}_\omega^{\mathcal{P}}(s_0),\mathcal{L}^{\mathcal{P}}]=[\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s_0),\mathcal{L}^{\mathcal{P}}(1)]\subset\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s_0+1)$ and applying (c1) we obtain $\mathcal{L}_\omega^{\mathcal{P}}(s_0+1)\subset\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s_0+1)$.

For the opposite direction, note that by the inductive assumption,

$$\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s_0+1)\cap C_2(\mathcal{L}^{\mathcal{P}})=\sum_{s_1+s_2=s_0+1}\left[\mathcal{L}_\omega^{\mathcal{P}}(s_1),\mathcal{L}_\omega^{\mathcal{P}}(s_2)\right]\subset\mathcal{L}_\omega^{\mathcal{P}}(s_0+1)$$

and then (c2) implies that $\mathcal{L}_{\mathrm{lin}}^{\mathcal{P}}\cap\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s_0+1)\subset\mathcal{L}_\omega^{\mathcal{P}}(s_0+1)$. As a result, $\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s_0+1)\subset\mathcal{L}_\omega^{\mathcal{P}}(s_0+1)$ and our proposition is proved. □

Proof of Lemma 4.3. Let

$$\mathcal{N}=\sum_{s\geqslant 1}t^{-\bar{c}^0 s}\mathcal{L}_{\bar{c}^0}^{\mathcal{P}}(s)_{\mathrm{m}_{\mathcal{K}}},$$

where $\mathrm{m}_{\mathcal{K}}$ is the maximal ideal of the N-valuation ring $\mathcal{O}_{\mathcal{K}}$ of $\mathcal{K}$. Clearly, $\mathcal{N}$ has an induced structure of a Lie algebra over $\mathbb{F}_p$ and $e\in\mathcal{N}$.

Let $e^{(m)}:=(\mathrm{Ad}_k^{(m)}\otimes\mathrm{id}_{\mathcal{K}})e=\sum\limits_{a\in\mathbb{Z}_N^+(p)}t^{-a}D_{a\,0}^{(m)}+\alpha_{\bar{0}}D_{\bar{0}}^{(m)}$.

The recovering of $C^{0(m)}\in G(\mathcal{L}^{\mathcal{P}}_{\mathcal{K}})$ and $e^{(m)}$ from relation

$$(\mathrm{id}_{\mathcal{L}^{\mathcal{P}}}\otimes h^{(m)}_{\omega})e\circ C^{0(m)}=(\sigma C^{0(m)})\circ e^{(m)}\,, \tag{4.5}$$

is a part of the recurrent procedure from Sect. 4.2. Clearly, the operators $\mathcal{S}$ and $\mathcal{R}$ from Sect. 3.1 map $\mathcal{N}$ to itself. Therefore, when following the recurrent procedure we remain at each step in $\mathcal{N}$. As a result, all $e^{(m)}, C^{0(m)}, \sigma C^{0(m)}\in\mathcal{N}$.

For any $j\geqslant 0$, introduce the ideals $\mathcal{N}(j):=t^{\bar c^0 j}\mathcal{N}$ of $\mathcal{N}$. The operators $\mathcal{R}$ and $\mathcal{S}$ also map the ideals $\mathcal{N}(j)$ to itself.

The following properties are obtained by direct calculations:

(i) $(\mathrm{id}_{\mathcal{L}^{\mathcal{P}}}\otimes h^{(m)}_{\omega})e=e+e^{(m)}_1 \bmod \mathcal{N}(2)$, $e^{(m)}_1=e^{(m)+}_1+e^{(m)-}_1\in\mathcal{N}(1)$,

$$e^{(m)-}_1=-\sum_{\substack{\iota\geqslant\bar 0\\ a\in\mathbb{Z}^+_N(p)}} t^{-a}a^{(m)}A_\iota D_{a+\bar c^0+p\iota,0},\quad e^{(m)+}_1=-\sum_{\substack{\iota\geqslant\bar 0\\ \bar 0<a<\bar c^0+p\iota}} a^{(m)}A_\iota t^{-a+\bar c^0+p\iota}D_{a\,0}$$

(note that $e^{(m)+}_1\in\mathcal{L}^{\mathcal{P}}_{\mathrm{m}\mathcal{K}}$ and, therefore, $\mathcal{S}(e^{(m)+}_1)=0$);

(ii) the congruence $(\mathrm{id}_{\mathcal{L}^{\mathcal{P}}}\otimes h^{(m)}_{\omega})e\equiv e \bmod \mathcal{N}(1)$ implies that $e^{(m)}\equiv e \bmod \mathcal{N}(1)$ and $C^{0(m)},\sigma C^{0(m)}\in\mathcal{N}(1)$. Indeed, in the procedure of specification of $\hat h^{0(m)}_{\omega}$ it holds that for all s, $C^{(m)}_s,\sigma C^{(m)}_s\in\mathcal{N}(1)$ and $(\mathcal{A}^{(m)}_s\otimes \mathrm{id}_{\mathcal{K}})e\equiv e \bmod \mathcal{N}(1)$;

(iii) $e^{(m)}=(-\sigma C^{0(m)})\circ(\mathrm{id}_{\mathcal{L}^{\mathcal{P}}}\otimes h^{(m)}_{\omega})e\circ C^{0(m)}\equiv(C^{0(m)}-\sigma C^{0(m)})+e+e^{(m)}_1 \bmod \mathcal{N}(2)+t^{\bar c^0}\widetilde{\mathcal{N}}^{(2)}$, where $\widetilde{\mathcal{N}}^{(2)}:=\sum\limits_{s\geqslant 2}t^{-s\bar c^0}(\mathcal{L}^{\mathcal{P}}_{\bar c^0}(s)\cap C_2(\mathcal{L}^{\mathcal{P}}))_{\mathrm{m}\mathcal{K}}$ (use that $[\mathcal{N}(1),\mathcal{N}(1)]\subset\mathcal{N}(2)$ and $[\mathcal{N}(1),\mathcal{N}]\subset t^{\bar c^0}\widetilde{\mathcal{N}}^{(2)}$);

(iv) $\mathcal{S}(\mathcal{N}(2)+t^{\bar c^0}\widetilde{\mathcal{N}}^{(2)})\subset\mathcal{N}(2)+t^{\bar c^0}\widetilde{\mathcal{N}}^{(2)}$, $\mathcal{S}(e^{(m)}-e-e^{(m)-}_1)=e^{(m)}-e-e^{(m)-}_1$, $\mathcal{S}(C^{(m)}-\sigma C^{(m)}+e^{(m)+}_1)=0$. Therefore, item (iii) implies

$$e^{(m)}\equiv e+e^{(m)-}_1 \bmod \mathcal{N}(2)+t^{\bar c^0}\widetilde{\mathcal{N}}^{(2)}\,.$$

More explicitly,

$$e^{(m)}\equiv\sum_{a\in\mathbb{Z}^+_N(p)}t^{-a}\left(D_{a\,0}-a^{(m)}\sum_{\iota\geqslant\bar 0}A_\iota D_{a+\bar c^0+p\iota,0}\right)+\alpha_{\bar 0}D_{\bar 0} \tag{4.6}$$

modulo $\mathcal{N}(2)+t^{\bar c^0}\widetilde{\mathcal{N}}^{(2)}$.

Deduce from this congruence statements a) and b) of our lemma. Consider the presentation of an element $l\in\mathcal{L}^{\mathcal{P}}_{\mathcal{K}}$ in the form of a $\mathcal{P}$-convergent series $l=\sum\limits_{b\in\mathbb{Z}_N}t^b l_b$, with all $l_b\in\mathcal{L}^{\mathcal{P}}_k$.

Suppose $s\geqslant 1$ and $-(s-1)\bar c^0\geqslant b>-s\bar c^0$.

It follows directly from definitions that:

— if $l\in\mathcal{N}$ then $l_b\in\mathcal{L}^{\mathcal{P}}_{\bar c^0}(s)_k$;

— if $l\in\mathcal{N}(2)$ then $l_b\in\mathcal{L}^{\mathcal{P}}_{\bar c^0}(s+2)_k$;

— if $l\in t^{\bar c^0}\widetilde{\mathcal N}^{(2)}$ then $l_b\in\mathcal L^{\mathcal P}_{\bar c^0}(s+1)_k\cap C_2(\mathcal L^{\mathcal P}_k)$.

As a result, the parts a) and b) of lemma are obtained by comparing coefficients in (4.6).

Now note that for any m_1,

$$(\mathrm{id}_{\mathcal L^{\mathcal P}}\otimes h^{(m_1)}_\omega)e\equiv e+e^{(m_1)}_1 \operatorname{mod}\mathcal N(2)\,. \tag{4.7}$$

Let $\mathcal N_{<p}=\sum_{s\geqslant 1}t^{-\bar c^0s}\mathcal L^{\mathcal P}_{\bar c^0}(s)_{\mathrm m_{<p}}$, where $\mathrm m_{<p}$ is the maximal ideal of the N-valuation ring $\mathcal O_{\mathcal K_{<p}}$. Again, $\mathcal N_{<p}$ has the induced structure of a Lie $\mathbb F_p$-algebra and for any $j\geqslant 0$, $\mathcal N_{<p}(j)=t^{j\bar c_0}\mathcal N_{<p}$ is ideal in $\mathcal N_{<p}$.

As earlier, $f,\sigma f\in\mathcal N_{<p}$, and congruence (4.7) implies that

$$(\mathrm{id}_{\mathcal L^{\mathcal P}}\otimes h^{(m_1)}_\omega h^{(m))}_\omega)e\equiv e+e^{(m)}_1+e^{(m_1)}_1 \operatorname{mod}\mathcal N_{<p}(2)\,.$$

where $f^{(m_1)}_1\in\mathcal N_{<p}(1)$ is such that $\sigma f^{(m_1)}_1-f^{(m_1)}_1=e^{(m_1)}_1$.

This implies

$$(\mathrm{id}_{\mathcal L^{\mathcal P}}\otimes\hat h^{0(m)}_\omega\hat h^{0(m_1)}_\omega)f\equiv(\mathrm{id}_{\mathcal L^{\mathcal P}}\otimes\hat h^{0(m_1)}_\omega\hat h^{0(m)}_\omega)f\equiv f+f^{(m_1)}_1+f^{(m)}_1 \operatorname{mod}\mathcal N_{<p}(2)$$

and, therefore, $(\mathrm{id}_{\mathcal L^{\mathcal P}}\otimes(\hat h^{0(m)}_\omega,\hat h^{0(m_1)}_\omega))f\equiv f\operatorname{mod}\mathcal N_{<p}(2)$.

On the other hand, the commutator $(\hat h^{0(m)}_\omega,\hat h^{0(m_1)}_\omega)$ is a lift of $\mathrm{id}_{\mathcal K}$, i.e. it coincides with $\pi^{-1}(l_{mm_1})\in\mathcal G^{\mathcal P}_{<p}$. Therefore, $l_{mm_1}\in\mathcal L^{\mathcal P}\cap\mathcal N_{<p}(2)=\mathcal L^{\mathcal P}_{\bar c^0}(2)$. The part c) is proved. $\square$

4.4. **The group $\Gamma^{\mathcal P}_\omega$.** Let $\bar{\mathcal L}=\mathcal L/\mathcal L_{\bar c_0}(p)$ and $\bar{\mathcal L}^{\mathcal P}=\mathcal L^{\mathcal P}/\mathcal L^{\mathcal P}_{\bar c_0}(p)$. Then $\bar{\mathcal L}^{\mathcal P}$ is dense in $\bar{\mathcal L}$. If $\mathcal K(p)=\mathcal K_{<p}^{G(\mathcal L_{\bar c^0}(p))}$ then $\bar{\mathcal G}:=\mathrm{Gal}(\mathcal K(p)/\mathcal K)$ and the identification $\pi_f(e)$ induces the identification $\bar\pi:\bar{\mathcal G}\simeq G(\bar{\mathcal L})$. This identification can be obtained via nilpotent Artin-Schreier theory: for $\bar e\in\bar{\mathcal L}_{\mathcal K}$ and $\bar f\in\bar{\mathcal L}_{\mathcal K(p)}$, we have $\sigma\bar f=\bar e\circ\bar f$ and $\bar\pi=\pi_{\bar f}(\bar e)$. However, the algebra $\bar{\mathcal L}_{\mathcal K}$ is too big for the process of linearization, cf. below. This motivates the following definitions.

Let

$$\mathcal M:=\sum_{1\leqslant s<p}t^{-s\bar c^0}\mathcal L_{\bar c^0}(s)_{\mathrm m_{\mathcal K}}+\mathcal L_{\bar c^0}(p)_{\mathcal K}$$

$$\mathcal M^{\mathcal P}:=\sum_{1\leqslant s<p}t^{-s\bar c^0}\mathcal L^{\mathcal P}_{\bar c^0}(s)_{\mathrm m_{\mathcal K}}+\mathcal L^{\mathcal P}_{\bar c^0}(p)_{\mathcal K}$$

$$\mathcal M_{<p}:=\sum_{1\leqslant s<p}t^{-s\bar c^0}\mathcal L_{\bar c^0}(s)_{\mathrm m_{<p}}+\mathcal L_{\bar c^0}(p)_{\mathcal K_{<p}}$$

where (as earlier) $\mathrm m_{<p}$ is the maximal ideal of the N-valuation ring $\mathcal O_{\mathcal K_{<p}}$. Then $\mathcal M$ and $\mathcal M^{\mathcal P}$ have the induced structure of Lie $\mathbb F_p$-algebras (use the Lie bracket from $\mathcal L_{\mathcal K}$). For $s\geqslant 0$, $\mathcal M(s):=t^{s\bar c^0}\mathcal M$ and $\mathcal M^{\mathcal P}(s):=t^{s\bar c^0}\mathcal M^{\mathcal P}$ form a decreasing central filtrations of ideals in $\mathcal M$ and $\mathcal M^{\mathcal P}$. Similarly, $\mathcal M_{<p}$ is a Lie $\mathbb F_p$-algebra (containing $\mathcal M$ as its subalgebra), for $s\geqslant 0$, $\mathcal M_{<p}(s):=t^{s\bar c^0}\mathcal M_{<p}$ is a decreasing central filtration of ideals in $\mathcal M_{<p}$ and $\mathcal M_{<p}(s)\cap\mathcal M^{\mathcal P}=\mathcal M^{\mathcal P}(s)$. It can be easily seen that $e\in\mathcal M^{\mathcal P}$ and $f,\sigma f\in\mathcal M_{<p}$.

There is a natural embedding

$$\bar{\mathcal{M}}^{\mathcal{P}} := \mathcal{M}^{\mathcal{P}}/\mathcal{M}^{\mathcal{P}}(p-1) \subset \bar{\mathcal{M}}_{<p} := \mathcal{M}_{<p}/\mathcal{M}_{<p}(p-1)\,,$$

and the induced decreasing filtrations of ideals $\bar{\mathcal{M}}^{\mathcal{P}}(s)$ and $\bar{\mathcal{M}}_{<p}(s)$ (where $\bar{\mathcal{M}}^{\mathcal{P}}(p-1) = \bar{\mathcal{M}}_{<p}(p-1) = 0$) are compatible with this embedding. For all $s \geqslant 0$, $(\mathrm{id}_{\bar{\mathcal{L}}} \otimes h_\omega^{(m)} - \mathrm{id}_{\bar{\mathcal{M}}^{\mathcal{P}}})^s \bar{\mathcal{M}}^{\mathcal{P}} \subset \bar{\mathcal{M}}^{\mathcal{P}}(s)$.

The algebras $\bar{\mathcal{M}}^{\mathcal{P}}$ and $\bar{\mathcal{M}}_{<p}$ are naturally identified with the following subquotients of $\bar{\mathcal{L}}^{\mathcal{P}}_{\mathcal{K}}$ and $\bar{\mathcal{L}}_{<p}$:

$$\bar{\mathcal{M}}^{\mathcal{P}} = \left(\sum_{1\leqslant s<p} t^{-s\bar{c}^0}\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}(s)_{\mathrm{m}}\right) \otimes \mathcal{O}_{\mathcal{K}}/t^{(p-1)\bar{c}^0}$$

$$\bar{\mathcal{M}}_{<p} = \left(\sum_{1\leqslant s<p} t^{-s\bar{c}^0}\bar{\mathcal{L}}_{\bar{c}^0}(s)_{\mathrm{m}_{<p}}\right) \otimes \mathcal{O}_{\mathcal{K}_{<p}}/t^{(p-1)\bar{c}^0}\,.$$

We can see easily that $\bar{e}\otimes 1 \in \bar{\mathcal{M}}^{\mathcal{P}}$, $\bar{f}\otimes 1, \sigma\bar{f}\otimes 1 \in \bar{\mathcal{M}}_{<p}$ and $\sigma\bar{f}\otimes 1 = (\bar{e}\otimes 1)\circ(\bar{f}\otimes 1)$. The following property shows that we still have full control of the identification $\bar{\pi}$.

Proposition 4.5. *The correspondence* $\tau \mapsto (-\bar{f}\otimes 1)\circ\tau(\bar{f}\otimes 1)$ *induces the natural projections* $\mathcal{G}_{<p} \longrightarrow \bar{\mathcal{G}} \simeq G(\bar{\mathcal{L}})$ *and* $\mathcal{G}^{\mathcal{P}}_{<p} \longrightarrow \bar{\mathcal{G}}^{\mathcal{P}} \simeq G(\bar{\mathcal{L}}^{\mathcal{P}})$.

Proof. $(-\bar{f}\otimes 1)\circ\tau\bar{f}\otimes 1$ comes from $(-f)\circ\tau f \in G(\mathcal{L})$. It remains to notice that $\mathcal{L}\cap\mathcal{M}(p-1) = \mathcal{L}_{\bar{c}_0}(p)$ and $\mathcal{L}\cap\mathcal{M}^{\mathcal{P}}(p-1) = \mathcal{L}^{\mathcal{P}}_{\bar{c}_0}(p)$. □

Remark. In the above setting we can replace $\bar{\mathcal{M}}_{<p}$ by its analogue $\bar{\mathcal{M}}_{\mathcal{K}(p)}$, where the field $\mathcal{K}(p)$ is used instead of $\mathcal{K}_{<p}$ (because $\bar{f} \in \bar{\mathcal{L}}_{\mathcal{K}(p)}$).

Let $\Gamma^{\mathcal{P}}_\omega := \mathcal{G}^{\mathcal{P}}_\omega/(\mathcal{G}^{\mathcal{P}}_\omega)^p C_p(\mathcal{G}^{\mathcal{P}}_\omega)$. Then $\Gamma_\omega := \mathcal{G}_\omega/\mathcal{G}^p_\omega C_p(\mathcal{G}_\omega)$ can be recovered as the pro-finite completion of $\Gamma^{\mathcal{P}}_\omega$.

Proposition 4.6. *Exact sequence* (4.3) *induces the following exact sequence of profinite p-groups*

$$1 \longrightarrow \bar{\mathcal{G}}^{\mathcal{P}} \longrightarrow \Gamma^{\mathcal{P}}_\omega \longrightarrow \prod_{1\leqslant m\leqslant N} \langle h_\omega^{(m)}\rangle/\langle h_\omega^{(m)p}\rangle \longrightarrow 1\,.$$

Proof. Consider the orbit of $\bar{f}\otimes 1$ with respect to the natural action of $\mathcal{G}_\omega \subset \mathrm{Aut}\,\mathcal{K}_{<p}$ on f (recall that all "values" of f belong to $\mathcal{K}_{<p} \subset \mathcal{K}_{sep}$). Then the stabilizer $\mathcal{H}$ of $\bar{f}\otimes 1$ equals $\mathcal{G}^p_\omega C_p(\mathcal{G}_\omega)$. This fact and the remaining part of the proof go along the lines of Prop. 3.5 from [11]. □

Suppose $\pi_\omega : \Gamma^{\mathcal{P}}_\omega \simeq G(L^{\mathcal{P}}_\omega)$ extends $\bar{\pi} = \pi_{\bar{f}}(\bar{e})$ for a suitable Lie $\mathbb{F}_p$-algebra $L^{\mathcal{P}}_\omega$ containing $\bar{\mathcal{L}}^{\mathcal{P}}$. Then the automorphisms $h_\omega^{(m)}$ give rise to Lie elements $l_\omega^{(m)}$ and we obtain the following property.

Corollary 4.7. *There is a natural exact sequence of Lie* $\mathbb{F}_p$*-algebras*

$$0 \longrightarrow \bar{\mathcal{L}}^{\mathcal{P}} \longrightarrow L^{\mathcal{P}}_\omega \longrightarrow \prod_{1\leqslant m\leqslant N} \mathbb{F}_p l_\omega^{(m)} \longrightarrow 0\,. \tag{4.8}$$

We recover the structure of $L^{\mathcal{P}}_\omega$ below by analyzing the orbit of $\bar{f}$.

4.5. **Filtered module $\bar{\mathcal{M}}^f$ and the procedure of linearization.** Let $\bar{h}_\omega^{(m)}\in\Gamma_\omega^{\mathcal{P}}$ be a lift of $h_\omega^{(m)}$ to $\mathcal{K}(p)$. We use below the notation $\bar{l}_\omega^{(m)}$ for the corresponding element $\pi_\omega(\bar{h}_\omega^{(m)})\in L_\omega^{\mathcal{P}}$. For example, cf. Sect. 4.3, if $\bar{l}_\omega^{0(m)}=\pi_\omega(\hat{h}_\omega^{0(m)})|_{\mathcal{K}(p)})$ then the notation $\mathrm{Ad}\,^{(m)}$ appears as $\mathrm{Ad}\,(\bar{l}_\omega^{0(m)})$.

Let $\Gamma_\omega^{(m)\mathcal{P}}$ be a subgroup in $\Gamma_\omega^{\mathcal{P}}$ generated by $\bar{h}_\omega^{(m)}$ and $\bar{\mathcal{G}}^{\mathcal{P}}=\bar{\pi}^{-1}G(\bar{\mathcal{L}}^{\mathcal{P}})$ (clearly, it does not depend on the choice of $\bar{h}_\omega^{(m)}$). Then we have the following exact sequence

$$1\longrightarrow\bar{\mathcal{G}}^{\mathcal{P}}\longrightarrow\Gamma_\omega^{(m)\mathcal{P}}\longrightarrow\langle h_\omega^{(m)}\rangle/\langle h_\omega^{(m)p}\rangle\longrightarrow 1\,.$$

Let $L_\omega^{(m)\mathcal{P}}$ be a Lie subalgebra in $L_\omega^{\mathcal{P}}$ such that $\pi_\omega(\Gamma_\omega^{(m)\mathcal{P}})=G(L_\omega^{(m)\mathcal{P}})$. We obtain the following exact sequence of $\mathbb{F}_p$-Lie algebras

$$0\longrightarrow\bar{\mathcal{L}}^{\mathcal{P}}\longrightarrow L_\omega^{(m)\mathcal{P}}\longrightarrow\mathbb{F}_p l_\omega^{(m)}\longrightarrow 0.$$

obtained from (4.8) via the natural embedding $\mathbb{F}_p l_\omega^{(m)}\longrightarrow\prod\limits_{1\leqslant m\leqslant N}\mathbb{F}_p l_\omega^{(m)}$.

The structure of the Lie algebras $L_\omega^{(m)\mathcal{P}}$ (as well as the groups $\Gamma_\omega^{(m)\mathcal{P}}$) can be studied via the "linearization techniques" from [13, 12].

Namely, the action of $\mathrm{id}_{\bar{\mathcal{L}}^{\mathcal{P}}}\otimes h_\omega^{(m)}$ on $\bar{\mathcal{L}}_{\mathcal{K}}^{\mathcal{P}}$ induces the action on $\bar{\mathcal{M}}^{\mathcal{P}}$, which can be presented in the form $\widetilde{\exp}(\mathrm{id}_{\bar{\mathcal{L}}^{\mathcal{P}}}\otimes dh_\omega^{(m)})$, where $\mathrm{id}_{\bar{\mathcal{L}}^{\mathcal{P}}}\otimes dh_\omega^{(m)}$ is a derivation on $\bar{\mathcal{M}}^{\mathcal{P}}$. Indeed, the elements of $\bar{\mathcal{M}}^{\mathcal{P}}$ can be written uniquely as sums of elements of the form $l\otimes t^{-a}$, where for some $1\leqslant s<p$, $(s-1)\bar{c}^0\leqslant a<s\bar{c}^0$ and $l\in\bar{\mathcal{L}}^{\mathcal{P}}(s)_k$. Then this derivation comes from the correspondences $l\otimes t^{-a}\mapsto l\otimes(-a^{(m)})t^{-a}\omega^p$.

Let $\bar{\mathcal{M}}^f$ be the minimal Lie subalgebra in $\bar{\mathcal{M}}_{\mathcal{K}(p)}$ obtained by joining to $\bar{\mathcal{M}}^{\mathcal{P}}$ the element $\bar{f}$ and its images under the action of the group generated by $\mathrm{id}_{\bar{\mathcal{L}}}\otimes\bar{h}_\omega^{(m)}$. The algebra $\bar{\mathcal{M}}^f$ still reflects all essential information about the structure of Γ_ω. Then

$$(\mathrm{id}_{\bar{\mathcal{L}}}\otimes\bar{h}_\omega^{(m)})\bar{f}=\bar{C}^{(m)}\circ(\bar{\mathcal{A}}^{(m)}\otimes\mathrm{id}_{\mathcal{K}(p)})\bar{f}\,,$$

where $\bar{C}^{(m)}\in\bar{\mathcal{M}}^{\mathcal{P}}$ and $\bar{\mathcal{A}}^{(m)}=\mathrm{Ad}\,\bar{l}_\omega^{(m)}\in\mathrm{Aut}\bar{\mathcal{L}}^{\mathcal{P}}$. So, this relation determines the action of the lift $\mathrm{id}_{\bar{\mathcal{L}}}\otimes\bar{h}_\omega^{(m)}$ on $\bar{\mathcal{M}}^f$.

For any $n\geqslant 1$, we have $(\mathrm{id}\otimes\bar{h}_\omega^{(m)n})\,\bar{f}=\bar{C}^{(m,n)}\circ(\bar{\mathcal{A}}^{(m)n}\otimes\mathrm{id})\bar{f}$, where the element $\bar{C}^{(m,n)}\in\bar{\mathcal{M}}^{\mathcal{P}}$ can be presented in the following form

$$(\mathrm{id}\otimes h_\omega^{(m)n-1})\bar{C}^{(m)}\circ(\bar{\mathcal{A}}^{(m)}\otimes h_\omega^{(m)n-2})\bar{C}^{(m)}\circ\ldots\circ(\bar{\mathcal{A}}^{(m)n-1}\otimes\mathrm{id})\bar{C}^{(m)}\,.$$

Let $\bar{c}_i^{(m)}\in\bar{\mathcal{M}}^{\mathcal{P}}$ be such that for all $1\leqslant n<p$, $\bar{C}^{(m,n)}=\sum\limits_{1\leqslant i<p}n^i\,\bar{c}_i^{(m)}$. (Such elements $\bar{c}_i^{(m)}$ are unique because $\det((n^i))_{1\leqslant n,i<p}\neq 0\bmod p$.) Summarizing our approach from [13, 12] we obtain:

Proposition 4.8.

a) $\bar{\mathcal{A}}^{(m)}=\widetilde{\exp}\,\bar{\mathcal{B}}^{(m)}$, *where* $\bar{\mathcal{B}}^{(m)}$ *is a derivation on* $\bar{\mathcal{L}}^{\mathcal{P}}$*;*

b) *if* $\alpha_p := \operatorname{Spec}\mathbb{F}_p[U]$ *with* $U^p = 0$*, is a finite group scheme with coaddition* $\Delta U = U\otimes 1 + 1\otimes U$ *then the correspondence*

$$\bar h^{(m)U} : \bar f \mapsto (U\otimes \bar c_1^{(m)} + \dots + U^{p-1}\otimes \bar c_{p-1}^{(m)})\circ(\bar{\mathcal{A}}^{(m)U}\otimes \mathrm{id})\bar f =$$

$$= (U\otimes \bar c_1^{(m)} + \dots + U^{p-1}\otimes \bar c_{p-1}^{(m)})\circ\left(\sum_{0\leqslant n<p} U^n\otimes(\bar{\mathcal{B}}^n/n!\otimes \mathrm{id})\right)\bar f$$

induces a coaction of α_p *on* $\bar{\mathcal{M}}^f$*;*

c) $\bar h_\omega^{(m)n}(\bar f) = \bar h_\omega^{(m)U}|_{U=n}$*;*

d) *if for all* $a\in\mathbb{Z}_N^0(p)$, $V_{a\,0}^{(m)} := \operatorname{ad}\bar l_\omega^{(m)}(D_{a\,0})$ *then*

$$\sigma\bar c_1^{(m)} - \bar c_1^{(m)} + \sum_{a\in\mathbb{Z}_N^0(p)} t^{-a}V_{a\,0}^{(m)} =$$

$$-\sum_{1\leqslant k<p}\frac{1}{k!}t^{-(a_1+\dots+a_k)}\omega^p a_1^{(m)}[\dots[D_{a_1\,0},D_{a_2\,0}],\dots,D_{a_k\,0}]$$

$$-\sum_{2\leqslant k<p}\frac{1}{k!}t^{-(a_1+\dots+a_k)}[\dots[V_{a_1\,0},D_{a_2\,0}],\dots,D_{a_k\,0}]$$

$$-\sum_{1\leqslant k<p}\frac{1}{k!}t^{-(a_1+\dots+a_k)}[\dots[\sigma\bar c_1^{(m)},D_{a_10}],\dots,D_{a_k0}] \tag{4.9}$$

(the indices $a_1,\dots,a_k$ *in all above sums run over* $\mathbb{Z}_N^0(p)$*);*

e) *the solutions* $\{\bar c_1^{(m)}, V_{a0}^{(m)}\mid a\in\mathbb{Z}_N^0(p)\}$ *of* (4.9) *are in bijection with the lifts* $\bar h_\omega^{(m)}$ *of* $h_\omega^{(m)}$ *to* $\mathcal{K}(p)$*;*

f) *suppose* $\bar c_1^{(m)} = \sum_{\iota\in\mathbb{Z}^N}\bar c_1^{(m)}(\iota)t^\iota$*, where all* $\bar c_1^{(m)}(\iota)\in\bar{\mathcal{L}}_k^{\mathcal{P}}$*; then different solutions of* (4.9) *have different* $\bar c_1^{(m)}(\bar 0)$*, i.e.* $\bar c_1^{(m)}(\bar 0)\in\bar{\mathcal{L}}_k^{\mathcal{P}}$ *are strict invariants of the lifts* $\bar h_\omega^{(m)}$*.*

Proof. – a) is just a general fact about the structure of unipotent automorphisms on modules with filtration of length $<p$;

– b) this is also a sufficiently general interpretation of unipotent additive action on modules with filtration of length $<p$ (Sect.3 of [13] contains necessary background of the specification of this situation to the Campbell-Hausdorff composition law.);

– c) this follows obviously from b);

– d) note that the relations

$$(\mathrm{id}\otimes h_\omega^{(m)n})\bar e = (\sigma\bar C^{(m,n)})\circ(\operatorname{Ad}{}^n\bar l_\omega^{(m)}\otimes\mathrm{id})\bar e\circ(-\bar C^{(m,n)})$$

imply that

$$(\mathrm{id}\otimes h_\omega^{(m)U})\bar e = (\sigma\bar C^{(m)U})\circ(\operatorname{Ad}{}^U\bar l_\omega^{(m)}\otimes\mathrm{id})\bar e\circ(-\bar C^{(m)U}),$$

where $\bar{C}^{(m)U}=U\bar{c}_1^{(m)}+\dots+U^{p-1}\bar{c}_{p-1}^{(m)}$. This implies

$$(\bar{c}_1^{(m)}U)\circ(\mathrm{id}\otimes h_\omega^{(m)}U)\bar{e}\equiv(\sigma\bar{c}_1^{(m)}U)\circ(\mathrm{Ad}^U\bar{l}_\omega^{(m)}\otimes\mathrm{id})\bar{e}\,\mathrm{mod}\,U^2\,,$$

and we need just to follow the coefficents for U;

Remark. Relation (4.9) can be uniquely lifted to $\bar{\mathcal{L}}_{\mathcal{K}}^{\mathcal{P}}=\mathcal{L}_{\mathcal{K}}^{\mathcal{P}}\,\mathrm{mod}\,\mathcal{L}_{\bar{c}_0}^{\mathcal{P}}(p)_{\mathcal{K}}$ by taking suitable unique lifts of $\bar{c}_1^{(m)}$ (use that σ is nilpotent on $\mathcal{M}(p-1)_{\mathcal{K}}\,\mathrm{mod}\,\mathcal{L}_{\bar{c}_0}^{\mathcal{P}}(p)_{\mathcal{K}}$). In other words, we have unique lifts of $\bar{c}_1^{(m)}$ to $\bar{\mathcal{L}}_{\mathcal{K}}$ such that (4.9) is still an equality in $\bar{\mathcal{L}}_{\mathcal{K}}^{\mathcal{P}}$. We will use the same notation $\bar{c}_1^{(m)}$ for these lifts.

– e) note, first, the corresponding data $\{\bar{c}_1^{(m)},V_{\bar{a}0}^{(m)}\mid\bar{a}\in\mathbb{Z}_N^0(p)\}$ are in a bijection with the lifts $\bar{h}_\omega^{(m)}$ of $h_\omega^{(m)}$ to $\mathcal{K}(p)$, cf. Sect. 1.6. Therefore, we should verify that $\bar{c}_1^{(m)}$ determines uniquely the whole vector $\bar{c}^{(m)}$. This follows formally from b) and can be verified as follows (we used a different approach in [12], cf. Remark in Sect. 3.5).

Since $\bar{h}_\omega^{(m)U}:\bar{\mathcal{M}}^f\longrightarrow\mathbb{F}_p[U]\otimes\bar{\mathcal{M}}^f$ is the coaction of the group scheme α_p, we have in $\mathbb{F}_p[U_1,U_2]\otimes\bar{\mathcal{M}}^f$ that

$$(\mathrm{id}_{U_1}\otimes\bar{h}_\omega^{(m)U_2})\cdot\bar{h}_\omega^{(m)U_1}=\bar{h}_\omega^{(m)U_1+U_2}\,.$$

Then we obtain in $\bar{\mathcal{L}}_{\mathcal{K}(p)}$,

$$(\mathrm{id}\otimes h_\omega^{(m)U_2})\bar{C}^{(m)U_1}\circ(\mathrm{Ad}^{U_1}\bar{l}_\omega^{(m)}\otimes\mathrm{id})\bar{C}^{(m)U_2}\circ(\mathrm{Ad}^{U_1+U_2}\bar{l}_\omega^{(m)}\otimes\mathrm{id})\bar{f}$$
$$=\bar{C}^{(m)U_1+U_2}\circ(\mathrm{Ad}^{U_1+U_2}\bar{l}_\omega^{(m)}\otimes\mathrm{id})\bar{f}$$

implies

$$\left(\sum_{n\geqslant1}U_1^n(\mathrm{id}\otimes h_\omega^{(m)U_2})\bar{c}_n^{(m)}\right)\circ\left(\sum_{n\geqslant1}U_2^n(\bar{\mathcal{A}}^{U_1}\otimes\mathrm{id})\bar{c}_n^{(m)}\right)$$
$$=\sum_{n_1,n_2}\bar{c}_{n_1+n_2}^{(m)}(U_1+U_2)^{n_1+n_2}$$

For $n\geqslant1$, the coefficient for $U_1U_2^n$ in the RHS equals $(n+1)\bar{c}_{n+1}^{(m)}$. The corresponding coeffcient in the LHS coincides with the coeffcient in

$$\left(U_1(\mathrm{id}\otimes h_\omega^{(m)U_2})\bar{c}_1^{(m)}\right)\circ\left((\mathrm{id}+U_1\bar{\mathcal{B}}^{(m)}\otimes\mathrm{id})\sum_{n\geqslant0}U_2^n\bar{c}_n^{(m)}\right)$$

and, therefore, equals $(\bar{\mathcal{B}}^{(m)}\otimes\mathrm{id})\bar{c}_n^{(m)}$ plus $\mathbb{F}_p$-linear combination of the elements of the following form (cf. Remark below)

$$[\dots[(\mathrm{id}\otimes d^{n_1}h_\omega^{(m)}/n_1!)c_1^{(m)},\bar{c}_{n_2}^{(m)}],\cdots],\bar{c}_{n_s}^{(m)}]\,,$$

where $n_1+\dots+n_s=n$, $s\geqslant1$ and $n_1\geqslant0$. As a result, $(n+1)\bar{c}_{n+1}^{(m)}$ can be uniquely recovered from $\bar{c}_1^{(m)},\dots,\bar{c}_n^{(m)}$.

Remark. We used well-known relation,

$$X + UY \equiv X \circ \left(U \sum_{k\geqslant 1} \frac{1}{k!}[\dots[Y, \underbrace{X], \dots, X]}_{k-1 \text{ times}} \right) \bmod U^2$$

with $U = U_1$ and $X = \sum_n U_2^n \bar{c}_n^{(m)}$, cf. references in [13], Sect. 3.2.

f) follows by induction on $\bmod C_i(\bar{\mathcal{L}}_{\mathcal{K}}^{\mathcal{P}})$, $i \geqslant 1$, from relation (4.9). $\square$

4.6. **The structure of $\Gamma_\omega^{\mathcal{P}}$.** We are going to determine the structure of the Lie algebra $L_\omega^{\mathcal{P}}$. In the above section we indicated the way how to specify the lifts $\bar{h}_\omega^{(m)} = \pi_\omega^{-1}(\bar{l}_\omega^{(m)})$, $1 \leqslant m \leqslant N$. This can be done by applying recurrent procedure (4.9) to find the elements $\bar{c}_1^{(m)}$ and $V_{a\,0}^{(m)} = \mathrm{ad}_k\, \bar{l}_\omega^{(m)}(D_{a0})$, $a \in \mathbb{Z}_N^0(p)$. In addition, we should specify the commutators $[\bar{l}_\omega^{(i)}, \bar{l}_\omega^{(j)}] := \bar{l}_\omega[i,j] \in \bar{\mathcal{L}}^{\mathcal{P}}$, $1 \leqslant i, j \leqslant N$.

Remark. From Lemma 4.3 c) it follows that all $\bar{l}_\omega[i,j] \in C_2(L_\omega^{\mathcal{P}}) = \bar{\mathcal{L}}_{\bar{c}^0}(2)$. It would be very interesting to find explicit expression for the elements $\bar{l}_\omega[i,j]$ in terms of the involved parameters $A_\iota = A_\iota(\omega)$ (recall that $E(\omega^p) = 1 + \sum_\iota A_\iota t^{\bar{c}^0 + p\iota}$). We verified by a direct calculation with relations (4.9) that for any lifts $\bar{h}_\omega^{(i)}$ and $\bar{h}_\omega^{(j)}$, the corresponding elements $\bar{l}_\omega[i,j] \in C_4(L_\omega^{\mathcal{P}})$.

The following property could be useful to study the properties of the elements $\bar{l}_\omega[i,j]$. To simplify the notation set for all m, $\mathrm{id} \otimes dh_\omega^{(m)} = d^{(m)}$ and $\mathrm{ad}\, \bar{l}_\omega^{(m)} \otimes \mathrm{id} = \mathrm{ad}^{(m)}$.

Proposition 4.9. *For all* $1 \leqslant i, j \leqslant N$,

$$\bar{l}_\omega[i,j] = (d^{(j)} - \mathrm{ad}^{(j)})\bar{c}_1^{(i)} - (d^{(i)} - \mathrm{ad}^{(i)})\bar{c}_1^{(j)} + [\bar{c}_1^{(i)}, \bar{c}_1^{(j)}] \,.$$

Proof. With above assumption we have in $\bar{\mathcal{L}}_{\mathcal{K}}^{\mathcal{P}}[U]$ for all m,

$$(\mathrm{id} + U d^{(m)})\bar{e} \equiv (U \sigma \bar{c}_1^{(m)}) \circ (\mathrm{id} + U \mathrm{ad}^{(m)})\bar{e} \circ (-U \bar{c}_1^{(m)}) \bmod U^2.$$

Let $\bar{E} = \widetilde{\exp}\, \bar{e}$ in the enveloping algebra $\bar{\mathfrak{A}}_{\mathcal{K}}^{\mathcal{P}}$ of $\bar{\mathcal{L}}_{\mathcal{K}}^{\mathcal{P}}$. Then we have the following congruence modulo $(U^2, (\bar{\mathcal{J}}_{\mathcal{K}}^{\mathcal{P}})^p)$, where $\bar{\mathcal{J}}_{\mathcal{K}}^{\mathcal{P}}$ is the augmentation ideal in $\bar{\mathfrak{A}}_{\mathcal{K}}^{\mathcal{P}}$,

$$(\mathrm{id} + U d^{(m)})\bar{E} \equiv (1 + \sigma \bar{c}_1^{(m)} U) \cdot (\mathrm{id} + U \mathrm{ad}^{(m)})\bar{E} \cdot (1 - \bar{c}_1^{(m)} U) \,.$$

Comparing the coefficients for U we obtain

$$(d^{(m)} - \mathrm{ad}^{(m)})\bar{E} \equiv \sigma \bar{c}_1^{(m)} \cdot \bar{E} - \bar{E} \cdot \bar{c}_1^{(m)} \bmod (\bar{\mathcal{J}}_{\mathcal{K}}^{\mathcal{P}})^p \,. \tag{4.10}$$

Note that $(d^{(i)} d^{(j)} - d^{(j)} d^{(i)})\bar{E} \equiv 0$ because

$$d^{(m)} \bar{E} = - \sum_{\bar{\iota} \geqslant \bar{0}} A_{\bar{\iota}} \sum_{s \geqslant 0} (1/s!)(a_1 + \dots + a_s)^{(m)} t^{-(a_1 + \dots + a_s) + \bar{c}^0 + p\bar{\iota}} D_{a\,0} \,.$$

In addition, for any i, j, $\mathrm{ad}^{(i)}$ and $d^{(j)}$ commute one with another.

Therefore, (4.10) implies

$$\operatorname{ad}\bar{l}_\omega[i,j](\bar{E}) \equiv (d^{(i)}-\operatorname{ad}^{(i)})(d^{(j)}-\operatorname{ad}^{(j)})\bar{E}-(d^{(j)}-\operatorname{ad}^{(j)})(d^{(i)}-\operatorname{ad}^{(i)})\bar{E}$$
$$\equiv \sigma(\mathcal{X})\bar{E}-\bar{E}\mathcal{X}\,,$$

where $\mathcal{X}=(d^{(i)}-\operatorname{ad}^{(i)})\bar{c}_1^{(j)}-(d^{(j)}-\operatorname{ad}^{(j)})\bar{c}_1^{(i)}+[\bar{c}_1^{(j)},\bar{c}_1^{(i)}]$.

Let $\mathcal{X}_0=\mathcal{X}-l_\omega[i,j]$, then

$$\sigma(\mathcal{X}_0)\bar{E}\equiv \bar{E}\mathcal{X}_0 \operatorname{mod}(\bar{\mathcal{J}}^{\mathcal{P}}_{\mathcal{K}})^p\,. \tag{4.11}$$

It remains to prove that □

Lemma 4.10. $\mathcal{X}_0=0$.

Proof of Lemma. As earlier, let $\bar{\mathfrak{A}}^{\mathcal{P}}$ be the enveloping algebra for $\bar{\mathcal{L}}^{\mathcal{P}}$. Let $\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(s)$, $s\geqslant 1$, be the ideal in $\bar{\mathfrak{A}}^{\mathcal{P}}$ generated by the monomials $D_{a_1n_1}\dots D_{a_un_u}$ of weight $\geqslant s$, i.e. such that if $s_i\in\mathbb{N}$ for $1\leqslant i\leqslant u$, are such that $(s_i-1)\bar{c}^0\leqslant a_i<s_i\bar{c}_0$ then $s_1+\dots+s_u\geqslant s$. For each s, $\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(s)\cap\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}=\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}(s)$, in particular, $\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(p)\cap\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}=0$.

Note that $\mathcal{X}_0\in\bar{\mathcal{L}}^{\mathcal{P}}_{\mathcal{K}}$ and let $\mathcal{X}_0=\mathcal{X}^{(1)}+\dots+\mathcal{X}_0^{(p-1)}$, where each $\mathcal{X}_0^{(s)}$ is a $\mathcal{K}$-linear combination of the Lie monomials of weight s,

$$[\dots[D_{a_1n_1},D_{a_2n_2}],\dots,D_{a_un_u}]\,.$$

Clearly, $\mathcal{X}_0^{(s)}\in\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}(s)$ and if $\mathcal{X}_0^{(s)}\in\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(s+1)$ then $\mathcal{X}_0^{(s)}=0$. It will be enough to prove by induction on $1\leqslant s\leqslant p$ that

$$\mathcal{X}_0\in\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}(s)+\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(s+1)_{\mathcal{K}}\,.$$

If $s=1$ then (4.11) implies that $\sigma(\mathcal{X}_0)\equiv\mathcal{X}_0\operatorname{mod}\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(2)_{\mathcal{K}}$ (use that $\bar{E}\equiv 1\operatorname{mod}\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(1)_{\mathcal{K}})$. So, $\sigma\mathcal{X}_0^{(1)}=\mathcal{X}_0^{(1)}$ in $\bar{\mathcal{L}}^{\mathcal{P}}_{\mathcal{K}}$, i.e. $\mathcal{X}_0^{(1)}\in\bar{\mathcal{L}}^{\mathcal{P}}=\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}(1)$ and $\mathcal{X}_0\in\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}(1)+\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(2)_{\mathcal{K}}$.

Suppose the lemma is proved for some $1\leqslant s\leqslant p-1$.

So, $\mathcal{X}_0=\mathcal{X}_0^{(s)}+\dots+\mathcal{X}_0^{(p-1)}\in\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}(s)+\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(s+1)_{\mathcal{K}}$, and (4.11) implies

$$\sigma\mathcal{X}_0^{(s+1)}-\mathcal{X}_0^{(s+1)}\equiv\sum_{a\in\mathbb{Z}^0_N(p)}t^{-a}[D_{a0},\mathcal{X}_0^{(s)}]\operatorname{mod}\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(s+2)_{\mathcal{K}}\,.$$

Thus (compare with Prop. 3.1) all $[D_{a0},\mathcal{X}_0^{(s)}]\in\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(s+2)_{\mathcal{K}}$ and

$$\mathcal{X}_0^{(s+1)}\in\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}(s+1)+\bar{\mathfrak{A}}^{\mathcal{P}}_{\bar{c}^0}(s+2)_{\mathcal{K}}\,.$$

If weight of $D_{a0}=1$ then $[D_{a0},\mathcal{X}_0^{(s)}]$ has weight $s+1$, therefore, $[D_{a0},\mathcal{X}_0^{(s)}]=0$ and $\mathcal{X}_0^{(s)}=0$. Similarly, $\mathcal{X}_0^{(s+1)}\in\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}(s+1)$.

The lemma is proved. □

Using the notation from item f) of Prop. 4.8 we obtain

Corollary 4.11. *For* $1\leqslant i,j\leqslant N$,

$$\bar{l}_\omega[i,j]=\operatorname{ad}^{(i)}(\bar{c}_1^{(j)}(\bar{0}))-\operatorname{ad}^{(j)}(\bar{c}_1^{(i)}(\bar{0}))+\sum_{\iota\in\mathbb{Z}^N}[\bar{c}_1^{(i)}(\iota),\bar{c}_1^{(j)}(-\iota)]\,.$$

4.7. **The structure of $\Gamma_\omega^{\mathcal{P}}$ modulo third commutators.** Consider the lift of relation (4.9) to $\bar{\mathcal{L}}_{\mathcal{K}}^{\mathcal{P}}$ taken modulo $C_2(L_\omega^{\mathcal{P}})_{\mathcal{K}} = \bar{\mathcal{L}}_{\bar{c}^0}^{\mathcal{P}}(2)_{\mathcal{K}}$ (cf. Remark in the part d) of the proof of Prop. 4.8).

$$\sigma\bar{c}_1^{(m)} - \bar{c}_1^{(m)} + \sum_{a\in\mathbb{Z}_N^0(p)} t^{-a}V_{a0}^{(m)} \equiv - \sum_{\substack{a\in\mathbb{Z}_N^0(p)\\ \iota\geqslant 0}} A_\iota(\omega)t^{\bar{c}^0+p\iota-a}a^{(m)}D_{\bar{a}0}\,. \tag{4.12}$$

Here $V_{a0}^{(m)} = \operatorname{ad}\bar{l}_\omega^{(m)}(D_{a0})$ and $a^{(m)}$ is the m-th component of $a\in\mathbb{Z}_N^0(p)$. Applying to (4.12) the operator $\mathcal{R}$ from Sect. 3.1 we obtain:

1) $V_{\bar{0}}^{(m)} = \operatorname{ad}\bar{l}_\omega^{(m)}(D_{\bar{0}}) \in C_2(L_\omega^{\mathcal{P}})$;

2) for all $a\in\mathbb{Z}_N^+(p)$,

$$V_{a0}^{(m)} = \operatorname{ad} l_\omega^{(m)}(D_{a0}) \in \bmod C_2(L_{\omega,k}^{\mathcal{P}})\,.$$

Property 2) means that all generators of $L_{\omega,k}^{(m)\mathcal{P}}$ of the form D_{an} with $a>\bar{c}^0$ and $a^{(m)}\neq 0 \bmod p$, can be eliminated from the system

$$\{D_{an}\mid a\in\mathbb{Z}_N^+(p)\}\cup\{D_{\bar{0}}\}\cup\{l_\omega^{(m)}\}$$

of generators of $L_{\omega,k}^{(m)\mathcal{P}}$. Indeed, because $A_{\bar{0}}\neq 0$ and $a^{(m)}\neq 0\bmod p$, all $D_{a+\bar{c}^0,0}$ belong to the ideal of second commutators

$$C_2(L_{\omega,k}^{(m)\mathcal{P}}) = (\operatorname{ad}\bar{l}_\omega^{(m)})\bar{\mathcal{L}}_k^{\mathcal{P}} + C_2(\bar{\mathcal{L}}_k^{\mathcal{P}})\,,$$

and for any $n\in\mathbb{Z}/N_0$, all $D_{a+\bar{c}^0,n} = \sigma^n D_{a+\bar{c}^0,0}$ also belong to $C_2(L_{\omega,k}^{(m)\mathcal{P}})$. Property 1) means that $L_\omega^{(m)\mathcal{P}}$ has only one relation with respect to any minimal $\mathcal{P}$-topological set of generators. Therefore, $\Gamma_\omega^{(m)\mathcal{P}}$ can be treated as a $\mathcal{P}$-topological group with one relation.

On the other hand, the Lie algebra $L_{\omega,k}^{\mathcal{P}}$ has the system of generators

$$\{D_{an}\mid a\in\mathbb{Z}_N^+(p)\}\cup\{D_{\bar{0}}\}\cup\{\bar{l}_\omega^{(m)}\mid 1\leqslant m\leqslant N\}$$

with the corresponding system of relations

$$[D_{an},\bar{l}_\omega^{(m)}] = V_{an}^{(m)},\ [D_{\bar{0}},\bar{l}_\omega^{(m)}] = V_{\bar{0}}^{(m)},\ [\bar{l}_\omega^{(i)},\bar{l}_\omega^{(j)}] = \bar{l}_\omega[i,j]\,,$$

where $a\in\mathbb{Z}_N^+(p)$ and $1\leqslant m,i,j\leqslant N$.

Choose for every $a\in\mathbb{Z}_N^+(p)$, $m_a\in[1,N]$ such that $a^{(m_a)}\neq 0\bmod p$. Then the relations $[D_{an},\bar{l}_\omega^{(m_a)}] = V_{an}^{(m_a)}$, $a\in\mathbb{Z}_N^+(p)$, can be used to eliminate extra generators $\{D_{an}\mid a>\bar{c}_0\}$ and to present the structure of $L_{\omega,k}^{\mathcal{P}}$ in terms of the corresponding minimal system of generators $\{D_{an}\mid a\in\mathbb{Z}_N^+(p), a<\bar{c}^0\}\cup\{D_{\bar{0}}\}\cup\{\bar{l}_\omega^{(m)}\}$.

Consider second central step to obtain explicitly the above relations modulo $C_3(L_{\omega,k}^{\mathcal{P}})$.

Proposition 4.12. *For $1 \leqslant m \leqslant N$ and $a \in \mathbb{Z}_N^+(p)$, there are the following congruences* $\operatorname{mod} C_3(L^{\mathcal{P}}_{\omega,k})$*:*

$$V_{\bar{0}}^{(m)} \equiv -\frac{1}{2} \sum_{\substack{\iota \geqslant \bar{0} \\ 0 \leqslant n < N_0}} \sigma^n \left(A_\iota \sum_{\substack{a_1+a_2= \\ \bar{c}^0+p\iota}} a_1^{(m)} [D_{a_1,0}, D_{a_2,0}] \right),$$

$$V_{a0}^{(m)} \equiv -\sum_{\substack{n \geqslant 1 \\ \iota \geqslant 0}} \sigma^n \left(A_\iota \sum_{\substack{a_1+a_2/p^n \\ =\bar{c}^0+p\iota+a/p^n}} a_1^{(m)} [D_{a_1,0}, D_{a_2,-n}] \right)$$

$$-\sum_{\iota \geqslant \bar{0}} A_\iota a^{(m)} D_{\bar{c}^0+p\iota+a,0} - \frac{1}{2} \sum_{\substack{n \geqslant 0 \\ \iota \geqslant \bar{0}}} \sigma^{-n} \left(A_\iota \sum_{\substack{a_1+a_2= \\ \bar{c}^0+p\iota+ap^n}} a_1^{(m)} [D_{a_1,0}, D_{a_2,0}] \right).$$

Proof. From (4.11) we obtain (apply the operator $\mathcal{S}$ from Sect. 3.1)

$$\bar{c}_1^{(m)} \equiv \sum_{\substack{a,\iota \geqslant \bar{0} \\ n \geqslant 0}} \sigma^n(A_\iota) t^{p^n(\bar{c}^0+p\iota-a)} a^{(m)} D_{an} \operatorname{mod} C_2(L^{\mathcal{P}}_{\omega,\mathcal{K}}).$$

Then the right-hand side of (4.9) modulo $C_3(L^{\mathcal{P}}_{\omega,\mathcal{K}})$ appears as

$$-\sum_{\iota} A_\iota t^{\bar{c}^0+p\iota-a} a^{(m)} D_{a0} - \frac{1}{2} \sum_{a_1,a_2,\iota} A_\iota t^{\bar{c}^0+p\iota-a_1-a_2} a_1^{(m)} [D_{a_10}, D_{a_20}]$$

$$-\sum_{\substack{a_1,a_2,\iota \\ n \geqslant 1}} \sigma^n(A_\iota) t^{p^n(\bar{c}^0+p\iota-a_1)-a_2} a_1^{(m)} [D_{a_1,n}, D_{a_20}]$$

Applying the operators $\mathcal{R}$ and $\mathcal{S}$ we obtain our proposition. □

Corollary 4.13. $L^{\mathcal{P}}_{\omega,k} \operatorname{mod} C_3(L^{\mathcal{P}}_{\omega,k})$ *is the maximal σ-invariant quotient of nilpotent class 2 of a free Lie k-algebra with generators*

$$\{D_{an} \mid a \in \mathbb{Z}_N^+(p), a < \bar{c}^0, n \in \mathbb{Z}/N_0\} \cup \{D_{\bar{0}}\} \cup \{\bar{l}_\omega^{(m)} \mid 1 \leqslant m \leqslant N\}$$

satisfying for $1 \leqslant m_1, m_2 \leqslant N$ and $a \in \mathbb{Z}_N^+(p)$, the relations:

$$\mathcal{R}(m_1, m_2) : [\bar{l}_\omega^{(m_1)}, \bar{l}_\omega^{(m_2)}] = 0,$$

$$\mathcal{R}_{\bar{0}}(m) : \quad [D_{\bar{0}}, \bar{l}_\omega^{(m)}] + \frac{1}{2} \sum_{\substack{\iota \geqslant \bar{0} \\ 0 \leqslant n < N_0}} \sigma^n \left(A_\iota \sum_{\substack{a_1+a_2= \\ \bar{c}^0+p\iota}} a_1^{(m)} [D_{a_1,0}, D_{a_2,0}] \right) = 0,$$

$$\mathcal{R}_a(m_1,m_2): \sum_{\substack{n\geqslant 1\\ \iota\geqslant\bar 0}}\sigma^n\left(A_\iota\sum_{\substack{a_1+a_2/p^n\\=\bar c^0+p\iota+a/p^n}}\left(a_1^{(m_1)}a_2^{(m_2)}-a_1^{(m_2)}a_2^{(m_1)}\right)[D_{a_1,0},D_{a_2,-n}]\right)$$

$$+\frac{1}{2}\sum_{\substack{n\geqslant 0\\ \iota\geqslant\bar 0}}\sigma^{-n}\left(A_\iota\sum_{\substack{a_1+a_2=\\ \bar c^0+p\iota+ap^n}}\left(a_1^{(m_1)}a_2^{(m_2)}-a_1^{(m_2)}a_2^{(m_1)}\right)[D_{a_1,0},D_{a_2,0}]\right)=0\,.$$

Remark. a) If $N=1$ then there is only one (Demushkin) relation $\mathcal{R}_{\bar 0}(1)$, cf. [13, 12].

b) The simplest example can be obtained by choosing $\bar c^0=(p,0,\dots,0)$, $\omega^p=t^{\bar c^0}$ and $N_0=1$, cf. Sect. 5.5.

c) The structure of $\Gamma^{\mathcal{P}}_\omega$ depends only on ω, more precisely, only on $\omega^p \operatorname{mod} t^{(p-1)\bar c^0}$, i.e. on the constants A_ι with $\iota<(p-2)\bar c^0/p$.

d) The structure of $\Gamma^{\mathcal{P}}_\omega$ modulo $C_s(\Gamma^{\mathcal{P}}_\omega)$, $s\leqslant p$, depends only on the constants A_ι with $\iota<(s-2)\bar c^0/p$.

4.8. **The simplest example.** $N=2$, $N_0=1$, $\bar c^0=(p,0)$, $A_{\bar 0}=1$, all remaining $A_\iota=0$.

The minimal generators:
$\{D_a\mid a\in\mathbb{Z}^+_2(p), a<(p,0)\}\cup\{D_{\bar 0}\}\cup\{\bar l^{(1)},\bar l^{(2)}\}$.

The relations:

$$\mathcal{R}(1,2):[\bar l^{(1)},\bar l^{(2)}]\,,$$

$$\mathcal{R}_{\bar 0}(1)=[D_{\bar 0},\bar l^{(1)}]+\sum_{\substack{1\leqslant\alpha\leqslant\frac{p-1}{2}\\ \gamma\in\mathbb{Z}}}\alpha[D_{(\alpha,\gamma)},D_{(p-\alpha,-\gamma)}]$$

$$\mathcal{R}_{\bar 0}(2)=[D_{\bar 0},\bar l^{(2)}]+\sum_{\substack{0\leqslant\alpha\leqslant\frac{p-1}{2}\\ \gamma\in\mathbb{Z}}}\gamma[D_{(\alpha,\gamma)},D_{(p-\alpha,-\gamma)}]$$

$$\mathcal{R}_a(1,2):\quad [D_a,a^{(2)}\bar l^{(1)}-a^{(1)}\bar l^{(2)}]-\delta_{a^{(1)},0}\sum_\gamma a^{(2)}[D_{(p-1,\gamma)},D_{(p,a^{(2)}-p\gamma)}]$$

$$+a^{(1)}\sum_{\substack{n\geqslant 1\\ \beta\in\mathbb{Z}^+(p)}}\beta[D_{(p,-\beta)},D_{a+(0,p^n\beta)}]+\frac{1}{2}\sum_{\substack{a_1+a_2=\\ (p,0)+a}}\left(a_1^{(1)}a_2^{(2)}-a_1^{(2)}a_2^{(1)}\right)[D_{a_1},D_{a_2}]\,.$$

5. Characteristic 0 case

In this section K is an N-dimensional local field of characteristic 0. We assume that the first residue field $K^{(1)}$ of K has characteristic p. The last residue field k of K is isomorphic to $\mathbb{F}_{p^{N_0}}$, $N_0 \in \mathbb{N}$. We also fix a system of local parameters $\pi = \{\pi_1, \cdots, \pi_N\}$ of K, denote by v^1 the first valuation of K such that $v^1(p) = 1$ and by O^1_K the corresponding valuation ring. Starting Sect. 5.2 we assume that K contains a primitive p-th root of unity ζ_1.

5.1. **The field-of-norms functor.** N-dimensional local fields are special cases of $(N-1)$-big fields used by Scholl [31] to construct a higher dimensional analogue of the Fontaine-Wintenberger field-of-norms functor, [**?**]. We can apply this construction to the case of higher local fields due to the fact that the structure of an N-dimensional field is uniquely extended to its finite field extensions. We don't use here the construction of the field-of-norms functor from [7]: it is based on essentially close ideas but works in the category of higher local fields with additional structure given by "subfields of constants" (because the whole theory in [7] is based only on the concept of ramification for higher local fields).

Let K_{alg} be an algebraic closure of K. Denote by the same symbol a unique extension of the valuation v^1 to K_{alg}. For any $0 < c \leqslant 1$, let $\mathfrak{p}^c = \{x \in K_{alg} \mid v^1(x) \geqslant c\}$. If L is a field extension of K in K_{alg}, we use the simpler notation $O^1_L/\mathfrak{p}^c$ instead of $O^1_L/(\mathfrak{p}^c \cap O^1_L)$.

An increasing fields tower $K_\bullet = (K_n)_{n\geqslant 0}$, where $K_0 = K$, is strictly deeply ramified (SDR) with parameters (n_0, c), if for $n \geqslant n_0$, we have $[K_{n+1} : K_n] = p^N$ and there is a surjective map $\Omega^1_{O^1_{F_{n+1}}/O^1_{F_n}} \longrightarrow (O^1_{F_{n+1}}/\mathfrak{p}^c)^N$ or, equivalently, the p-th power map induces epimorphic maps

$$i^1_n(K_\bullet) : O^1_{K_{n+1}}/\mathfrak{p}^c \longrightarrow O^1_{K_n}/\mathfrak{p}^c.$$

This implies that for all $n \geqslant n_0$, the last residue fields $K_n^{(N)}$ are the same and there are systems of local parameters $\{\pi_{n1}, \dots, \pi_{nN}\}$ in K_n such that for all $1 \leqslant m \leqslant N$, $\pi^p_{n+1,m} \equiv \pi_{nm} \bmod \mathfrak{p}^c$, where $\pi_m = \pi_{0m}$. Equivalently, on the level of the N-valuation rings $\mathcal{O}_{K_n}$, the p-th power map induces epimorphic maps

$$i_n(K_\bullet) : \mathcal{O}_{K_{n+1}}/\mathfrak{p}^c \longrightarrow \mathcal{O}_{K_n}/\mathfrak{p}^c. \tag{5.1}$$

Let $\mathcal{O} = \varprojlim_n \mathcal{O}_{K_n}/\mathfrak{p}^c$. Then $\mathcal{O}$ is an integral domain and we can introduce the field of fractions $\mathcal{K}$ of $\mathcal{O}$. The field-of-norms functor X associates to the SDR tower $K_\bullet$ the field $\mathcal{K} = X(K_\bullet)$. This field has characteristic p, it inherits a structure of N-dimensional local field such that the elements $t_m := \varprojlim_n \pi_{nm}$, $1 \leqslant m \leqslant N$, form a system of local parameters in $\mathcal{K}$. Then the N-dimensional valuation ring $\mathcal{O}_{\mathcal{K}}$

of $\mathcal{K}$ coincides with $\mathcal{O}$, and for $n \geqslant n_0$, the last residue fields of $\mathcal{K}$ and K_n coincide. Since the identification $\mathcal{O}_{\mathcal{K}} = \varprojlim \mathcal{O}_{K_n}/\mathfrak{p}^c$ relates the appropriate power series in given systems of local parameters the field-of-norms functor is compatible with $\mathcal{P}$-topological structures on the fields K_n and $\mathcal{K}$.

Suppose L is a finite extension of K in K_{alg}. Then the tower $L_\bullet = (LK_n)_{n\geqslant 0}$ is again SDR and $X(L_\bullet) = \mathcal{L}$ is a separable extension of $\mathcal{K}$ of degree $[LK_n : K_n]$, where $n \gg 0$. The extension $\mathcal{L}/\mathcal{K}$ is Galois if and only if for $n \gg 0$, LK_n/K_n is Galois. From the definition of $\mathcal{L}$ and $\mathcal{K}$ it follows that we have a natural identification of groups $\mathrm{Gal}(\mathcal{L}/\mathcal{K}) = \mathrm{Gal}(LK_n/K_n)$. As a result, $X(K_{alg}) := \varinjlim_{L} X(L_\bullet)$ is a separable closure $\mathcal{K}_{sep}$ of $\mathcal{K}$ and the functor X identifies $\mathrm{Gal}(\mathcal{K}_{sep}/\mathcal{K})$ with $\mathrm{Gal}(K_{alg}/K_\bullet^\infty)$, where $K_\bullet^\infty = \varinjlim_{n} K_n$.

Similarly to the classical 1-dimensional situation there is the following interpretation of the functor X. Let $\mathbb{C}_p(N)$ be the v^1-adic completion of K_{alg} and let $R_0(N) = \varprojlim_{n\geqslant 0} \mathbb{C}_p(N)$ with respect to the p-th power maps on $\mathbb{C}_p(N)$. The operations on $R_0(N)$ are defined as follows: if $a = \{a_n\}_{n\geqslant 0}$ and $b = \{b_n\}_{n\geqslant 0}$ belong to $R_0(N)$ then $ab = \{a_nb_n\}_{n\geqslant 0}$ and $a + b = \{c_n\}_{n\geqslant 0}$, where $c_n = \lim_{m\to\infty} (a_{n+m} + b_{n+m})^{p^n}$. So, $R_0(N)$ is a field of characteristic p, there is a natural embedding $\mathcal{K}_{sep} \subset R_0(N)$, and $R_0(N)$ appears as the completion of $\mathcal{K}_{sep}$ with respect to the first valuation.

We denote by $R(N) \subset R_0(N)$ the completion of the N-valuation ring of $\mathcal{K}_{sep}$, and let $\mathrm{m}_{R(N)}$ be the maximal ideal of $R(N)$. Clearly,

$$R(N) = \varprojlim_{n\geqslant 0} \mathcal{O}_{\mathbb{C}_p(N)} = \varprojlim_{n\geqslant 0} \mathcal{O}_{K_{alg}}/p\,.$$

Note that the $\mathcal{P}$-topology on K is uniquely extended to $R_0(N)$ and this extension coincides with the extension of the $\mathcal{P}$-topological structure of $\mathcal{K}$ to $R_0(N)$ (as the completion of the $\mathcal{P}$-topology on $\mathcal{K}_{sep}$).

5.2. Suppose $u \in \mathbb{N}$ and $\phi_1, \dots, \phi_u \in \mathrm{m}_K \setminus \{0\}$ are independent modulo p-th powers, i.e. for any $n_1, \dots, n_u \in \mathbb{Z}$, the product $\phi_1^{n_1} \dots \phi_u^{n_u} \in K^{*p}$ iff all $n_i \equiv 0 \bmod p$. Let $K^\phi = \bigcup_{n\geqslant 0} K(\phi_{1n}, \dots, \phi_{un})$, where for $1 \leqslant i \leqslant u$, $\phi_i = \phi_{i0}$ and for all $n \in \mathbb{N}$, $\phi_{i,n-1} = \phi_{in}^p$.

For $n \in \mathbb{N}$, let $\zeta_{n+1} \in K_{alg}$ be such that $\zeta_{n+1}^p = \zeta_n$. Here $\zeta_1 \in K$ is our given p-th primitive root of unity. Let $\widetilde{K}^\phi = K^\phi(\{\zeta_n \mid n \in \mathbb{N}\})$. Then $\widetilde{K}^\phi/K$ is normal and let Γ^ϕ be its Galois group.

Lemma 5.1. *If $\Gamma^\phi_{<p}$ is the maximal quotient of Γ^ϕ of period p and nilpotent class $< p$ then there is a natural exact sequence of groups*

$$\mathrm{Gal}(\widetilde{K}^\phi/K^\phi) \longrightarrow \Gamma^\phi_{<p} \longrightarrow \mathrm{Gal}(K(\phi_{11}, \dots, \phi_{u1})/K) \longrightarrow 1\,.$$

Proof. Clearly, $\Gamma_{\widetilde{K}_\phi/K}=\langle\sigma,\tau_1,\dots,\tau_u\rangle$, where for any $1\leqslant i,m\leqslant u$ and some $s_0\in\mathbb{Z}$, $\sigma\zeta_n=\zeta_n^{1+ps_0}$, $\sigma\phi_{in}=\phi_{in}$, $\tau_m(\zeta_n)=\zeta_n$, $\tau_m(\phi_{in})=\phi_{in}\zeta_n^{\delta_{mi}}$, and $\sigma^{-1}\tau_m\sigma=\tau_m^{(1+ps_0)^{-1}}$.

Therefore, $(\Gamma^\phi)^p=\langle\sigma^p,\tau_1^p,\dots,\tau_u^p\rangle$ and for the subgroup of second commutators we have $C_2(\Gamma^\phi)\subset\langle\tau_1^p,\dots,\tau_u^p\rangle\subset(\Gamma^\phi)^p$. As a result, it holds $(\Gamma^\phi)^pC_p(\Gamma^\phi)=\langle\sigma^p,\tau_1^p,\dots,\tau_u^p\rangle$ and the lemma is proved. □

We are going to apply the above lemma to our field K and the set of local parameters $\pi=\{\pi_1,\dots,\pi_n\}$. The lemma provides us with the field extensions $\widetilde{K}^\pi\supset K^\pi\supset K$. Let $\Gamma_{<p}:=\Gamma/\Gamma^pC_p(\Gamma)$ and $\Gamma_{K^\pi}=\mathrm{Gal}(K_{alg}/K^\pi)$. The embedding $\Gamma_{K^\pi}\subset\Gamma$ induces a continuous homomorphism $\iota^\pi:\Gamma_{K^\pi}\longrightarrow\Gamma_{<p}$. Denote by κ^π the natural surjection $\Gamma_{<p}\longrightarrow\mathrm{Gal}(K(\sqrt[p]{\pi_1},\dots,\sqrt[p]{\pi_N})/K)=\prod\limits_{1\leqslant m\leqslant N}\langle\tau_m\rangle^{\mathbb{Z}/p}$, where $\tau_m(\sqrt[p]{\pi_i})=\sqrt[p]{\pi_i}\zeta_1^{\delta_{im}}$.

Proposition 5.2. *The following sequence of profinite groups*

$$\Gamma_{K^\pi}\xrightarrow{\iota^\pi}\Gamma_{<p}\xrightarrow{\kappa^\pi}\prod_{1\leqslant m\leqslant N}\langle\tau_m\rangle^{\mathbb{Z}/p}\longrightarrow 1$$

is exact.

Proof. Note that the elements of the group $\Gamma_{\widetilde{K}^\pi}=\mathrm{Gal}(K_{alg}/\widetilde{K}^\pi)$ together with a lift $\hat\sigma\in\Gamma_{K^\pi}$ of σ generate the group Γ_{K^π}. Now the exact sequence from Lemma 5.1 implies that $\mathrm{Ker}\,\kappa^\pi$ is generated by $\hat\sigma$ and the image of $\Gamma_{\widetilde{K}^\pi}\subset\Gamma_{K^\pi}$. As a result, this kernel coincides with the image of Γ_{K^π} in $\Gamma_{<p}$. □

Let $R(N)$ be the ring from Sect. 5.1. Recall, there is a natural embedding $k\subset R$ and for $1\leqslant m\leqslant N$, $t_m:=\varprojlim_n\{\pi_{mn}\}_{n\geqslant 0}\in R(N)$, where $\pi_{m0}=\pi_m$ and $\pi_{m,n+1}^p=\pi_{mn}$.

Following Sect. 2.3 set $t=(t_1,\dots,t_N)$ and $\mathcal{K}=k((t))$. Then $\mathcal{K}$ is a closed subfield of $R_0(N)=\mathrm{Frac}\,R(N)$ with a system of local parameters t. The tower $K_\bullet^\pi=\{K(\pi_{1n},\dots,\pi_{Nn})\}_{n\geqslant 0}$ is SDR and $K^\pi=K_\bullet^{\pi\,\infty}$. Therefore, the field-of-norms functor X from Sect. 5.1 identifies $X(K_\bullet^\pi)$ with $\mathcal{K}$ and $R_0(N)$ with the completion of $\mathcal{K}_{sep}$. In particular, there is a natural inclusion $\Gamma\longrightarrow\mathrm{Aut}\,R_0(N)$ which induces the identification of $\mathcal{G}=\mathrm{Gal}(\mathcal{K}_{sep}/\mathcal{K})$ and Γ_{K^π}.

We are going to apply below the results of the previous sections and will use the appropriate notation related to our field $\mathcal{K}$, e.g. $\mathcal{G}_{<p}=\mathrm{Gal}(\mathcal{K}_{<p}/\mathcal{K})$, where $\mathcal{K}_{<p}=\mathcal{K}_{sep}^{\mathcal{G}^pC_p(\mathcal{G})}$. The field-of-norms identification $\mathcal{G}\simeq\Gamma_{K^\pi}$ composed with the morphism ι^π from Prop. 5.2 induces a group homomorphism $\iota^\pi_{<p}:\mathcal{G}_{<p}\longrightarrow\Gamma_{<p}$ and we obtain the following property.

Proposition 5.3. *The following sequence of profinite groups is exact*

$$\mathcal{G}_{<p} \xrightarrow{\iota^{\pi}_{<p}} \Gamma_{<p} \xrightarrow{j} \prod_{1\leqslant m\leqslant N} \langle \tau_m \rangle^{\mathbb{Z}/p} \longrightarrow 1\,.$$

5.3. Isomorphism $\kappa_{<p}$. Let $\bar{c}^1 \in \mathbb{Z}^N_{>\bar{0}}$ be such that $p = \pi^{\bar{c}^1}u$, where $u \in \mathcal{O}_K^*$ (as earlier, $\pi^{\bar{c}^1} = \pi_1^{c_1^1}\dots\pi_N^{c_N^1}$ with $\bar{c}^1 = (c_1^1, \dots, c_N^1)$).

Set $\bar{c}^0 = p\bar{c}^1/(p-1)$. Note that $\bar{c}^0 \in \mathbb{Z}^N_{>\bar{0}}$, because $\zeta_1 \in K$, $\zeta_1 - 1 \in \pi^{\bar{c}^2}\mathcal{O}_K^*$ with $\bar{c}^2 \in \mathbb{Z}^N_{>\bar{0}}$ and $(p-1)\bar{c}^2 = \bar{c}^1$.

Consider the auxillary Lie algebra $\bar{\mathcal{M}}_{<p}$ from Sect. 4.4 and its analogue

$$\bar{\mathcal{M}}_{R_0(N)} = \left(\sum_{1\leqslant s<p} t^{-s\bar{c}^0}\bar{\mathcal{L}}_{\bar{c}^0}(s)_{\mathrm{m}_{R(N)}}\right) \otimes R(N)/t^{(p-1)\bar{c}^0}\,.$$

Clearly, $\bar{f} \in \bar{\mathcal{L}}_{R_0(N)}$ and $\bar{f} \otimes 1 \in \bar{\mathcal{M}}_{R_0(N)}$. (Recall that $\bar{\pi} = \pi_{\bar{f}}(\bar{e}) : \bar{\mathcal{G}} \simeq G(\bar{\mathcal{L}})$.)

Consider a $v_{\mathcal{K}}^1$-continuous embedding η of the field $\mathcal{K}$ into $R_0(N)$ such that the image of $(\mathrm{id}_{\bar{\mathcal{L}}} \otimes \eta)\bar{e}$ in $\bar{\mathcal{M}}_{R_0(N)}$ coincides with $\bar{e} \otimes 1$.

Such field embedding satisfies $\eta|_k = \mathrm{id}$ and is uniquely determined by a choice of the elements $\eta(t_m) \in R(N)_0$, where $1 \leqslant m \leqslant N$, such that $\eta(t_m) \equiv t_m \bmod t^{(p-1)\bar{c}^0}\mathrm{m}_{R(N)}$.

Proposition 5.4. *There is a unique lift $\bar{\eta}$ of η to $\mathcal{K}(p)$ such that*

$$(\mathrm{id}_{\bar{\mathcal{L}}} \otimes \bar{\eta})\bar{f} \otimes 1 = \bar{f} \otimes 1\,.$$

Proof. Let $\hat{\eta}$ be an extension of η to $\mathcal{K}_{sep}$. Clearly, $\sigma((\mathrm{id}_{\bar{\mathcal{L}}} \otimes \hat{\eta})\bar{f} \otimes 1) = (\bar{e} \otimes 1) \circ ((\mathrm{id}_{\bar{\mathcal{L}}} \otimes \hat{\eta})\bar{f} \otimes 1)$. So,

$$(-(\bar{f} \otimes 1)) \circ ((\mathrm{id}_{\bar{\mathcal{L}}} \otimes \hat{\eta})\bar{f} \otimes 1) \in \bar{\mathcal{M}}_{R_0(N)}|_{\sigma=\mathrm{id}} = \bar{\mathcal{L}}\,.$$

In other words, there is $l \in \bar{\mathcal{L}}$ such that for $g = \bar{\pi}^{-1}(l)$, it holds

$$(\mathrm{id}_{\bar{\mathcal{L}}} \otimes \hat{\eta})\bar{f} \otimes 1 = (\bar{f} \otimes 1) \circ l = g(\bar{f}) \otimes 1\,.$$

As a result, we can take $\bar{\eta} = \hat{\eta} \cdot g^{-1}$. The uniqueness of $\bar{\eta}$ is obvious because any two such lifts differ by an element of $\bar{\mathcal{G}}$ but $\bar{\mathcal{G}}$ acts strictly on $\bar{f} \otimes 1$. □

Let $\varepsilon = (\zeta_n \bmod p)_{n\geqslant 0} \in R \subset R(N)$ be Fontaine's element (here $\zeta_1 \in K$ is our p-th root of unity and for all n, $\zeta_n^p = \zeta_{n-1}$).

Let $\zeta_1 = 1 + \pi^{\bar{c}^2}\sum_{\iota\geqslant\bar{0}}[\beta_\iota]\pi^\iota$, where all $[\beta_\iota]$ are the Teichmuller representatives of $\beta_\iota \in k$ and $\beta_{\bar{0}} \neq 0$. Here $\pi^{\bar{c}^2}\mathcal{O}_K = (\zeta_1 - 1)\mathcal{O}_K = p^{1/(p-1)}\mathcal{O}_K$, i.e. $p\bar{c}^2 = \bar{c}_0$.

Consider the identification of rings $R(N)/t^{\bar{c}^1} \simeq \mathcal{O}_{K_{alg}}/p$, coming from the projection $R(N) = \varprojlim_{n\geqslant 0}(\mathcal{O}_{K_{alg}})_n$ to $(\mathcal{O}_{K_{alg}})_0 \bmod p$. This implies

$\sigma^{-1}\varepsilon\equiv 1+\sum_{\iota\geqslant\bar 0}\beta_\iota t^{\bar c^2+\iota}\operatorname{mod} t^{\bar c^1}R(N)$ and, therefore,

$$\varepsilon\equiv 1+\sum_{\iota\geqslant\bar 0}\beta_\iota^p t^{\bar c^0+p\iota}\operatorname{mod} t^{(p-1)\bar c^0}R(N)\,.$$

Assume the morphisms $h_\omega^{(m)}\in\mathrm{Aut}\mathcal K$ from Sect. 4.1 are determined by ω such that $E(\omega^p)=1+\sum_{\iota\geqslant\bar 0}\beta_\iota^p t^{\bar c^0+p\iota}$, i.e. for all ι, $A_\iota(\omega)=\beta_\iota^p$. As a result, for any m, $\tau_m(t)\equiv h_\omega^{(m)}(t)\operatorname{mod} t^{(p-1)\bar c^0}\mathrm m_{R(N)}$.

Suppose $\tau\in\Gamma$ (recall that $\Gamma\subset\mathrm{Aut}\,R_0(N)$). Then for some integers $m_1,\dots,m_N$, we have the following congruence modulo $t^{(p-1)\bar c^0}\mathrm m_{R(N)}$

$$\tau(t)=\{t_1\varepsilon^{m_1},\dots,t_N\varepsilon^{m_N}\}\equiv\{h_\omega^{(1)m_1}(t_1),\dots,h_\omega^{(N)m_N}(t_N)\}\,.$$

Let $h_\tau=h_\omega^{(1)m_1}\dots h_\omega^{(N)m_N}\in\mathrm{Aut}\,\mathcal K$. Then (use Prop. 4.1)

$$\tau|_{\mathcal K}(t)\equiv h_\tau(t)\operatorname{mod} t^{(p-1)\bar c^0}\mathrm m_{R(N)}\,.$$

This means that $\eta:=\tau^{-1}|_{\mathcal K}\cdot h_\tau:\mathcal K\longrightarrow R_0(N)$ satisfies the assumption of Prop. 5.4 and we can consider the corresponding lift $\bar\eta:\mathcal K(p)\longrightarrow R_0(N)$. Let $\hat\eta$ be a lift of $\bar\eta$ to $\mathcal K_{sep}$.

Set $\bar h_\tau:=(\tau\cdot\hat\eta)|_{\mathcal K(p)}$.

Then $\bar h_\tau|_{\mathcal K}=(\tau\cdot\hat\eta)|_{\mathcal K}=h_\tau$ and by Galois theory $\bar h_\tau\in\Gamma_\omega\subset\mathrm{Aut}\mathcal K(p)$.

As a result, we obtained the map (of sets) $\kappa:\Gamma\longrightarrow\Gamma_\omega$ uniquely characterized by the property $(\mathrm{id}_{\bar{\mathcal L}}\otimes\tau)\bar f=(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau))\bar f$.

Proposition 5.5. *κ induces a group isomorphism $\kappa_{<p}:\Gamma_{<p}\longrightarrow\Gamma_\omega$.*

Proof. Suppose $\tau_1,\tau\in\Gamma$. Let $\bar C\in\bar{\mathcal L}_{\mathcal K}$ and $\bar{\mathcal A}\in\mathrm{Aut}_{\bar{\mathcal L}}$ be such that $(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau))\bar f=\bar C\circ(\bar{\mathcal A}\otimes\mathrm{id}_{\mathcal K(p)})\bar f$. Then

$$(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau_1\tau))\bar f=(\mathrm{id}_{\bar{\mathcal L}}\otimes\tau_1\tau)\bar f=(\mathrm{id}_{\bar{\mathcal L}}\otimes\tau_1)(\mathrm{id}_{\bar{\mathcal L}}\otimes\tau)\bar f$$

$$=(\mathrm{id}_{\bar{\mathcal L}}\otimes\tau_1)(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau))\bar f=(\mathrm{id}_{\bar{\mathcal L}}\otimes\tau_1)(\bar C\circ(\bar{\mathcal A}\otimes\mathrm{id}_{\mathcal K(p)})\bar f)=$$

$$(\mathrm{id}_{\bar{\mathcal L}}\otimes\tau_1)\bar C\circ(\bar{\mathcal A}\otimes\tau_1)\bar f=(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau_1))\bar C\circ(\bar{\mathcal A}\otimes\mathrm{id}_{\mathcal K(p)})(\mathrm{id}_{\bar{\mathcal L}}\otimes\tau_1)\bar f=$$

$$(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau_1))\bar C\circ(\bar{\mathcal A}\otimes\mathrm{id}_{\mathcal K(p)})(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau_1))\bar f=(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau_1))(\bar C\circ(\bar{\mathcal A}\otimes\mathrm{id}_{\mathcal K(p)})\bar f$$

$$=(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau_1))(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau))\bar f=(\mathrm{id}_{\bar{\mathcal L}}\otimes\kappa(\tau_1)\kappa(\tau))\bar f$$

and, therefore, $\kappa(\tau_1\tau)=\kappa(\tau_1)\kappa(\tau)$ (use that Γ_ω acts strictly on the orbit of $\bar f$). In particular, κ factors through the natural projection $\Gamma\to\Gamma_{<p}$ and defines the group homomorphism $\kappa_{<p}:\Gamma_{<p}\to\Gamma_\omega$.

Recall that we have the identification of $\Gamma_{K^\pi}=\mathrm{Gal}(K_{alg}/K^\pi)$ with $\mathcal G=\mathrm{Gal}(\mathcal K_{sep}/\mathcal K)$ and, therefore, $\kappa_{<p}$ identifies the groups $\kappa(\Gamma_{K^\pi})$ and $G(\bar{\mathcal L})\subset\Gamma_\omega$. Besides, $\kappa_{<p}$ induces a group isomorphism of the group $\mathrm{Gal}(K(\pi_{11},\dots,\pi_{1N})/K)=\langle\tau_1\rangle^{\mathbb Z/p}\times\dots\times\langle\tau_m\rangle^{\mathbb Z/p}$ and the quotient $\langle h_\omega^{(1)}\rangle^{\mathbb Z/p}\times\dots\times\langle h_\omega^{(N)}\rangle^{\mathbb Z/p}$ of Γ_ω. Now Proposition 5.3 implies that $\kappa_{<p}$ is a group isomorphism. ∎

5.4. Groups $\Gamma^{\mathcal{P}}_{<p}$. Consider the group isomorphism $\kappa_{<p}:\Gamma_{<p}\longrightarrow\Gamma_\omega$ from Sect. 5.3.

Definition. $\mathrm{cl}^{\mathcal{P}}\Gamma_{<p}$ is the class of conjugated subgroups in $\Gamma_{<p}$ containing $\Gamma^{\mathcal{P}}_{<p}:=\kappa^{-1}(\Gamma^{\mathcal{P}}_\omega)$.

This definition involves a choice of local parameters in K.

Proposition 5.6. *The class $\mathrm{cl}^{\mathcal{P}}\Gamma_{<p}$ does not depend on a choice of a system of local parameters in K.*

Proof. Let $\pi=\{\pi_1,\dots,\pi_N\}$ and $\pi'=\{\pi'_1,\dots,\pi'_{1N}\}$ be two systems of local parameters in K. Let $K_1^\pi=K(\pi_{11},\dots,\pi_{1N})$ and $K_1^{\pi'}=K(\pi'_{11},\dots,\pi'_{1N})$, where (as earlier) for all m, $\pi_{1m}^p=\pi_m$ and $\pi_{1m}'^p=\pi'_m$. Denote by $K^{(1)}$ the composite of K_1^π and $K_1^{\pi'}$.

Consider the SDR-towers $K_\bullet^\pi$ and $K_\bullet^{\pi'}$, the fields-of-norms $\mathcal{K}=X(K_\bullet^\pi)$ and $\mathcal{K}'=X(K_\bullet^{\pi'})$ with the corresponding systems of local parameters $t=\{t_1,\dots,t_N\}$ and $t'=\{t'_1,\dots,t'_N\}$.

Then we have the appropriate field extensions

$$\mathcal{K}\subset\mathcal{K}^{(1)}\subset\mathcal{K}(p)\subset\mathcal{K}_{<p}\subset R_0(N)$$

$$\mathcal{K}'\subset\mathcal{K}'^{(1)}\subset\mathcal{K}'(p)\subset\mathcal{K}'_{<p}\subset R_0(N)$$

where $\mathcal{K}^{(1)}=X(K^{(1)}K_\bullet^\pi)$, $\mathcal{K}'^{(1)}=X(K^{(1)}K_\bullet^{\pi'})$, the fields $\mathcal{K}_{<p}$ and $\mathcal{K}(p)$ were defined earlier, $\mathcal{K}'_{<p}$ and $\mathcal{K}'(p)$ are their analogs if $\mathcal{K}$ is replaced by $\mathcal{K}'$. The exact sequence from Sect. 4.5

$$1\longrightarrow\bar{\mathcal{G}}\longrightarrow\Gamma_\omega\longrightarrow H_\omega\longrightarrow 1\,,$$

with $H_\omega:=\prod\limits_{1\leqslant m\leqslant N}\langle h_\omega^{(m)}\rangle/\langle h_\omega^{(m)p}\rangle$, gives rise to the exact sequence

$$1\longrightarrow\bar{\mathcal{G}}^{(1)}\longrightarrow\Gamma_\omega\longrightarrow\mathrm{Gal}(\mathcal{K}^{(1)}/\mathcal{K})\times H_\omega\longrightarrow 1\,, \tag{5.2}$$

where $\bar{\mathcal{G}}^{(1)}=\mathrm{Gal}(\mathcal{K}(p)/\mathcal{K}^{(1)})$. We have a similar sequence for $\mathcal{K}'$

$$1\longrightarrow\bar{\mathcal{G}}'^{(1)}\longrightarrow\Gamma'_{\omega'}\longrightarrow\mathrm{Gal}(\mathcal{K}'^{(1)}/\mathcal{K}')\times H'_{\omega'}\longrightarrow 1\,, \tag{5.3}$$

where $H'_{\omega'}$ is an analog of H_ω. The isomorphisms $\kappa_{<p}:\Gamma_\omega\simeq\Gamma_{<p}$ and $\kappa'_{<p}:\Gamma'_{\omega'}\simeq\Gamma_{<p}$ induce the isomorphisms

$$\mathrm{Gal}(K^{(1)}/K)\simeq\mathrm{Gal}(\mathcal{K}^{(1)}/\mathcal{K})\times H_\omega\simeq\mathrm{Gal}(\mathcal{K}'^{(1)}/\mathcal{K}')\times H'_{\omega'}\,,$$

$$\mathrm{Gal}(K_{<p}/K^{(1)})\simeq\bar{\mathcal{G}}^{(1)}\simeq\bar{\mathcal{G}}'^{(1)}\,. \tag{5.4}$$

We want to study relation between the $\mathcal{P}$-structures on $\bar{\mathcal{G}}^{(1)}$ and $\bar{\mathcal{G}}'^{(1)}$ (induced from $\bar{\mathcal{G}}$ and $\bar{\mathcal{G}}'$).

Recall, cf. Sect. 4.4, there is a Lie algebra

$$\bar{\mathcal{M}}=\left(\sum_s t^{-s\bar{c}^0}\mathrm{m}_{\mathcal{K}}\bar{\mathcal{L}}_{\bar{c}^0}(s)\right)\otimes\mathcal{O}_{\mathcal{K}}/t^{(p-1)\bar{c}^0}$$

and its analogue $\bar{\mathcal{M}}_{\mathcal{K}(p)}$, where $\mathcal{K}$ is replaced by $\mathcal{K}(p)$. There are $\bar{e}\otimes 1\in\bar{\mathcal{M}}$ and $\bar{f}\otimes 1\in\bar{\mathcal{M}}_{\mathcal{K}(p)}$ such that $\sigma(\bar{f}\otimes 1)=(\bar{e}\otimes 1)\circ(\bar{f}\otimes 1)$ and the identification $\pi_{\bar{f}}(\bar{e}):\bar{\mathcal{G}}\simeq G(\bar{\mathcal{L}})$ is given by $\tau\mapsto(-\bar{f}\otimes 1)\circ(\tau\bar{f}\otimes 1)$.

The algebra $\bar{\mathcal{L}}^{\mathcal{P}}$ (as well as the group $\bar{\mathcal{G}}^{\mathcal{P}}$) is defined in terms of the $\mathcal{P}$-topological structure on $\bar{\mathcal{M}}$ coming from the corresponding structures on $\mathcal{O}_{\mathcal{K}}/t^{(p-1)\bar{c}^0}$ and $\bar{\mathcal{L}}$. Note that $\bar{\mathcal{L}}$ is the image of $\mathcal{L}$ and the $\mathcal{P}$-topology on $\mathcal{L}$ is induced from $L/C_2(L)=\mathrm{Hom}(\bar{\mathcal{K}},\mathbb{F}_p)$. Therefore, the $\mathcal{P}$-structure on $\bar{\mathcal{L}}$ comes from

$$\bar{\mathcal{L}}/C_2(\bar{\mathcal{L}})=\mathrm{Hom}(t^{-(p-1)\bar{c}^0}\mathrm{m}_{\mathcal{K}}/(\sigma-\mathrm{id})\mathcal{K},\mathbb{F}_p)\,.$$

As a result,

$$\bar{e}\otimes 1\in\bar{\mathcal{M}}^{\mathcal{P}}=\left(\sum_s t^{-s\bar{c}^0}\mathrm{m}_{\mathcal{K}}\bar{\mathcal{L}}^{\mathcal{P}}_{\bar{c}^0}(s)\right)\otimes^{\mathcal{P}}\mathcal{O}_{\mathcal{K}}/t^{(p-1)\bar{c}^0}$$

and $\pi_{\bar{f}}(\bar{e})^{-1}G(\bar{\mathcal{L}}^{\mathcal{P}})=\bar{\mathcal{G}}^{\mathcal{P}}$.

Similar construction is used to obtain (in the context of $\mathcal{K}'$) that

$$\bar{e}'\otimes 1\in\bar{\mathcal{M}}'^{\mathcal{P}}=\left(\sum_s t'^{-s\bar{c}'^0}\mathrm{m}_{\mathcal{K}'}\bar{\mathcal{L}}'^{\mathcal{P}}_{\bar{c}'^0}(s)\right)\otimes^{\mathcal{P}}\mathcal{O}_{\mathcal{K}'}/t'^{(p-1)\bar{c}'^0}$$

and $\pi_{\bar{f}'}(\bar{e}')^{-1}G(\bar{\mathcal{L}}'^{\mathcal{P}})=\bar{\mathcal{G}}'^{\mathcal{P}}$.

The $\mathcal{P}$-subgroups $\bar{\mathcal{G}}^{\mathcal{P}}$ and $\bar{\mathcal{G}}'^{\mathcal{P}}$ "live in different worlds", but the field-of-norms functor X identifies them with subgroups in $\Gamma_{<p}=\mathrm{Gal}(K_{<p}/K)$. This procedure can be specified as follows.

Let $\Gamma_{<p}=G(L)$, where L is a suitable Lie $\mathbb{F}_p$-algebra. Introduce

$$\bar{M}=\left(\sum_{1\leqslant s<p}(\zeta_1-1)^{-s}C_s(L)_{\mathrm{m}_K}\right)\otimes\mathcal{O}_K/p\,.$$

$$\bar{M}_{<p}=\left(\sum_{1\leqslant s<p}(\zeta_1-1)^{-s}C_s(L)_{\mathrm{m}_{K_{<p}}}\right)\otimes\mathcal{O}_{K_{<p}}/p\,.$$

The projection $\mathrm{pr}_1:R(N)=\varprojlim_{n\geqslant 0}(\mathcal{O}_{K_{alg}}/p)_n\longrightarrow(\mathcal{O}_{K_{alg}}/p)_1$ establishes the ring isomorphism $R(N)/t^{p\bar{c}^1}=R(N)/t^{(p-1)\bar{c}^0}\simeq\mathcal{O}_{K_{alg}}/p$, cf. Sect. 5.4 (note that $\mathrm{Ker}\,\mathrm{pr}_1$ is generated by $t^{(p-1)\bar{c}^0}$).

Consider the induced by pr_1 isomorphism $\mathcal{O}_{\mathcal{K}(p)}/t^{(p-1)\bar{c}^0}\simeq\mathcal{O}_{K_{<p}}/p$. This gives for each $1\leqslant s<p$, the compatible identifications of the $\mathcal{O}_{\mathcal{K}(p)}/t^{(p-1)\bar{c}^0}$-module $t^{-s\bar{c}^0}\mathrm{m}_{\mathcal{K}(p)}\otimes\mathcal{O}_{\mathcal{K}(p)}/t^{(p-1)\bar{c}^0}$ with $\mathcal{O}_{K_{<p}}/p$-module $(\zeta_1-1)^{-s}\mathrm{m}_{K_{<p}}\otimes\mathcal{O}_{K_{<p}}/p$. In addition, the field-of-norms functor identifies the group $\bar{\mathcal{G}}$ with $\Gamma_1^{\pi}=\mathrm{Gal}(K_{<p}/K_1^{\pi})\subset\Gamma$ and the Lie algebra $\bar{\mathcal{L}}$ with the Lie subalgebra $L_1^{\pi}\subset L$, where $G(L_1^{\pi})=\Gamma_1^{\pi}$. As a result, we obtain the embedding $F_{<p}:\bar{\mathcal{M}}_{\mathcal{K}(p)}\longrightarrow\bar{M}_{<p}$ and the induced embedding $F=F_{<p}|_{\bar{\mathcal{M}}}:\bar{\mathcal{M}}\longrightarrow\bar{M}$.

Set $e^{\pi}=F(\bar{e}\otimes 1)$ and $f^{\pi}=F_{<p}(\bar{f}\otimes 1)$. Then $\sigma f^{\pi}=e^{\pi}\circ f^{\pi}$ and the map $\tau\mapsto(-f^{\pi})\circ\tau(f^{\pi})$ recovers the identification $\Gamma_1^{\pi}\simeq\bar{\mathcal{G}}$ or,

equivalently, $\kappa^\pi : L_1^\pi \simeq \bar{\mathcal{L}}$. (We used it earlier when constructing the isomorphism $\kappa_{<p}$.)

The identification κ^π is compatible with the $\mathcal{P}$-topology.

Indeed, the $\mathcal{P}$-topological structure on L_1^π comes (via tensor topology) from $L_1^\pi/C_2(L_1^\pi) = \mathrm{Hom}(p^{-1}\mathrm{m}_K/(\sigma - \mathrm{id})K, \mathbb{F}_p)$ and the field-of-norms identification of $p^{-1}\mathrm{m}_K/(\sigma - \mathrm{id})K$ with $t^{-(p-1)\bar{c}^0}\mathrm{m}_{\mathcal{K}}/(\sigma - \mathrm{id})\mathcal{K}$.

Repeating the above argiments in the context of the system of parameters π' we obtain the $\mathcal{P}$-continuous identification $\kappa^{\pi'} : L_1^{\pi'} \simeq \bar{\mathcal{L}}'$.

Let $\bar{\mathcal{G}}^{(1)} = G(\bar{\mathcal{L}}^{(1)})$ and $\bar{\mathcal{G}}'^{(1)} = G(\bar{\mathcal{L}}'^{(1)})$. Then isomorphism (5.4) appears as compatible with $\mathcal{P}$-structures isomorphisms $L^{(1)} \simeq \mathcal{L}^{(1)} \simeq \mathcal{L}'^{(1)}$. Therefore, the conjugacy classes of $(\kappa^\pi)^{-1}\bar{\mathcal{L}}^{(1)\mathcal{P}}$ and $(\kappa^{\pi'})^{-1}\bar{\mathcal{L}}'^{(1)\mathcal{P}}$ coincide.

Finally, applying the interpretation from Sect.3.6 we obtain from exact sequences (5.2) and (5.3) that the conjugate classes of $\kappa_{<p}(\Gamma_\omega^{\mathcal{P}})$ and $\kappa'_{<p}(\Gamma'^{\mathcal{P}}_{\omega'})$ coincide. ☐

As a result, $\Gamma_{<p}^{\mathcal{P}}$ is provided with the $\mathcal{P}$-topology induced from the subgroup $\bar{\mathcal{G}}^{\mathcal{P}}$ and this topology does not depend on a choice of such subgroup (i.e. on the choice of local parameters in K.

For any (local N-dimensional) subfield $K \subset K' \subset K_{<p}$, set $H^{\mathcal{P}} = H \cap \Gamma_{<p}^{\mathcal{P}}$, where $H = \mathrm{Gal}(K_{<p}/K')$. As earlier (in the case of local fields of characteristic p), we easily obtain the following property.

Corollary 5.7. *a) The profinite completion of* $H^{\mathcal{P}}$ *is* H*;*

b) $(\Gamma_{<p} : H) = (\Gamma_{<p}^{\mathcal{P}} : H^{\mathcal{P}}) = [K' : K]$*;*

c) for all subfields K'*, the subgroups* $H^{\mathcal{P}}$ *of* $\Gamma_{<p}^{\mathcal{P}}$ *can be characterized as all* $\mathcal{P}$*-open subgroups of finite index in* $\Gamma_{<p}^{\mathcal{P}}$.

5.5. Explicit structure of $\Gamma_{<p}^{\mathcal{P}}$. Recall that K is an N-dimensional local field, with the first residue field of characteristic p and the last residue field $k \simeq \mathbb{F}_{p^{N_0}}$. Review the above results about $\Gamma_{<p}$.

Suppose $\pi = \{\pi_1, \dots, \pi_N\}$ is a system of local parameters in K and $\zeta_1 \in K$ is a primitive p-th root of unity. Then

$$\zeta_1 = E\left(\sum_{\iota \geqslant \bar{0}} [\beta_\iota]\pi^{\bar{c}_2 + \iota}\right),$$

where $E(X) \in \mathbb{Z}_p[[X]]$ is the Artin-Hasse exponential, all $\iota \in \mathbb{Z}^N_{\geqslant \bar{0}}$, $[\beta_\iota]$ are the Teichmuller representatives of $\beta_\iota \in k$, $\beta_{\bar{0}} \neq 0$. (Recall that $\pi^{\bar{c}^2} := \pi_1^{c_1} \dots \pi_N^{c_N}$, where $\bar{c}^2 = (c_1, \dots, c_N)$.)

Associate with K the topological $\mathbb{F}_p$-algebra $L^{\mathcal{P}}$ as follows.

Consider an N-dimensional local field $\mathcal{K}$ of characteristic p with finite residue field $k \simeq \mathbb{F}_{p^{N_0}}$. Then we have the toplogical module $\bar{\mathcal{K}}^{\mathcal{P}D} := \mathrm{Hom}_{\mathcal{P}\text{-cont}}(\mathcal{K}/(\sigma - \mathrm{id})\mathcal{K}, \mathbb{F}_p)$, which generate the "maximal" $\mathbb{F}_p$-Lie algebra $\mathcal{L}^{\mathcal{P}}$ of nilpotent class $< p$, i.e. the quotient of free Lie algebra with module of generators $\bar{\mathcal{K}}^{\mathcal{P}D}$ by the ideal of p-th commutators. (Note that $\mathcal{L}^{\mathcal{P}}$ appears as the projective limit of "maximal" Lie algebras $\mathcal{L}_{\bar{C}_\alpha}$ (of nilpotent class $< p$) generated by the elements of open subsets $\bar{C}_\alpha^D \subset \mathcal{M}$.) If $t = \{t_1, \dots, t_N\}$ is a system of local parameters in $\mathcal{K}$ then $\mathcal{L}_k^{\mathcal{P}}$ is provided with natural system of $\mathcal{P}$-topological generators $\{D_{an} \mid a \in \mathbb{Z}_N^+(p), n \in \mathbb{Z}/N_0\} \cup \{D_0\}$. Recall that

$$\mathbb{Z}_N^+(p) = \{a \in \mathbb{Z}_{>\bar{0}}^N \mid \gcd(a, p) = 1\}, \mathbb{Z}_N^0(p) = \mathbb{Z}^+(p) \cup \{\bar{0}\}.$$

Let $\bar{c}^0 = p\bar{c}^2$ and introduce the weight function wt on $\mathcal{L}^{\mathcal{P}}$ by setting $\mathrm{wt}(D_{an}) = s$ if $(s-1)\bar{c}^0 \leqslant a < s\bar{c}^0$, $\mathrm{wt}(D_0) = 1$. Let $\mathcal{L}_k^{\mathcal{P}}(p)$ be the ideal in $L_k^{\mathcal{P}}$ of elements of weight $\geqslant p$, $\bar{\mathcal{L}}_k = \mathcal{L}_k/\mathcal{L}(p)$. Let σ be the Frobenius automorphism on k and denote by the same symbol the σ-linear automorphism of $\bar{\mathcal{L}}_k$ such that $D_{an} \mapsto D_{a,n+1}$, $D_{\bar{0}} \mapsto D_{\bar{0}}$. (We also set $D_{\bar{0}n} = \sigma^n(\alpha_0)D_{\bar{0}}$, where α_0 is a fixed element from $W(k) \subset K$ with absolute trace 1.)

Let $\bar{\mathcal{L}}^{\mathcal{P}} = \mathcal{L}_k^{\mathcal{P}}/\mathcal{L}_k^{\mathcal{P}}(p)|_{\sigma=\mathrm{id}}$ with induced $\mathcal{P}$-topological structure.

Introduce the Lie $\mathbb{F}_p$-algebra $L^\flat$ as the maximal algebra of nilpotent class $< p$ containing $\bar{\mathcal{L}}$ and the generators $\{\bar{l}^{(m)} \mid 1 \leqslant m \leqslant N\}$. Introduce the ideal of relations $\mathcal{R}_k^\flat$ in $L_k^\flat$ as follows.

Specify recurrent relation (4.9) to our situation:

$$\sigma\bar{c}_1^{(m)} - \bar{c}_1^{(m)} + \sum_{a \in \mathbb{Z}_N^0(p)} t^{-a} V_{a0}^{(m)} =$$

$$-\sum_{\iota \geqslant \bar{0}} \beta_\iota^p \sum_{1 \leqslant k < p} \frac{1}{k!} t^{-(a_1 + \dots + a_k) + p\iota} a_1^{(m)} [\dots [D_{a_1 0}, D_{a_2 0}], \dots, D_{a_k 0}]$$

$$- \sum_{2 \leqslant k < p} \frac{1}{k!} t^{-(a_1 + \dots + a_k)} [\dots [V_{a_1 0}^{(m)}, D_{a_2 0}], \dots, D_{a_k 0}]$$

$$- \sum_{1 \leqslant k < p} \frac{1}{k!} t^{-(a_1 + \dots + a_k)} [\dots [\sigma\bar{c}_1^{(m)}, D_{a_1 0}], \dots, D_{a_k 0}] \tag{5.5}$$

(the indices $a_1, \dots, a_k$ in all above sums run over $\mathbb{Z}_N^0(p)$). Recall that $\bar{c}_1^{(m)} \in \bar{\mathcal{L}}_{\mathcal{K}}^{\mathcal{P}}$ and $V_{a0} \in \bar{\mathcal{L}}_k^{\mathcal{P}}$. Then the ideal $\mathcal{R}_k^\flat \subset L_k^\flat$ is generated by the following elements:

1) $[D_{an}, \bar{l}^{(m)}] - \sigma^n(V_{a0}^{(m)})$, $a \in \mathbb{Z}_N^0(p)$;

2) $[\bar{l}^{(i)}, \bar{l}^{(j)}] - \bar{l}[i,j]$, where $\bar{l}[i,j] \in \bar{\mathcal{L}}$ is given in notation of Cor. 4.11

$$\bar{l}[i,j] = \mathrm{ad}^{(i)}(\bar{c}_1^{(j)}(\bar{0})) - \mathrm{ad}^{(j)}(\bar{c}_1^{(i)}(\bar{0})) + \sum_{\iota \in \mathbb{Z}^N} [\bar{c}_1^{(i)}(\iota), \bar{c}_1^{(j)}(-\iota)].$$

Let $L_k = L_k^\flat/\mathcal{R}_k^\flat$ and $L = L_k|_{\sigma=\mathrm{id}}$.

Remark. Taking another solutions $\{\bar{c}_1^{(m)}, V_{a0}^{(m)} \mid a \in \mathbb{Z}_N^0(p), 1 \leqslant m \leqslant N\}$ of (5.5) is equivalent to replacing the generators $\bar{l}^{(m)}$ modulo $\bar{\mathcal{L}}^{\mathcal{P}}$.

Summarizing the above results we state the following theorem.

Theorem 5.8. *a)* $\Gamma^{\mathcal{P}}_{<p} \simeq G(L^{\mathcal{P}})$*;*

b) $\Gamma_{<p}$ *is the profinite completion of* $G(L^{\mathcal{P}})$*.*

5.6. Simplest example. The above description of $\Gamma_{<p}$ essentially uses the equivalence of the category of p-groups and $\mathbb{F}_p$-Lie algebras (there is no operation of extension of scalars in the category of p-groups). It could be also verified that the study of involved p-groups at the level of their Lie algebras gives much simpler form of the corresponding relations. At the same time the above presentation of $L^{\mathcal{P}}$ as the quotient of $L^{\flat}$ by appropriate relations is not the simplest one. Some relations, e.g. $[D_{an}, \bar{l}^{(m)}] - V_{an}$ with $a > \bar{c}^0$ can be used to exclude extra generators D_{an} with $a > \bar{c}^0$. Ideally, the whole description should be done in terms of, say, the minimal system of generators $\{D_{an} \mid a < \bar{c}^0\} \cup \{\bar{l}^{(m)} \mid 1 \leqslant m \leqslant N\}$. This was done in Sect. 4.7 where we presented our description modulo third commutators.

In the case of the last residue field $k \simeq \mathbb{F}_p$ we do not need the operation of extension of scalars and can express the answer directly in terms of groups. This will not give any simplifications, but can be easily obtained when working modulo third commutators.

Namely, if $k \simeq \mathbb{F}_p$ then $\Gamma^{\mathcal{P}}_{<p} \operatorname{mod} C_3(\Gamma^{\mathcal{P}}_{<p})$ appears as the group with $\mathcal{P}$-topological generators $\{\tau_a \mid a \in \mathbb{Z}^0_N(p), a < \bar{c}^0\} \cup \{\bar{h}^{(m)} \mid 1 \leqslant m \leqslant N\}$ and the subgroup of relations generated (as normal subgroup) by following relations:

- $R(i,j) = (\bar{h}^{(i)}, \bar{h}^{(j)})$, here $1 \leqslant i < j \leqslant N$;

- $$R_{\bar{0}}(m) = (\tau_{\bar{0}}, \bar{h}^{(m)}) \prod_{\iota \geqslant \bar{0}} \prod_{\substack{b+c= \\ \bar{c}^0+p\iota}} (\tau_b, \tau_c)^{b^{(m)}\beta_\iota/2};$$

- $$R_a(i,j) = (\tau_a, \bar{h}^{(i)a^{(j)}}/\bar{h}^{(j)a^{(i)}}) \prod_{\substack{n\geqslant 1 \\ \iota\geqslant 0}} \prod_{\substack{b+c/p^n= \\ \bar{c}^0+p\iota+a/p^n}} (\tau_b, \tau_c)^{(b^{(i)}c^{(j)}-b^{(j)}c^{(i)})\beta_\iota} \times$$
$$\times \prod_{\substack{n\geqslant 0 \\ \iota\geqslant \bar{0}}} \prod_{\substack{b+c= \\ \bar{c}^0+p\iota+ap^n}} (\tau_b, \tau_c)^{(b^{(i)}c^{(j)}-b^{(j)}c^{(i)})\beta_\iota/2};$$

here $1 \leqslant m \leqslant N$, $1 \leqslant i < j \leqslant N$, $a \in \mathbb{Z}^+_N(p)$.

The above example could be simplified if we take 2-dimensional $K = \mathbb{Q}_p(\zeta_1)\{\{\pi_2\}\}$ with the system of local parameters $\pi = \{\pi_1, \pi_2\}$, where $E(\pi_1) = \zeta_1$. In this case $\bar{c}^0 = (p, 0)$, $\beta_{\bar{0}} = 1$ and all remaining $\beta_\iota = 0$.

We have the system of minimal generators

$$\{\tau_a \mid a \in \mathbb{Z}^+_2(p), a < (p,0)\} \cup \{\tau_{\bar{0}}\} \cup \{\bar{h}^{(1)}, \bar{h}^{(2)}\}$$

and the following relations:

- $\mathcal{R}(1,2) = (\bar{h}^{(1)}, \bar{h}^{(2)})$;

- $\mathcal{R}_{\bar{0}}(1) = (\tau_{\bar{0}}, \bar{h}^{(1)}) \prod_{1\leqslant\alpha\leqslant\frac{p-1}{2}} \prod_{\gamma} (\tau_{(\alpha,\gamma)}, \tau_{(p-\alpha,-\gamma)})^{\alpha}$

- $\mathcal{R}_{\bar{0}}(2) = (\tau_{\bar{0}}, \bar{h}^{(2)}) \prod_{1\leqslant\alpha\leqslant\frac{p-1}{2}} \prod_{\gamma} (\tau_{(\alpha,\gamma)}, \tau_{(p-\alpha,-\gamma)})^{\gamma}$

- $\mathcal{R}_{a}(1,2):\ (\tau_a, \bar{h}^{(1)a^{(2)}}/\bar{h}^{(2)a^{(1)}}) \times \prod_{\gamma} (\tau_{(p-1,\gamma)}, \tau_{(p,a^{(2)}-p\gamma)})^{-a^{(2)}\delta_{0,a^{(1)}}}$

$$\times \prod_{n\geqslant 1,\beta} (\tau_{(p,-\beta)}, \tau_{a+(0,p^n\beta)})^{a^{(1)}\beta} \times \prod_{\substack{b+c=\\(p,0)+a}} (\tau_b, \tau_c)^{(b^{(1)}c^{(2)}-b^{(2)}c^{(1)})/2}$$

References

[1] V.A.Abrashkin, *Ramification filtration of the Galois group of a local field*, Proceedings of the St. Petersburg Mathematical Society, vol. III, 35-100, Amer. Math. Soc. Transl. Ser. 2, (1995) **166**, Amer. Math. Soc., Providence, RI

[2] V.A. Abrashkin, *Ramification filtration of the Galois group of a local field. II*, Proceedings of Steklov Math. Inst. **208** (1995), 18-69

[3] V.Abrashkin, *Ramification filtration of the Galois group of a local field. III*, Izvestiya RAN: Ser. Mat., **62**, no.5 (1998), 3-48; English transl. Izvestiya: Mathematics **62**, no.5, 857–900

[4] V. Abrashkin, *On a local analogue of the Grothendieck Conjecture*, Int. J. Math. (2000) **11**, no.1, 3–43

[5] V. Abrashkin, *Ramification theory for higher dimensional fields*, Contemp. Math. (2002) **300**, 1-16

[6] V. Abrashkin, *Characteristic p case of the Grothendieck conjecture for 2-dimensional local fields*, Proceedings of Steklov Institute (2003) **241**, 1-35

[7] V. Abrashkin *An analogue of the fields-of-norms functor and the Grothendieck Conjecture*, J. of Algebraic Geometry, **16** (2007), pp. 671-730

[8] V. Abrashkin, *Modified proof of a local analogue of the Grothendieck Conjecture*, Journal Théorie des Nombres de Bordeaux **22**, (2010), 1-50

[9] V.Abrashkin, *Galois groups of local fields, Lie algebras and ramification.* In: Arithmetic and Geometry, eds. Dieulefait, L., Faltings, G., Heath-Brown, D.R., Manin, Yu., Moroz, B.Z., Wintenberger, J.-P. Cambridge University Press. **420**: 1-23

[10] V.Abrashkin, R.Jenni *The field-of-norms functor and the Hilbert symbol for higher local fields* J.Théor. Nombres Bordeaux, **24** (2012), no.1, 1-39

[11] V. Abrashkin, *Groups of automorphisms of local fields of period p and nilpotent class < p, I*, Int. J. Math. (2017) **28**, no. 6: 1750043

[12] V. Abrashkin, *Groups of automorphisms of local fields of period p and nilpotent class < p, II.* Int. J. Math. (2017) **28**

[13] V. Abrashkin, *Groups of automorphisms of local fields of period p^M and nilpotent class < p*, Ann. Inst. Fourier (2017) **67**, no. 2, 605-635

[14] V. Abrashkin, *Ramification filtration via deformations*, Sbornik: Mathematics (2021) **212**, no.?, pp?

[15] D.G. Benois, S.V. Vostokov, *On P-Extensions of Multidimensional Local Fields*, Math. Nachr. **160** (1993), 59-68

[16] P. Berthelot, W. Messing, *Théorie de Deudonné Cristalline III: Theorémes d'Équivalence et de Pleine Fidelité*, The Grothendieck Festschrift (P.Cartier etc., eds.), A Collection of Articles Written in Honor of 60th Birthday of Alexander Grothendieck, vol. 1, Birkhauser, 1990, p. 173-247.

[17] F. Bogomolov, Y. Tschinkel, *Commuting elements in Galois groups of function fields*, Motives, Polylogarithms and Hodge theory, 2002, pp.75-120

[18] N. Bourbaki, *Elements of Mathematics. Lie Groups and Lie Algebras.*

[19] Colas Bordavid, *Profinite completion and double-dual: isomorphisms and counter-examples*, hal-00208000, version 1 - 18 Jan 2008

[20] U.Jannsen, K.Wingberg, *Die Struktur der absoluten Galoisgruppe p-adischer Zahlkörper*, Invent. math. (1982) **70**, 71-98

[21] K. Kato, *A generalization of local class field theory by using K-groups I*, J. Fac. Sci. Univ. Tokyo Sec. IA 26 No. 2 (1979). 303-376

[22] K. Kato, *A generalization of local class field theory*, J. Fac. Sci. Univ. Tokyo Sec. IA 27 No. 3 (1980). 603-683

[23] K. Kato, *Existence theorem for higher local class field theory*, (Münster, 1999), Geom. Topol. Monogr., (2000), no.3, 165-195

[24] H.Koch, *Galois theory of p-extensions*, Springer Monographs in Mathematics, 2002, XIII, 191 p

[25] M. Lazard, *Sur les groupes nilpotentes et les anneaux de Lie*, Ann. Ecole Norm. Sup. (1954) **71**, 101-190

[26] A. I. Madunts, I. B. Zhukov, *Multidimensional complete fields: topology and other basic constructions*, Trudy S.Peterb. Mat. Obshch. (1995); English translation in: Amer. Math. Soc. Transl., (Ser.2) **165**, 1-34

[27] Sh.Mochizuki, *A version of the Grothendieck conjecture for p-adic local fields*, Int. J. Math.,**8**, no.4 (1997), 499-506

[28] A. N. Parshin, *Class fields and algebraic K-theory. (Russian)*, Uspekhi Mat. Nauk **30**,(1975), 253-254; English translation in: Russian Math. Surveys.

[29] A. N. Parshin, *Local class field theory. (Russian)*, Algebraic Geometry and its applications, Trudy Mat. Inst. Steklov **165**,(1985), 143-170; English translation in: Proc. Steklov Inst. Math., 1985, issue 3, 157-185.

[30] A. N. Parshin, *Galois cohomology and Brauer group of local fields. (Russian)*, Trudy Mat. Inst. Steklov **4**,(1990), 143-170; English translation in: Proc. Steklov Inst. Math., 1991, issue 4, 191-201.

[31] A. Scholl, *Higher fields of norms and (phi, Gamma)-modules*, Documenta Mathematica, Extra Volume: John H. Coates' Sixties Birthday (2006), 685-709

[32] J.-P.Serre, *Cohomologie Galoisienne* Springer Verlag, Berlin-Gottingen-Heidelberg-New York, 1964

[33] J.-P.Serre Bourbaki *Structure de certains pro-p-groupes (d'aprés Demushkin).* Séminaire Bourbaki, **8**, Exp. No. 252, 145155, Soc. Math. France, Paris, 1995.

[34] S. V. Vostokov, *Explicit construction of class field theory for a multidimensional local field.* Izv. Akad. Nauk SSSR Ser. Mat. (1985) no. 2; English translation in Math. USSR Izv. 26 (1986) 263-288

[35] K. Wingberg, *Galois groups of Poincare type*, *Galois groups of Poincaré type over algebraic number fields*, in: Galois groups over $\mathbb{Q}$, ed. by Y.Ihara, K.Ribet, J.-P.Serre, MSRI Publ. Springer (1989), 439-449

[36] J.-P. Wintenberger, *Le corps des normes de certaines extensions infinies des corps locaux; application.* Ann. Sci. Ec. Norm. Super., IV. Ser, **16** (1983), 59–89

[37] I. Zhukov, *Higher dimensional local fields* (Münster, 1999), Geom. Topol. Monogr., (2000), no.3, 5-18

[38] I. B. Zhukov, *Milnor and topological K-groups of multidimensional complete fields*, Algebra i analiz, (1997); English translation in St. Petersburg Math. J. **9** (1998), 69-105.

[39] I. Zhukov, *On ramification theory in the case of an imperfect residue field.* Sb. Math., **194** (2003), no. 12, 1747-1774

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