RAMIFICATION FILTRATION VIA DEFORMATIONS

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Abstract. Let $\mathcal{K}$ be a field of formal Laurent series with coefficients in a finite field of characteristic p, $\mathcal{G}_{<p}$ — the maximal quotient of the Galois group of $\mathcal{K}$ of period p and nilpotent class $< p$ and $\{\mathcal{G}_{<p}^{(v)}\}_{v\geqslant 0}$ — the filtration by ramification subgroups in the upper numbering. Let $\mathcal{G}_{<p} = G(\mathcal{L})$ be the identification of nilpotent Artin-Schreier theory: here $G(\mathcal{L})$ is the group obtained from a suitable profinite Lie $\mathbb{F}_p$-algebra $\mathcal{L}$ via the Campbell-Hausdorff composition law. We develop a new technique to the description of the ideals $\mathcal{L}^{(v)}$ such that $G(\mathcal{L}^{(v)}) = \mathcal{G}_{<p}^{(v)}$ and the explicite construction of their generators. Given $v_0 \geqslant 1$ we construct epimorphism of Lie algebras $\bar{\eta}^\dagger : \mathcal{L} \longrightarrow \bar{\mathcal{L}}^\dagger$ and an action Ω_U of the formal group of order p, $\alpha_p = \operatorname{Spec} \mathbb{F}_p[U]$, $U^p = 0$, on $\bar{\mathcal{L}}^\dagger$. Suppose $d\Omega_U = B^\dagger U$, where $B^\dagger \in \operatorname{Diff} \bar{\mathcal{L}}^\dagger$, and $\bar{\mathcal{L}}^\dagger[v_0]$ is the ideal of $\bar{\mathcal{L}}^\dagger$ generated by the elements of $B^\dagger(\bar{\mathcal{L}}^\dagger)$. The main result of the paper states that $\mathcal{L}^{(v_0)} = (\bar{\eta}^\dagger)^{-1}\bar{\mathcal{L}}^\dagger[v_0]$. In the last sections we relate this result to the explicit construction of generators of $\mathcal{L}^{(v_0)}$ obtained earlier by the author, develop its more efficient version and apply it to the recovering of the whole ramification filtration of $\mathcal{G}_{<p}$ from the set of its jumps.

Introduction

Let $\mathcal{K}$ be a complete discrete valuation field of characteristic p with finite residue field $k \simeq \mathbb{F}_{p^{N_0}}$, $N_0 \in \mathbb{N}$. Let $\mathcal{K}_{<p}$ be a maximal p-extension of $\mathcal{K}$ with the Galois group $\operatorname{Gal}(\mathcal{K}_{<p}/\mathcal{K}) := \mathcal{G}_{<p}$ of nilpotence class $< p$ and exponent p. The advantage of $\mathcal{G}_{<p}$ (compared to the whole Galois group $\mathcal{G}$ of $\mathcal{K}$) comes from the following fact: any p-group G of nilpotence class $s_0 < p$ and exponent p can be presented in the form $G(L)$, where L is a Lie $\mathbb{F}_p$-algebra of nilpotence class s_0 and the set $G(L) := L$ is provided with a group structure via the Campbell-Hausdorff composition law, cf. Sect. 1.2.

Consider the decreasing filtration by ramification subgroups in the upper numbering $\{\mathcal{G}_{<p}^{(v)}\}_{v\geqslant 0}$ of $\mathcal{G}_{<p}$. This filtration substantially reflects arithmetic structure of the field $\mathcal{K}$, cf. [7]. First results about the structure of these ramification subgroups were obtained by the author in [1]. This approach included:

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a) a construction of the identification $\mathcal{G}_{<p}=G(\mathcal{L})$, where $\mathcal{L}$ is explicitly defined Lie $\mathbb{F}_p$-algebra (nilpotent Artin-Shreier theory);

b) a construction of ideals $\mathcal{L}^{(v)}$ such that $\mathcal{G}_{<p}^{(v)}=G(\mathcal{L}^{(v)})$.

Namely, we constructed explicit elements $\mathcal{F}_{\alpha,-N}\in\mathcal{L}\otimes k$, where $\alpha\geqslant 1$ and $N\in\mathbb{Z}_{\geqslant 0}$, allowing us to characterize the ideals $\mathcal{L}^{(v)}$ as follows. Given $v_0\geqslant 1$ there is $N(v_0)\in\mathbb{Z}_{\geqslant 0}$ such that $\mathcal{L}^{(v_0)}$ is the minimal ideal in $\mathcal{L}$ satisfying the condition: if $\alpha\geqslant v_0$ and $N\geqslant N(v_0)$ then $\mathcal{F}_{\alpha,-N}\in\mathcal{L}^{(v_0)}\otimes k$.

For a generalization of these results cf. [2, 3] and for their application to an analogue of the Grothendieck conjecture cf. [4, 5]. For the study of an analogue $\Gamma_{<p}=G(L)$ of the group $\mathcal{G}_{<p}$ in the case of local fields K of mixed characteristic containing p-th roots of unity cf. [8, 9]. In these two papers we obtained a description of the corresponding ramification ideals $L^{(v)}$ and their interpretation in terms of the Demushkin relation for $\Gamma_{<p}$. Our method was based on a new technique (a linearization procedure) which allowed us to work with arithmetic properties of local fields in terms of Lie algebras. The statement of final results in terms of Lie algebras looked quite natural. We believe that it would be difficult to achieve such results exclusively via group theoretic means. To some extent this phenomenon could be treated as an evidence of the existence of a hidden "analytic structure" on the Galois group which shows up on the level of Lie algebras in our case. However, the above mentioned study of the mixed characteristic case is based quite substantially on the characteristic p results from the papers [1], [2] and [3]. It should be pointed out that in [1] the proof of the main result was not done completely in terms of Lie algebras. We could not "linearize" the verification of the criterion describing the ramification ideals $\mathcal{L}^{(v)}$. As a result, we proceeded with non-trivial calculations in the enveloping algebra of $\mathcal{L}$. In later papers [2] and [3] we managed to generalize our approach to the case of groups of period p^M, $M>1$ (but still of nilpotence class $<p$). At the same time it became clear that we should develop new techniques and methods when working with more complicated objects, e.g. higher local fields, cf. e.g. [10].

In this paper we develop a linearization procedure which allows us to obtain the results from [1] exclusively in terms of Lie theory. For a given $v_0>0$, we characterize the ramification ideal $\mathcal{L}^{(v_0)}$ in terms of deformations of some auxilliary Lie $\mathbb{F}_p$-algebra $\bar{\mathcal{L}}^{\dagger}$ with a suitably chosen module of coefficients. This algebra is provided with an action of a formal group of order p which comes from a derivation of a higher order. The appearance of such derivations is quite a new phenomenon. Note that in [8, 9] we also used the action of formal group of order p but it came from usual derivations.

Let us sketch briefly the main steps of our approach.

We start with a choice of an (sufficiently general) epimorphism $\eta_e:\mathcal{G}\longrightarrow G(\mathcal{L})$ which induces identification $\mathcal{G}_{<p}\simeq G(\mathcal{L})$ given by the

nilpotent Artin-Shreier theory. Here $\mathcal{L}$ is a profinite Lie $\mathbb{F}_p$-algebra such that its extension of scalars $\mathcal{L}_k := \mathcal{L}\otimes k$ has a fixed set of profinite generators. The map η_e depends on a choice of an element $e\in\mathcal{L}_{\mathcal{K}} := \mathcal{L}\otimes\mathcal{K}$ specified below.

Choose $v_0\in\mathbb{R}$, $v_0>0$. We aim to characterize the ideal $\mathcal{L}^{(v_0)}\subset\mathcal{L}$ such that $\eta_e(\mathcal{G}^{(v_0)})=\mathcal{L}^{(v_0)}$. For this reason we:

a) define a decreasing central filtration of $\mathcal{L}$ by its ideals $\mathcal{L}=\mathcal{L}(1)\supset\dots\supset\mathcal{L}(s)\supset\dots$, and set $\bar{\mathcal{L}}=\mathcal{L}/\mathcal{L}(p)$ with the induced filtration $\{\bar{\mathcal{L}}(s)\}_{s\geqslant 1}$ (note that $\bar{\mathcal{L}}(p)=0$);

b) introduce a lift $\mathcal{V}:\bar{\mathcal{L}}^{\dagger}\longrightarrow\bar{\mathcal{L}}$ where $\bar{\mathcal{L}}^{\dagger}$ is a Lie $\mathbb{F}_p$-algebra of nilpotent class $<p$ together with its central filtration $\bar{\mathcal{L}}^{\dagger}(s)$ such that $\mathcal{V}(\bar{\mathcal{L}}^{\dagger}(s))=\bar{\mathcal{L}}(s)$ and $\bar{\mathcal{L}}^{\dagger}(p)=0$;

c) specify a group epimorphism $\eta_{\bar{e}^{\dagger}}:\mathcal{G}\longrightarrow G(\bar{\mathcal{L}}^{\dagger})$ such that

$$\mathcal{V}\eta_{\bar{e}^{\dagger}}=\eta_{\bar{e}}:=\eta_e \bmod G(\mathcal{L}(p));$$

d) introduce the actions $\Omega_\gamma:\bar{\mathcal{L}}^{\dagger}\longrightarrow\bar{\mathcal{L}}^{\dagger}$ of the elements $\gamma\in\mathbb{Z}/p$;

e) introduce the ideal $\bar{\mathcal{L}}[v_0]$ in $\bar{\mathcal{L}}$ as the minimal ideal such that for any $\gamma\in\mathbb{Z}/p$, $\mathcal{V}^{-1}\bar{\mathcal{L}}[v_0]\supset\Omega_\gamma(\mathrm{Ker}\mathcal{V})$ (this condition is not easy to study because the action of $\mathbb{Z}/p$ appears in terms of complicated Campbell-Hausdorff group law);

f) establish that the actions Ω_γ can be defined in terms of some co-action $\Omega_U:\bar{\mathcal{L}}^{\dagger}\longrightarrow\bar{\mathcal{L}}^{\dagger}\otimes\mathbb{F}_p[U]$ of the formal group scheme $\alpha_p=\mathbb{F}_p[U]$, $U^p=0$, with coaddition $\Delta U=U\otimes 1+1\otimes U$;

g) if $d\Omega_U=B^{\dagger}U$ is the differential of Ω_U (here $B^{\dagger}\in\mathrm{Diff}\bar{\mathcal{L}}^{\dagger}$) then $\bar{\mathcal{L}}[v_0]$ appears as the minimal ideal in $\bar{\mathcal{L}}$ containing $\mathcal{V}B^{\dagger}(\bar{\mathcal{L}}^{\dagger})$;

h) verify that $\mathcal{L}^{(v_0)}=\bar{\mathrm{pr}}^{-1}\bar{\mathcal{L}}[v_0]$, where $\bar{\mathrm{pr}}$ is the natural projection from $\mathcal{L}$ to $\bar{\mathcal{L}}$.

The above characterization of $\mathcal{L}^{(v_0)}$ can be used for a considerable simplification of the process of recovering of explicit generators. These generators appeared in [1] as "linear" components of some elements from $\mathcal{L}^{(v_0)}$. Our method allows us to skip the verification that these linear components generate the ideal $\mathcal{L}^{(v_0)}$.

In the final Section we relate the description of ramification ideals with their description in [1], discuss the problem of effective construction of their generators, and show how the knowledge of the jumps of ramification filtration in $\mathcal{G}_{<p}$ allows us to recover the structure of this filtration.

The methods of this paper admit a generalization to the Galois groups of period p^M as well as to the case of higher dimensional local fields in the characteristic p case. In particular, the "p^M-version" [3] of [1] required much more complicated study of "non-linear" components, which can be now avoided due to our approach (the paper in preparation). This also will provide us with much better background for the papers [8, 9] and their upcoming "p^M-versions" including the case of higher dimensional local fields.

Notation. Suppose $s \in \mathbb{N}$. For any topological group G, we denote by $C_s(G)$ the closure of the subgroup of G generated by the commutators of order $\geqslant s$. If L is a toplogical Lie algebra then $C_s(L)$ is the closure of the ideal generated by commutators of degree $\geqslant s$. For any topological A-modules M and B we use the notation $M_B := M\hat{\otimes}_A B$.

1. Preliminaries

Suppose $\mathcal{K}$ is a field of characteristic p, $\mathcal{K}_{sep}$ is a separable closure of $\mathcal{K}$ and $\mathcal{G} = \mathrm{Gal}(\mathcal{K}_{sep}/\mathcal{K})$. We assume that $\mathcal{G}$ acts on $\mathcal{K}_{sep}$ as follows: if $g_1, g_2 \in \mathcal{G}$ and $a \in \mathcal{K}_{sep}$ then $g_1(g_2 a) = (g_1 g_2)a$. Denote by σ the morphism of taking p-th power in $\mathcal{K}_{sep}$.

In [1, 2] we developed a nilpotent analogue of the classical Artin-Schreier theory of cyclic field extensions of characteristic p. We are going to use the covariant analog of this theory, cf. the discussion in [7], for explicit description of the group $\mathcal{G}_{<p} = \mathcal{G}/\mathcal{G}^p C_p(\mathcal{G})$ as follows.

1.1. **Lie algebra $\mathcal{L}$.** Suppose $\mathcal{K} = k((t))$ where t is a fixed uniformizer and $k \simeq \mathbb{F}_{p^{N_0}}$ with $N_0 \in \mathbb{N}$. Fix $\alpha_0 \in k$ such that $\mathrm{Tr}_{k/\mathbb{F}_p}(\alpha_0) = 1$.

Let $\mathbb{Z}^+(p) = \{a \in \mathbb{N} \mid \gcd(a, p) = 1\}$ and $\mathbb{Z}^0(p) = \mathbb{Z}^+(p) \cup \{0\}$.

Let $\widetilde{\mathcal{L}}$ be a profinite free Lie $\mathbb{F}_p$-algebra with the (topological) module of generators $\mathcal{K}^*/\mathcal{K}^{*p}$ and $\mathcal{L} = \widetilde{\mathcal{L}}/C_p(\widetilde{\mathcal{L}})$. We can obtain the set

$$\{D_0\} \cup \{D_{an} \mid a \in \mathbb{Z}^+(p), n \in \mathbb{Z}/N_0\}$$

of topological generators of $\mathcal{L}_k$ via the following identifications:

$$(\mathcal{K}^*/\mathcal{K}^{*p})\hat{\otimes}_{\mathbb{F}_p} k = \mathrm{Hom}_{\mathbb{F}_p}(\mathcal{K}/(\sigma - \mathrm{id})\mathcal{K}\,, k) =$$

$$\mathrm{Hom}_{\mathbb{F}_p}(\oplus_{a\in\mathbb{Z}^+(p)} kt^{-a} \oplus \mathbb{F}_p\,\alpha_0\,, k) = \prod_{a\in\mathbb{Z}^+(p)} \mathrm{Hom}_{\mathbb{F}_p}(kt^{-a}, k) \times kD_0$$

and $\mathrm{Hom}_{\mathbb{F}_p}(kt^{-a}, k) = \prod_{n\in\mathbb{Z}/N_0} kD_{an}$, where for any $\alpha \in k$ and $a, b \in \mathbb{Z}^+(p)$, $D_{an}(\alpha t^{-b}) = \delta_{ab}\,\sigma^n(\alpha)$. Note also that the first identification uses the Witt pairing [11, 6] and D_0 comes from $t\otimes 1 \in (\mathcal{K}^*/\mathcal{K}^{*p})\hat{\otimes}_{\mathbb{F}_p} k$.

For any $n \in \mathbb{Z}/N_0$, set $D_{0n} = t \otimes (\sigma^n\alpha_0) = (\sigma^n\alpha_0)D_0$.

1.2. **Groups and Lie algebras of nilpotent class $< p$.** The basic ingredient of the nilpotent Artin-Schreier theory is the equivalence of the category of p-groups of nilpotent class $s_0 < p$ and the category of Lie $\mathbb{Z}_p$-algebras of the same nilpotent class s_0, [13, 12]. In the case of objects killed by p, this equivalence can be explained as follows.

Let L be a Lie $\mathbb{F}_p$-algebra of nilpotent class $< p$, i.e. $C_p(L) = 0$.

Let A be an enveloping algebra of L. Then there is a natural embedding $L \subset A$, the elements of L generate the augmentation ideal J

of A and we have a morphism of algebras $\Delta : A \longrightarrow A\otimes A$ uniquely determined by the condition $\Delta(l) = l\otimes 1 + 1\otimes l$ for all $l\in L$.

Applying the Poincare-Birkhoff-Witt Theorem as in [1] Sect. 1.3.3, we obtain that:

— $L\cap J^p = 0$;

— $L \bmod J^p = \{a \bmod J^p \mid \Delta(a) \equiv a\otimes 1 + 1\otimes a \bmod (J\otimes 1 + 1\otimes J)^p\}$;

— the set $\widetilde{\exp}(L) \bmod J^p$ is identified with the set of all "diagonal elements modulo degree p", i.e. with the set of $a\in 1+J \bmod J^p$ such that $\Delta(a)\equiv a\otimes a \bmod (J\otimes 1+1\otimes J)^p$. (Here $\widetilde{\exp}(x) = \sum_{0\leqslant i<p} x^i/i!$ is the truncated exponential.)

In particular, there is a natural embedding $L\subset A/J^p$ and in terms of this embedding the Campbell-Hausdorff formula appears as

$$(l_1,l_2)\mapsto l_1\circ l_2 = l_1 + l_2 + \frac{1}{2}[l_1,l_2] + \dots ,\quad l_1,l_2\in L\,,$$

where $\widetilde{\exp}(l_1)\widetilde{\exp}(l_2)\equiv \widetilde{\exp}(l_1\circ l_2) \bmod J^p$. This composition law provides the set L with a group structure and we denote this group by $G(L)$. Note that a subset $I\subset L$ is an ideal in L iff $G(I)$ is a normal subgroup in $G(L)$. Clearly, $G(L)$ has exponent p and nilpotent class $<p$. Then the correspondence $L\mapsto G(L)$ is the above mentioned equivalence of the categories of p-groups of exponent p and nilpotent class $s<p$ and Lie $\mathbb{F}_p$-algebras of the same nilpotent class s. This equivalence can be naturally extended to the categories of pro-finite Lie algebras and pro-finite p-groups.

1.3. **Epimorphism $\eta_e : \mathcal{G}\longrightarrow G(\mathcal{L})$.** Let L be a finite Lie $\mathbb{F}_p$-algebra of nilpotent class $<p$ and set $L_{sep} := L_{\mathcal{K}_{sep}}$. The elements of $\mathcal{G} = \mathrm{Gal}(\mathcal{K}_{sep}/\mathcal{K})$ and σ act on L_{sep} through the second factor, $L_{sep}|_{\sigma=\mathrm{id}} = L$ and $(L_{sep})^{\mathcal{G}} = L_{\mathcal{K}}$. The covariant nilpotent Artin-Schreier theory states that for any $e\in G(L_{\mathcal{K}})$, the set

$$\mathcal{F}(e) = \{f\in G(L_{sep}) \mid \sigma(f) = e\circ f\}$$

is not empty and for any fixed $f\in\mathcal{F}(e)$, the map $\tau\mapsto(-f)\circ\tau(f)$ is a continuous group homomorphism $\pi_f(e):\mathcal{G}\longrightarrow G(L)$. The correspondence $e\mapsto\pi_f(e)$ has the following properties:

a) if $f'\in\mathcal{F}(e)$ then $f' = f\circ l$, where $l\in G(L)$, and $\pi_f(e)$ and $\pi_{f'}(e)$ are conjugated via l;

b) for any continuous group homomorphism $\pi:\mathcal{G}\longrightarrow G(L)$, there are $e\in G(L_{\mathcal{K}})$ and $f\in\mathcal{F}(e)$ such that $\pi_f(e) = \pi$;

c) for appropriate elements $e,e'\in G(L_{\mathcal{K}})$ and $f,f'\in G(L_{sep})$, we have $\pi_f(e) = \pi_{f'}(e')$ iff there is an $x\in G(L_{\mathcal{K}})$ such that $f' = x\circ f$ and, therefore, $e' = \sigma(x)\circ e\circ(-x)$.

In [1, 2, 3] we applied this theory to the Lie algebra $\mathcal{L}$ from Sect.1.1 via a special choice of $e\in\mathcal{L}_{\mathcal{K}}$. Now we just assume that

$$e \equiv \sum_{a\in\mathbb{Z}^0(p)} t^{-a} D_{a0} \operatorname{mod} C_2(\mathcal{L}_{\mathcal{K}})\,. \tag{1.1}$$

Under this assumption the map $\pi_f(e) \operatorname{mod} \mathcal{G}^p C_2(\mathcal{G})$ induces a group isomorphism of $\mathcal{G}^{ab}\hat{\otimes}\mathbb{F}_p$ and $G(\mathcal{L})/C_2(G(\mathcal{L})) = \mathcal{L}^{ab} = \mathcal{K}^*/\mathcal{K}^{*p}$, which coincides with the inverse to the reciprocity map of local class field theory, cf. [6]. This also implies that $\pi_f(e)$ (when taken modulo $\mathcal{G}^p C_p(\mathcal{G})$) induces a group isomorphism $\mathcal{G}_{<p} \simeq G(\mathcal{L})$. We agree to fix a choice of $f \in \mathcal{F}(e)$ and use the notation $\eta_e = \pi_f(e)$. So, at this stage, η_e is just an arbitrary lift of the canonical isomorphism of local class field theory.

1.4. **Auxiliary fields $\mathcal{K}'_\gamma$.** Our approach to the ramification filtration in $\mathcal{G}_{<p}$ substantially uses the construction of a totally ramified extension $\mathcal{K}'$ of $\mathcal{K}$ such that $[\mathcal{K}' : \mathcal{K}] = q$ and the Herbrand function $\varphi_{\mathcal{K}'/\mathcal{K}}$ has only one edge point (r^*, r^*). Here $q = p^{N^*}$ with $N^* \in \mathbb{N}$, and $r^* = b^*/(q-1)$, where $b^* \in \mathbb{Z}^+(p)$. For simplicity, we assume that $N^* \equiv 0 \operatorname{mod} N_0$, i.e. σ^{N^*} acts as identity on the residue field k of $\mathcal{K}$. More substantial restrictions on these parameters will be introduced in Sect.2.1.

For a detailed explanation of the construction of $\mathcal{K}'$ cf. e.g. [3], Sect.1.5. We just recall that if $r^* = m/n$ with coprime $m, n \in \mathbb{N}$, then $\mathcal{K}' = \mathcal{K}(U^n) \subset \mathcal{K}(u)(U)$, where $u^n = t$ and $U^q + r^*U = u^{-m}$. We can apply Hensel's Lemma to choose a uniformizer t_1 in $\mathcal{K}'$ such that $t = t_1^q E(t_1^{b^*})^{-1}$, where $E(X) = \exp(X + X^p/p + \dots + X^{p^n}/p^n + \dots) \in \mathbb{Z}_p[[X]]$ is the Artin-Hasse exponential.

We need the following generalization of the construction of $\mathcal{K}'$.

For $\gamma \in \mathbb{Z}/p \setminus \{0\}$, let the field $\mathcal{K}'_\gamma = k((t_\gamma))$ be such that:

a) $[\mathcal{K}'_\gamma : \mathcal{K}] = q$;

b) $\varphi_{\mathcal{K}'_\gamma/\mathcal{K}}(x)$ has only one edge point (r^*, r^*);

c) $\mathcal{K}'_\gamma = k((t_\gamma))$, where $t = t_\gamma^q E(\gamma t_\gamma^{b^*})^{-1}$.

The fields $\mathcal{K}'_\gamma$ appear in the same way as the field $\mathcal{K}'$. More precisely, $\mathcal{K}'_\gamma = \mathcal{K}(U_\gamma^n) \subset \mathcal{K}(u)(U_\gamma)$, where $u^n = t$ and $U_\gamma^q + \gamma r^* U_\gamma = u^{-m}$. Note that $\mathcal{K}'_\gamma$ is separable over $\mathcal{K}$ (but generally is not a p-extension over $\mathcal{K}$).

1.5. **The criterion.** Suppose $\mathcal{K}'_\gamma$ is the field from Sect.1.4. Consider the field isomorphism $\iota_\gamma : \mathcal{K} \longrightarrow \mathcal{K}'_\gamma$ such that $\iota_\gamma : t \mapsto t_\gamma$ and $\iota_\gamma|_k = \mathrm{id}_k$. Let $e_\gamma = (\mathrm{id}_{\mathcal{L}} \otimes \iota_\gamma)e$. Then $\sigma^{N^*} e_\gamma(t_\gamma) = e(t_\gamma^q)$.

Choose $f_\gamma \in \mathcal{F}(e_\gamma)$ and consider $\pi_{f_\gamma}(e_\gamma) : \mathrm{Gal}(\mathcal{K}_{sep}/\mathcal{K}'_\gamma) \longrightarrow G(\mathcal{L})$.

For $Y \in \mathcal{L}_{sep}$ and an ideal $\mathcal{I}$ in $\mathcal{L}$, define the field of definition of $Y \operatorname{mod} \mathcal{I}_{sep}$ over, say, $\mathcal{K}$ as

$$\mathcal{K}(Y \operatorname{mod} \mathcal{I}_{sep}) := \mathcal{K}_{sep}^{\mathcal{H}}\,,$$

where $\mathcal{H} = \{g \in \mathcal{G} \mid (\mathrm{id}_{\mathcal{L}} \otimes g)Y \equiv Y \operatorname{mod} \mathcal{I}_{sep}\}$.

For any field extension $\mathcal{E}'/\mathcal{E}$ in $\mathcal{K}_{sep}$, define the biggest ramification number

$$v(\mathcal{E}'/\mathcal{E}) = \max\{v \mid \mathrm{Gal}(\mathcal{K}_{sep}/\mathcal{E})^{(v)} \text{acts non-trivially on } \mathcal{E}'\}\,.$$

The methods from [1, 2, 3] are based on the following criterion.

Suppose $v_0 > 0$, $r^* < v_0$ and the auxiliary fields $\mathcal{K}'_\gamma$ correspond to the parameters r^* and N^* (with $q = p^{N^*}$).

Proposition 1.1. *Suppose $f = X_\gamma \circ \sigma^{N^*}(f_\gamma)$. Then $\mathcal{L}^{(v_0)}$ is the minimal ideal in the family of all ideals $\mathcal{I}$ of $\mathcal{L}$ such that*

$$v(\mathcal{K}'_\gamma(X_\gamma \bmod \mathcal{I}_{sep})/\mathcal{K}'_\gamma) < qv_0 - b^*.$$

The proof goes along the lines of the proof for $\gamma = 1$, cf. e.g. [3], Sect.1.6. It is based just on the following elementary properties of the upper ramification numbers:

if $v = v(\mathcal{K}(f \bmod \mathcal{I}_{sep})/\mathcal{K})$ then:

- $v(\mathcal{K}'_\gamma(f_\gamma \bmod \mathcal{I}_{sep})/\mathcal{K}'_\gamma) = v$;
- $v(\mathcal{K}'_\gamma(f_\gamma \bmod \mathcal{I}_{sep})/\mathcal{K}) = \varphi_{\mathcal{K}'_\gamma/\mathcal{K}}(v)$;
- *if $v > r^*$ then $\varphi_{\mathcal{K}'_\gamma/\mathcal{K}}(v) = r^* + (v - r^*)/q < v$.*

Note that $f = X_\gamma \circ \sigma^{N^*} f_\gamma$ implies that $e(t) = \sigma X_\gamma \circ \sigma^{N^*} e_\gamma \circ (-X_\gamma)$.

Inversely, suppose $X \in \mathcal{L}_{sep}$ and

$$e(t) = \sigma X \circ \sigma^{N^*} e_\gamma \circ (-X)\,. \tag{1.2}$$

Then $l = (-\sigma^{N^*} f_\gamma) \circ (-X) \circ f \in \mathcal{L}_{sep}|_{\sigma=\mathrm{id}} = \mathcal{L}$ and replacing f_γ by $f_\gamma \circ l \in \mathcal{F}(e_\gamma)$ we obtain $f = X \circ \sigma^{N^*} f_\gamma$. Therefore, in Prop.1.1 we can use identity (1.2) instead of the identity $f = X_\gamma \circ \sigma^{N^*} f_\gamma$.

Note that for any γ, there is a unique field isomorphism $\iota'_\gamma : \mathcal{K}'_\gamma \longrightarrow \mathcal{K}$ such that $\iota'_\gamma(t_\gamma) = t$ and $\iota'_\gamma|_k = \mathrm{id}$. Therefore, if we set $e^{(q)} := e(t^q)$ and $\gamma * e^{(q)} := e(t^q E(\gamma t^{b^*})^{-1})$ then Prop.1.1 can be stated in the following equivalent form.

Proposition 1.2. *If $X_\gamma \in \mathcal{L}_{sep}$ is such that*

$$\gamma * e^{(q)} = \sigma X_\gamma \circ e^{(q)} \circ (-X_\gamma)$$

then $\mathcal{L}^{(v_0)}$ is the minimal ideal in the set of all ideals $\mathcal{I}$ of $\mathcal{L}$ such that

$$v(\mathcal{K}(X_\gamma \bmod \mathcal{I}_{sep})/\mathcal{K}) < qv_0 - b^*\,.$$

Suppose $\widetilde{\mathcal{J}} \subset \mathcal{L}$ is a closed ideal and $\pi : \mathcal{L} \longrightarrow L := \mathcal{L}/\widetilde{\mathcal{J}}$ is a natural projection. Then we can use $e_L = \pi_{\mathcal{K}}(e) \in L_{\mathcal{K}}$, $f_L := \pi_{sep}(f) \in L_{sep}$, $\eta_{e_L} = \pi\eta_e : \mathcal{G} \longrightarrow G(L)$ and $X_{\gamma L} := \pi_{sep}(X_\gamma)$ to state the following analog of Prop.1.2.

Proposition 1.3. *$L^{(v_0)} := \eta_{e_L}(\mathcal{G}^{(v_0)})$ is the minimal ideal in the set of all ideals $\mathcal{I}$ of L such that $v(\mathcal{K}(X_{\gamma L} \bmod \mathcal{I}_{sep})/\mathcal{K}) < qv_0 - b^*$.*

1.6. **Lie algebra $\bar{\mathcal{L}}$ and epimorphism $\eta_{\bar{e}}$.** Introduce a weight function $\mathrm{wt}:\mathcal{L}_k\longrightarrow\mathbb{N}$ on $\mathcal{L}_k$ by setting on its generators $\mathrm{wt}(D_{an})=s$ if $(s-1)v_0\leqslant a<sv_0$. We obtain a decreasing central filtration by the ideals $\mathcal{L}(s)=\{l\in\mathcal{L}\mid \mathrm{wt}(l)\geqslant s\}$ of $\mathcal{L}$ such that $\mathcal{L}(1)=\mathcal{L}$. This weight function gives us also a decreasing filtration of ideals $\mathcal{J}(s)$ in the enveloping algebra $\mathcal{A}$ such that $\mathcal{J}(1)=\mathcal{J}$ and for any s, $(\mathcal{J}(s)+\mathcal{J}^p)\cap\mathcal{L}=\mathcal{L}(s)$ (use the Poincare-Birkhoff-Witt theorem).

Consider a k-submodule $\mathcal{N}$ in $\mathcal{L}_{\mathcal{K}}$ generated by all $t^{-b}l$, where for some $s\geqslant 1$, $l\in\mathcal{L}(s)_k$ and $b<sv_0$. Then $\mathcal{N}$ has a natural structure of a Lie algebra over k. For any $i\geqslant 0$, let $\mathcal{N}(i)$ be the k-submodule in $\mathcal{L}_{\mathcal{K}}$ generated by all $t^{-b}l$ where $l\in\mathcal{L}(s)$ and $b<(s-i)v_0$. Then $\mathcal{N}(i)$ is ideal in $\mathcal{N}$.

Let $\bar{\mathrm{pr}}:\mathcal{L}\longrightarrow\bar{\mathcal{L}}:=\mathcal{L}/\mathcal{L}(p)$ be a natural projection. Then $\bar{\mathcal{L}}(s)=\bar{\mathrm{pr}}(\mathcal{L}(s))$ is a decreasing central filtration in $\bar{\mathcal{L}}$ such that $\bar{\mathcal{L}}(p)=0$. Let $\bar{\mathcal{N}}\subset\bar{\mathcal{L}}_{\mathcal{K}}$ be an analog of $\mathcal{N}$ (where the algebra $\bar{\mathcal{L}}$ is used instead of $\mathcal{L}$).

For $i\geqslant 0$, let $\bar{\mathcal{N}}(i)$ be the appropriate ideals in $\bar{\mathcal{N}}$. Note that $\bar{\mathcal{N}}(p-1)\subset\bar{\mathcal{L}}_{\mathrm{m}}$, where $\mathrm{m}=tk[[t]]$ (use that $\bar{\mathcal{L}}(p)=0$), and introduce the Lie algebra $\widetilde{\mathcal{N}}=\bar{\mathcal{N}}/\bar{\mathcal{N}}(p-1)$.

We assume (in addition to (1.1)) that:

$$e\in\mathcal{N}\tag{1.3}$$

(now η_e is not an arbitrary lift of the reciprocity map of class field theory but it is still quite general).

Let $\bar{e}:=\bar{\mathrm{pr}}_{\mathcal{K}}e\in\bar{\mathcal{N}}$ and $\bar{f}:=(\bar{\mathrm{pr}}_{sep})f\in\bar{\mathcal{L}}_{sep}$. If $\eta_{\bar{e}}:=\bar{\mathrm{pr}}\cdot\eta_e$ then for any $\tau\in\mathcal{G}$, $\bar{\eta}_{\bar{e}}(\tau)=(-\bar{f})\circ\tau\bar{f}$. Verify that $\eta_{\bar{e}}$ depends only on $\widetilde{e}:=\bar{e}\,\mathrm{mod}\,\bar{\mathcal{N}}(p-1)\in\widetilde{\mathcal{N}}$.

Proposition 1.4. *Let $\bar{e}'\in\bar{\mathcal{L}}_{\mathcal{K}}$ and $\bar{e}'\equiv\bar{e}\,\mathrm{mod}\,\bar{\mathcal{N}}(p-1)$. Then there is a unique $\bar{f}'\in\bar{\mathcal{L}}_{sep}$ such that $\sigma\bar{f}'=\bar{e}'\circ\bar{f}'$ and $\bar{f}'\circ(-\bar{f})\in\bar{\mathcal{N}}(p-1)$.*

Proof. Note that σ is topologically nilpotent on $\bar{\mathcal{N}}(p-1)\subset\bar{\mathcal{L}}_{\mathrm{m}}$. Prove the existence of $\bar{x}\in\bar{\mathcal{N}}(p-1)$ such that $\bar{e}'=(\bar{\sigma}x)\circ\bar{e}\circ(-\bar{x})$ by induction on $s\geqslant 1$ modulo the ideals $\bar{\mathcal{L}}(s)_{\mathcal{K}}$ as follows:

– if $s=1$ there is nothing to prove;

– if $s\geqslant 1$ and $\bar{x}_s\in\bar{\mathcal{N}}(p-1)$ is such that $\bar{e}'=(\sigma\bar{x}_s)\circ\bar{e}\circ(-\bar{x})+A_s$ with $A_s\in\bar{\mathcal{L}}(s)_{\mathcal{K}}$, then $A_s\in\bar{\mathcal{N}}(p-1)\cap\bar{\mathcal{L}}(s)_{\mathcal{K}}$. If $\delta=-\sum\limits_{m\geqslant 0}\sigma^m(A_s)$ then $\bar{x}_{s+1}:=\bar{x}_s+\delta\in\bar{\mathcal{N}}(p-1)\cap\bar{\mathcal{L}}(s)_{\mathcal{K}}$, $\sigma\delta-\delta=A_s$ and

$$\bar{e}'\equiv(\sigma\bar{x}_{s+1})\circ\bar{e}\circ(-\bar{x}_{s+1})\,\mathrm{mod}\,\bar{\mathcal{L}}(s+1)_{\mathcal{K}}\,.$$

Clearly, $\bar{x}=\bar{x}_p$.

Now $\bar{f}'=\bar{x}\circ\bar{f}\in\bar{\mathcal{L}}_{sep}$ satisfies the requirements of proposition. If $\bar{f}''\in\bar{\mathcal{L}}_{sep}$ also has such properties then $\bar{f}''\circ(-\bar{f}')\in\bar{\mathcal{N}}(p-1)\cap\bar{\mathcal{L}}=0$ and $\bar{f}''=\bar{f}'$. ∎

2. Lie algebra $\bar{\mathcal{L}}^\dagger$ and ideal $\bar{\mathcal{L}}[v_0] \subset \bar{\mathcal{L}}$

In this section we introduce the Lie $\mathbb{F}_p$-algebra $\bar{\mathcal{L}}^\dagger$ together with the epimorphism of Lie algebras $\mathcal{V} : \bar{\mathcal{L}}^\dagger \longrightarrow \bar{\mathcal{L}}$ and its section $(j^0)^{-1} : \bar{\mathcal{L}} \simeq \bar{\mathcal{L}}^\dagger[0] \subset \bar{\mathcal{L}}^\dagger$. Let $\alpha_p = \operatorname{Spec} \mathbb{F}_p[U]$, $U^p = 0$, be the formal group scheme over $\mathbb{F}_p$ with the coaddition $\Delta(U) = U \otimes 1 + 1 \otimes U$. We introduce the coaction $\Omega_U : \bar{\mathcal{L}}^\dagger \longrightarrow \mathbb{F}_p[U] \otimes \bar{\mathcal{L}}^\dagger$ of α_p on $\bar{\mathcal{L}}^\dagger$ and use it to define and characterize the ideal $\bar{\mathcal{L}}[v_0]$ of $\bar{\mathcal{L}}$.

2.1. **Parameters r^* and N^*.** Fix $u^* \in \mathbb{N}$ and $w^* > 0$. (Below we will specify $u^* = (p-1)(p-2)+1$ and $w^* = (p-1)v_0$.)

For $1 \leqslant s < p$, denote by $\delta_0(s)$ the minimum of positive values of

$$v_0 - \frac{1}{s}(a_1 + a_2/p^{n_2} + \dots + a_u/p^{n_u}),$$

where $u \leqslant u^*$, all $n_i \in \mathbb{Z}_{\geqslant 0}$ and $a_i \in [0, w^*) \cap \mathbb{Z}$. The existence of such $\delta_0(s)$ can be proved easily by induction on u for any fixed s.

Set $\delta_0 := \min\{\delta_0(s) \mid 1 \leqslant s < p\}$.

Let $r^* \in \mathbb{Q}$ be such that $r^* = b_0^*/(q_0^* - 1)$, where $q_0^* = p^{N_0^*}$ with $N_0^* \geqslant 2$, $b_0^* \in \mathbb{N}$ and $\gcd(b_0^*, p(q_0^* - 1)) = 1$. The set of such r^* is dense in $\mathbb{R}_{>0}$ and we can assume that $r^* \in (v_0 - \delta_0, v_0)$.

For $1 \leqslant u \leqslant u^*$, introduce the following subsets in $\mathbb{Q}$:

— $A[u]$ is the set of all

$$a_1 p^{-n_1} + a_2 p^{-n_2} + \dots + a_u p^{-n_u},$$

where $0 = n_1 \leqslant \dots \leqslant n_u$, all $a_i \in [0, w^*) \cap \mathbb{Z}$. If $M \in \mathbb{Z}_{\geqslant 0}$ we denote by $A[u, M]$ the subset of $A[u]$ consisted of the elements satisfying the additional restriction $n_u \leqslant M$. Note that $A[u, M]$ is finite.

— $B[u]$ is the set of all numbers

$$r^*(b_1 p^{-m_1} + b_2 p^{-m_2} + \dots + b_u p^{-m_u}),$$

where all $0 = m_1 \leqslant \dots \leqslant m_u$, $b_i \in \mathbb{Z}_{\geqslant 0}$, $b_1 \neq 0$ and $b_1 + \dots + b_u < p$. (In particular, $0 \notin B[u]$.) For $M \in \mathbb{Z}_{\geqslant 0}$, $B[u, M]$ is the subset of $B[u]$ consisted of the elements satisfying the additional restrictions $m_u \leqslant M$. The set $B[u, M]$ is also finite.

Lemma 2.1. *For any u, $A[u] \cap B[u] = \emptyset$.*

Proof. Note that $A[u] \subset \mathbb{Z}[1/p]$. Prove that $B[u] \cap \mathbb{Z}[1/p] = \emptyset$.

It will be sufficient to verify that for any $n_1, \dots, n_u, b_1, \dots, b_u \in \mathbb{Z}_{\geqslant 0}$ such that $0 < b_1 + \dots + b_u < p$, we have

$$b_1 p^{n_1} + \dots + b_u p^{n_u} \not\equiv 0 \bmod (q_0^* - 1).$$

Since $q_0^* \equiv 1 \bmod (q_0^* - 1)$ we can assume that all $n_i < N_0^*$. But then $0 < b_1 p^{n_1} + \dots + b_u p^{n_u} \leqslant (p-1)p^{N_0^*-1} < q_0^* - 1$. The lemma is proved. □

For $\alpha, \beta \in \mathbb{Q}$, set $\rho(\alpha, \beta) = |\alpha - \beta|$.

Lemma 2.2. *If $\alpha \notin B[u]$ then*

$$\rho(\alpha, B[u]) := \inf\{\rho(\alpha, \beta) \mid \beta \in B[u]\} \neq 0\,.$$

Proof. Use induction on u.

If $u = 1$ there is nothing to prove because $B[1]$ is finite.

Suppose $u \geqslant 1$ and $\rho(\alpha, B[u]) > 0$.

Choose $M_u \in \mathbb{Z}_{\geqslant 0}$ such that $r^*(p-1)/p^{M_u+1} < \rho(\alpha, B[u])/2$.

If $\beta \in B[u+1] \setminus B[u+1, M_u]$ then there is $\beta' \in B[u]$ such that $\rho(\beta, \beta') < \rho(\alpha, B[u])/2$. Then

$$\rho(\alpha, \beta) \geqslant \rho(\alpha, \beta') - \rho(\beta', \beta) \geqslant \rho(\alpha, B[u]) - \rho(\alpha, B[u])/2 = \rho(\alpha, B[u])/2\,,$$

and we obtain

$$\rho(\alpha, B[u+1]) \geqslant \min\{\rho(\alpha, B[u+1, M_u]), \rho(\alpha, B[u])/2\} > 0\,.$$

The lemma is proved. □

Lemma 2.3. *If $\beta \notin A[u]$ then $\rho(\beta, A[u]) \neq 0$.*

Proof. The proof is similar to the proof of above Lemma 2.2. □

Lemma 2.4. *For all $u_1, u_2 \leqslant u^*$, $\rho(A[u_1], B[u_2]) > 0$.*

Proof. If $u_1 = 1$ this follows from Lemma 2.2 because $A[1]$ is finite.

Suppose $u_1 \geqslant 1$ and $\rho(A[u_1], B[u_2]) = \delta > 0$.

Choose $M_1 \in \mathbb{Z}_{\geqslant 0}$ such that $w^*/p^{M_1} < \delta/2$.

If $\alpha \in A[u_1 + 1] \setminus A[u_1 + 1, M_1]$ then there is $\alpha' \in A[u_1]$ such that $\rho(\alpha, \alpha') < \delta/2$. Then for any $\beta \in B[u_2]$, we have

$$\rho(\alpha, \beta) \geqslant \rho(\alpha', \beta) - \rho(\alpha, \alpha') > \delta/2\,.$$

Therefore, for any $\alpha \in A[u_1 + 1]$,

$$\rho(\alpha, B[u_2]) \geqslant \min\{\, \rho(A[u_1 + 1, M_1], B[u_2]),\ \delta/2\,\}\} > 0\,.$$

Lemma is proved. □

Fix the values $u^* = (p-1)(p-2) + 1$ and $w^* = (p-1)v_0$ (since $u^* \geqslant p - 1$, $B[u^*] = B[p-1]$).

Choose $N^* \in \mathbb{N}$ satifying the following conditions:

C1) $N^* \equiv 0 \bmod N_0^*$;

C2) $p^{N^*} \rho(A[u^*], B[u^*]) \geqslant 2r^*(p-1)$;

C3) $r^*(1 - p^{-N^*}) \in (v_0 - \delta_0, v_0)$.

Introduce $q = p^{N^*}$ and $b^* = b_0^*(q-1)/(q_0 - 1) \in \mathbb{N}$.

Note that $r^* = b^*/(q-1)$ and $b^* \in \mathbb{Z}^+(p)$.

Proposition 2.5. *If $\alpha \in A[u^*]$ and $\beta \in B[u^*]$ then*

$$q\,|q\alpha - (q-1)\beta| > b^*(p-1)\,.$$

Proof. Indeed, the left-hand side of our inequality equals

$$q\,|q\alpha-(q-1)\beta| = q^2|\alpha-\beta+\beta/q| \geqslant q^2|\alpha-\beta|-\beta q \geqslant q^2\rho(A[u^*],\, B[u^*]) \\ -r^*(p-1)q \geqslant 2r^*(p-1)q - r^*(p-1)q = r^*(p-1)q > b^*(p-1)\,.$$

□

2.2. **The set $\mathfrak{A}^0$.** Use the above parameters r^*, N^*, $q=p^{N^*}$.

Definition. $\mathfrak{A}^0$ is the set of all $\iota = p^m(q\alpha-(q-1)\beta)$, where $m\in\mathbb{Z}_{\geqslant 0}$, $\alpha\in A[u^*,m]$, $\beta\in B[u^*,m]\cup\{0\}$ and $|\iota|\leqslant b^*(p-1)$. (Note that $p^m\alpha\in\mathbb{Z}_{\geqslant 0}$ and $p^m\beta/r^*\in\mathbb{N}$.)

Let $\mathfrak{A}^0_0 := \{\iota\in\mathfrak{A}^0 \mid \beta=0\}$.

Lemma 2.6. *Suppose $\iota = p^m(q\alpha-(q-1)\beta)\in\mathfrak{A}^0$. Then:*

a) $\mathfrak{A}^0_0 = \{qa \mid a\in[0,(p-1)v_0)\cap\mathbb{Z}\,\}$*;*

b *if $\beta\neq 0$ then $m<N^*$ (in particular, $\mathfrak{A}^0$ is finite);*

c) *the integers $p^m\alpha$ and $p^m\beta/r^*$ do not depend on the presentation of ι in the form $p^m(q\alpha-(q-1)\beta)$ from the definition of $\mathfrak{A}^0$.*

Proof. a) If $\iota\in\mathfrak{A}^0_0$ then $\iota = qp^m\alpha\in\mathfrak{A}^0_0\subset\mathfrak{A}^0$ means that $p^m\alpha/(p-1)\leqslant b^*/q = r^*(1-q^{-1})\in(v_0-\delta_0,v_0)$. By the choice of δ_0 from Sect.2.1, the inequalities $p^m\alpha/(p-1)<v_0$ and $p^m\alpha/(p-1)\leqslant v_0-\delta_0$ are equivalent. Therefore, $\mathfrak{A}^0_0\subset\{qa\mid a\in[0,(p-1)v_0)\cap\mathbb{Z}\}$. The opposite embedding is obvious.

b) If $\beta\in B[u^*,m]$ and $m\geqslant N^*$ then by Prop.2.5, $|\iota|>b^*(p-1)$ i.e. $\iota\notin\mathfrak{A}^0$.

c) If $\iota = p^{m'}(q\alpha'-(q-1)\beta')$ is another presentation of ι then $p^m\beta/r^*$ and $p^{m'}\beta'/r^*$ are non-negative congruent modulo q integers and the both are smaller than q. Indeed, if $\beta/r^* = b_1+b_2p^{-m_2}+\dots+b_up^{-m_u}$, where all $0\leqslant m_i\leqslant m$ and $u\leqslant u^*$, then

$$p^m\beta/r^*\leqslant p^m(b_1+\dots+b_u)\leqslant p^m(p-1)<p^{m+1}\leqslant q$$

because $m<N^*$. Similarly, $p^{m'}\beta'/r^*<q$. Therefore, they coincide and this implies also that $p^m\alpha = p^{m'}\alpha'$. □

Corollary 2.7. *Suppose that $\iota = p^m(q\alpha-(q-1)\beta)\in\mathfrak{A}^0$. Then the sum of the "p-digits" $b_1+\dots+b_u$ of the appropriate $\beta/r^* = b_1+b_2p^{-m_2}+\dots+b_up^{-m_u}$ depends only on ι.*

Definition. $\mathrm{ch}(\iota) := b_1+\dots+b_u\,.$

In the notation from Sect.2.2 suppose $\iota = p^m(q\alpha-(q-1)\beta)\in\mathfrak{A}^0$. By Lemma 2.6 $p^m\alpha$ depends only on ι and can be presented (non-uniquely) in the form $a_1p^{n_1}+a_2p^{n_2}+\dots+a_up^{n_u}$ where all coefficients $a_i\in[0,(p-1)v_0))\cap\mathbb{Z}$, $0\leqslant n_i\leqslant m$, $n_1=m$ and $u\leqslant u^*$.

Definition. $\kappa(\iota)$ is the maximal natural number such that for any above presentation of $p^m\alpha$, $\kappa(\iota)\leqslant u$.

Remark. a) If $\iota \in \mathfrak{A}^0$ then $\kappa(\iota) \leqslant u^*$ and $\mathrm{ch}(\iota) \leqslant p-1$;

b) if $\iota \in \mathfrak{A}^0_0$ then $\mathrm{ch}(\iota) = 0$;

c) if $\iota \in \mathfrak{A}^0_0$ and $\iota \neq 0$ then $\kappa(\iota) = 1$.

2.3. **Lie algebras $\mathcal{L}^\dagger$ and $\bar{\mathcal{L}}^\dagger$.** Suppose $\iota = p^m(q\alpha - (q-1)\beta) \in \mathfrak{A}^0$ is given in the standard notation from Sect.2.2. Let $w^0(\iota)$ be the minimal natural number such that $\iota < w^0(\iota)b^*$.

Definition. The subset $\mathfrak{A}^+(p)$ consists of $\iota \in \mathfrak{A}^0$ such that

— $\iota > 0$;

— $\gcd(p^m\alpha\, , p^m\beta/r^*,\, p) = 1$;

— $w^0(\iota) + \mathrm{ch}(\iota) \leqslant p-1$;

— $\kappa(\iota) \leqslant (p-2)\mathrm{ch}(\iota) + w^0(\iota)$.

Remark. For any $\iota \in \mathfrak{A}^+(p)$, $(p-2)\mathrm{ch}(\iota) + w^0(\iota) \leqslant (p-2)^2 + p - 1 = u^*$.

The elements of $\{t^{-\iota} \mid \iota \in \mathfrak{A}^+(p)\}$ behave "well" modulo $(\sigma - \mathrm{id})\mathcal{K}$, i.e. the natural map $\sum\limits_{\iota \in \mathfrak{A}^+(p)} kt^{-\iota} \longrightarrow \mathcal{K}/(\sigma - \mathrm{id})\mathcal{K}$ is injective. This is implied by the following proposition.

Proposition 2.8. *Let v_p be the p-adic valuation such that $v_p(p) = 1$.*

a) *Then all $\iota p^{-v_p(\iota)}$, where $\iota \in \mathfrak{A}^+(p)$, are pairwise different.*

b) *If $\iota \in \mathfrak{A}^+(p)$ and $\mathrm{ch}(\iota) = 1$ then $\iota p^{-v_p(\iota)} \geqslant qv_0 - b^*$.*

Proof. a) Suppose $\iota = p^m(q\alpha - (q-1)\beta) \in \mathfrak{A}^+(p)$.

If $\mathrm{ch}(\iota) = 0$ then $\iota \mapsto \iota p^{-v_p(\iota)}$ identifies $\{\iota \in \mathfrak{A}^+(p) \mid \mathrm{ch}(\iota) = 0\}$ with $\mathbb{Z}^+(p) \cap [0, (p-1)v_0)$, cf. Lemma 2.6a).

Remark. For similar reasons, if $1 \leqslant s < p$ and $a \in \mathbb{Z}^+(p) \cap [0, (p-1)v_0)$ then $a < sv_0$ iff $qa < sb^*$.

If $\mathrm{ch}(\iota) \geqslant 1$ then $\iota p^{-m} \notin p\mathbb{N}$, i.e. $m \geqslant v_p(\iota)$.

Indeed, $\iota p^{-m} = q\alpha - (q-1)\beta \in p\mathbb{N}$ implies (use that $q\alpha \in p\mathbb{N}$ because $m < N^*$) that $p^{-m}(b_1 + b_2p^{m_2} + \cdots + b_up^{m_u}) \in p\mathbb{N}$ where all $m_i \in [0, m]$. But this number is $\leqslant \mathrm{ch}\,\iota < p$. The contradiction.

Then by Prop.2.5, $\iota p^{-v_p(\iota)} \geqslant \iota p^{-m} = |q\alpha - (q-1)\beta| > (b^*/q)(p-1) = r^*(1 - q^{-1})(p-1) > (v_0 - \delta_0)(p-1)$ (use property **C3** from Sect.2.1).

Finally, if $\iota \in \mathfrak{A}^0_0$ then $\iota p^{-v_p(\iota)} = a < (p-1)v_0$ implies that $a < (v_0 - \delta_0)(p-1)$ by the choice of δ_0, cf. Sect.2.1. On the other hand, for all $\iota \in \mathfrak{A}^+(p)$ with $\mathrm{ch}(\iota) \geqslant 1$, the values $\iota p^{-v_p(\iota)}$ are different (use that $\gcd(p^m\alpha, p^m\beta/r^*) \not\equiv 0 \bmod p$) and bigger than $(v_0 - \delta_0)(p-1)$.

b) Here $\iota p^{-\iota_p(\iota)} = \iota p^{-m} = q\alpha - b^*$. If $\alpha \geqslant v_0$ then $\iota p^{-v_p(\iota)} \geqslant qv_0 - b^*$. If $\alpha < v_0$ then $\iota p^{-v_p(\iota)} \leqslant q(v_0 - \delta_0 - r^*(q-1)/q)) < 0$, cf. condition C3) from Sect.2.1. The contradiction.

The proposition is completely proved. ∎

Definition. $\mathfrak{A}^0(p)=\mathfrak{A}^+(p)\cup\{0\}$.

Let $\widetilde{\mathcal{L}}_k^{\dagger}$ be the Lie algebra over k with the set of free generators

$$\{\ D_{\iota n}^{\dagger}\mid \iota\in\mathfrak{A}^+(p), n\in\mathbb{Z}/N_0\}\cup\{D_0^{\dagger}\}\,.$$

Set (compare with Sect.1.1) $D_{0n}^{\dagger}=\sigma^n(\alpha_0)D_0^{\dagger}$, use the notation σ for the σ-linear automorphism of $\widetilde{\mathcal{L}}_k^{\dagger}$ such that $\sigma: D_{\iota n}^{\dagger}\mapsto D_{\iota,n+1}^{\dagger}$, and introduce the Lie $\mathbb{F}_p$-algebras $\widetilde{\mathcal{L}}^{\dagger}:=\widetilde{\mathcal{L}}_k^{\dagger}|_{\sigma=\mathrm{id}}$ and $\mathcal{L}^{\dagger}=\widetilde{\mathcal{L}}^{\dagger}/C_p(\widetilde{\mathcal{L}}^{\dagger})$. Note that $\widetilde{\mathcal{L}}^{\dagger}\otimes k=\widetilde{\mathcal{L}}_k^{\dagger}$, and this matches the agreement about extensions of scalars from the end of Introduction.

Introduce the w^0-weights, $w^0(D_{\iota n}^{\dagger}):=w^0(\iota)$.

Denote by $\{\mathcal{L}^{\dagger}(s)\}_{s\geqslant 1}$ the minimal central filtration of $\mathcal{L}^{\dagger}$ such that all $D_{\iota n}^{\dagger}$ with $w^0(D_{\iota n}^{\dagger})\geqslant s$ belong to $\mathcal{L}^{\dagger}(s)_k$. This means that $\mathcal{L}^{\dagger}(s)_k$ is an ideal in $\mathcal{L}_k^{\dagger}$ generated as k-module by all $[\dots[D_{\iota_1 n_1}^{\dagger},D_{\iota_2 n_2}^{\dagger}],\dots,D_{\iota_r n_r}^{\dagger}]$ such that $w^0(\iota_1)+\dots+w^0(\iota_r)\geqslant s$. Note that $C_s(\mathcal{L}^{\dagger})\subset\mathcal{L}^{\dagger}(s)$.

Let $\mathcal{A}^{\dagger}$ be the enveloping algebra for $\mathcal{L}^{\dagger}$.

For $m\in\mathbb{Z}_{\geqslant 0}$, let $\mathcal{A}^{\dagger}[m]_k$ be the k-submodule in $\mathcal{A}_k^{\dagger}$ generated by all monomials $D_{\iota_1 n_1}^{\dagger}\dots D_{\iota_r n_r}^{\dagger}$ such that $\mathrm{ch}(\iota_1)+\dots+\mathrm{ch}(\iota_r)=m$.

By setting $\mathcal{A}^{\dagger}[m]=\mathcal{A}^{\dagger}\cap\mathcal{A}^{\dagger}[m]_k$ we obtain a grading in the category of $\mathbb{F}_p$-algebras $\mathcal{A}^{\dagger}=\oplus_{m\geqslant 0}\mathcal{A}^{\dagger}[m]$ and the induced grading $\mathcal{L}^{\dagger}=\oplus_{m\geqslant 0}\mathcal{L}^{\dagger}[m]$ in the category of Lie algebras.

For $s\geqslant 1$, set $\mathcal{L}^{\dagger}(s)[m]=\mathcal{L}^{\dagger}(s)\cap\mathcal{L}^{\dagger}[m]$. Then $\mathcal{L}^{\dagger}(s)=\oplus_{m\geqslant 0}\mathcal{L}^{\dagger}(s)[m]$.

Let $\bar{\mathcal{L}}^{\dagger}$ be the quotient of $\mathcal{L}^{\dagger}$ by the ideal $\sum\limits_{s+m\geqslant p}\mathcal{L}^{\dagger}(s)[m]$. We have the induced central filtration $\{\bar{\mathcal{L}}^{\dagger}(s)\}_{s\geqslant 1}$ in $\bar{\mathcal{L}}^{\dagger}$ such that $\bar{\mathcal{L}}^{\dagger}(p)=0$.

We also have the induced gradings $\bar{\mathcal{L}}^{\dagger}=\oplus_{m\geqslant 0}\bar{\mathcal{L}}^{\dagger}[m]$ and $\bar{\mathcal{L}}^{\dagger}(s)=\oplus_{m\geqslant 0}\bar{\mathcal{L}}^{\dagger}(s)[m]$, where $\bar{\mathcal{L}}^{\dagger}(s)[m]:=\bar{\mathcal{L}}^{\dagger}(s)\cap\bar{\mathcal{L}}^{\dagger}[m]$.

Definition. If $l\in\mathcal{L}^{\dagger}[m]_k$ or $l\in\bar{\mathcal{L}}^{\dagger}[m]_k$, $l\neq 0$, we set $\mathrm{ch}(l)=m$.

Clearly, for any m_1,m_2, $[\bar{\mathcal{L}}^{\dagger}[m_1],\bar{\mathcal{L}}^{\dagger}[m_2]]\subset\bar{\mathcal{L}}^{\dagger}[m_1+m_2]$.

2.4. **Lie algebra $\widetilde{\mathcal{N}}^{sp}$.** Let $\mathcal{N}^{\dagger}$ be the k-submodule in $\mathcal{L}_{\mathcal{K}}^{\dagger}$ generated by the elements of the form $t^{-b}l$, where $l\in\mathcal{L}^{\dagger}(s)[m]_k$ and $b<(s+m)b^*$; $\mathcal{N}^{\dagger}$ is a Lie k-subalgebra in $\mathcal{L}_{\mathcal{K}}^{\dagger}$ and for $j\geqslant 0$, $t^{jb^*}\mathcal{N}^{\dagger}$ are ideals in $\mathcal{N}^{\dagger}$.

Introduce similarly the k-subalgebra $\bar{\mathcal{N}}^{\dagger}$ in $\bar{\mathcal{L}}_{\mathcal{K}}^{\dagger}$ (generated by all $t^{-b}l$, where $l\in\bar{\mathcal{L}}^{\dagger}[m](s)_k$ and $b<(s+m)b^*$) and its ideals $t^{jb^*}\bar{\mathcal{N}}^{\dagger}$. Note that $t^{(p-1)b^*}\bar{\mathcal{N}}^{\dagger}\subset\bar{\mathcal{L}}_{\mathrm{m}}^{\dagger}$ (use that $\bar{\mathcal{L}}^{\dagger}(p)=0$).

Set $\widetilde{\mathcal{N}}^{\dagger}=\bar{\mathcal{N}}^{\dagger}/t^{(p-1)b^*}\bar{\mathcal{N}}^{\dagger}$.

The grading from Sect.2.3 induces the gradings $\bar{\mathcal{N}}^{\dagger}=\oplus_{m\geqslant 0}\bar{\mathcal{N}}^{\dagger}[m]$ and $\widetilde{\mathcal{N}}^{\dagger}=\oplus_{m\geqslant 0}\widetilde{\mathcal{N}}^{\dagger}[m]$.

Definition. Let $\widetilde{\mathcal{N}}^{sp}$ be the k-submodule in $\widetilde{\mathcal{N}}^{\dagger}$ generated by the elements of the form $t^{-\iota}l$ with $l\in\bar{\mathcal{L}}^{\dagger}(s)[m]_k$ such that:

a) $\iota\in\mathfrak{A}^0$;

b) $\iota + \mathrm{ch}(\iota)b^* < (s+m)b^*$;

c) $\mathrm{ch}(\iota) \geqslant m$, $\kappa(\iota) \leqslant (p-2)m + s$.

Remark. Condition b) means that $t^{-\iota}l \in t^{\mathrm{ch}(\iota)b^*}\widetilde{\mathcal{N}}^{\dagger}$.

It will be convenient to introduce the following modules.

Let $\mathcal{N}^{sp}$ be a k-submodule in $\mathcal{N}^{\dagger} \subset \mathcal{L}^{\dagger}_{\mathcal{K}}$ generated by the elements $t^{-\iota}l$ such that $l \in \mathcal{L}^{\dagger}(s)[m]$, $\iota \in \mathfrak{A}^0$, $\iota + \mathrm{ch}(\iota)b^* < (s+m)b^*$, $\mathrm{ch}(\iota) \geqslant m$ and $\kappa(\iota) \leqslant (p-2)m + s$. Then the image of $\mathcal{N}^{sp}$ under the natural map $\mathcal{N}^{\dagger} \longrightarrow \bar{\mathcal{N}}^{\dagger} \longrightarrow \widetilde{\mathcal{N}}^{\dagger}$ coincides with $\widetilde{\mathcal{N}}^{sp}$.

Let $\mathcal{I}^{sp}$ be the submodule in $\mathcal{N}^{\dagger}$ generated by $t^{-b}l$ such that $l \in \mathcal{L}^{\dagger}(s)[m]$ and either $s+m \geqslant p$ or $b < (s+m)b^* - (p-1)b^*$. Then the image of $\mathcal{I}^{sp}$ under the above natural map $\mathcal{N}^{sp} \longrightarrow \widetilde{\mathcal{N}}^{\dagger}$ is 0.

Lemma 2.9. a) *$\widetilde{\mathcal{N}}^{sp}$ is a Lie subalgebra in $\widetilde{\mathcal{N}}^{\dagger}$.*

b) *For any $j \geqslant 0$, $t^{jb^*}\widetilde{\mathcal{N}}^{sp}$ is an ideal in $\widetilde{\mathcal{N}}^{sp}$.*

Proof. a) Suppose $w_1 = t^{-\iota_1}l_1$ and $w_2 = t^{-\iota_2}l_2$ belong to $\mathcal{N}^{sp}$. Assume that for $j = 1, 2$, $l_j \in \mathcal{L}^{\dagger}(s_j)[m_j]$, where $s_j = w^0(l_j)$ and $m_j = \mathrm{ch}(l_j)$. We must prove that the image $\tilde{w}$ of $w = [w_1, w_2] \in \mathcal{N}^{\dagger}$ in $\widetilde{\mathcal{N}}^{\dagger}$ belongs to $\widetilde{\mathcal{N}}^{sp}$.

Let $s = s_1 + s_2$ and $m = m_1 + m_2$. Then $l = [l_1, l_2] \in \mathcal{L}^{\dagger}(s)[m]_k$. We can assume that $s + m < p$ (otherwise, $l \in \mathcal{I}^{sp}$ and $\tilde{w} = 0$).

Verify that $\iota = \iota_1 + \iota_2 \in \mathfrak{A}^0$. We can assume $\iota \geqslant (s+m)b^* - (p-1)b^*$ (otherwise, $w \in \mathcal{I}^{sp}$ and $\tilde{w} = 0$). This implies $\iota > -(p-1)b^*$.

Since $w_1, w_2 \in \mathcal{N}^{sp}$ we have also that

$$\iota + (\mathrm{ch}(\iota_1) + \mathrm{ch}(\iota_2))b^* < (s+m)b^* \leqslant (p-1)b^*\,,$$

and this implies $\iota < (p-1)b^*$.

We can assume that $m' = \mathrm{ch}(\iota_1) + \mathrm{ch}(\iota_2) < p$. (Otherwise, for $j = 1, 2$ $w_j \in t^{\mathrm{ch}(\iota_j)b^*}\mathcal{N}^{\dagger}$, $w \in t^{m'b^*}\widetilde{\mathcal{N}}^{\dagger} \subset \mathcal{I}^{sp}$ and $\tilde{w} = 0$.) In addition, $\kappa(\iota) \leqslant \kappa(\iota_1) + \kappa(\iota_2) \leqslant (p-2)m + s < u^*$. As a result, $\iota \in \mathfrak{A}^0$.

Finally, $\mathrm{ch}(\iota) = m' \geqslant m_1 + m_2 = m$, $[w_1, w_2] \in \mathcal{N}^{sp}$ and $w \in \widetilde{\mathcal{N}}^{sp}$.

b) Suppose $t^{-\iota}l \in \mathcal{N}^{sp}$ is given in terms of the above definition of $\mathcal{N}^{sp}$. We can assume that $s + m < p$ and $\iota \geqslant (s+m)b^* - (p-1)b^*$.

Prove that the image $\tilde{w}$ of $w = t^{-\iota + jb^*}l$ in $\widetilde{\mathcal{N}}^{\dagger}$ belongs to $\widetilde{\mathcal{N}}^{sp}$.

Let $\iota' = \iota - jb^*$. We can assume that $\iota' \geqslant (s+m)b^* - (p-1)b^*$ (otherwise, $w \in \mathcal{I}^{sp}$ and $\tilde{w} = 0$). Then $-(p-1)b^* < \iota' \leqslant \iota < (p-1)b^*$.

Suppose $\mathrm{ch}(\iota) + j < p$, then $\iota' = \iota - jb^* \in \mathfrak{A}^0$.

Indeed, $\mathrm{ch}(\iota') = \mathrm{ch}(\iota) + j < p$ and $\kappa(\iota') = \kappa(\iota) < u^*$. Therefore, $w = t^{-\iota'}l \in \widetilde{\mathcal{N}}^{sp}$, because $\iota' + \mathrm{ch}(\iota')b^* = \iota - jb^* + (\mathrm{ch}(\iota) + j)b^* < (s+m)b^*$, $\mathrm{ch}(\iota') \geqslant \mathrm{ch}(\iota) \geqslant m$ and $\kappa(\iota') = \kappa(\iota) \leqslant (p-2)m + s$.

If $\mathrm{ch}(\iota) + j \geqslant p$ then (as earlier) $w = t^{jb^*}t^{-\iota}l \in t^{(\mathrm{ch}(\iota)+j)b^*}\widetilde{\mathcal{N}}^{\dagger} \subset \mathcal{I}^{sp}$ and $\tilde{w} = 0$. The lemma is proved. □

Clearly, we have the induced grading $\widetilde{\mathcal{N}}^{sp} = \oplus_{m\geqslant 0}\widetilde{\mathcal{N}}^{sp}[m]$, where $\widetilde{\mathcal{N}}^{sp}[p-1]=0$. Any element from $\widetilde{\mathcal{N}}^{sp}[m]$ appears as a sum of elements of the form $t^{-\iota}l$, where for some $s\geqslant 1$, $l\in\bar{\mathcal{L}}^{\dagger}(s)[m]_k$, $\iota+\mathrm{ch}(\iota)b^*<(s+m)b^*$, $\mathrm{ch}(\iota)\geqslant m$ and $\kappa(\iota)\leqslant (p-2)m+s$.

Definition. For $j\geqslant 0$ and $s\geqslant 1$, let:

a) $\widetilde{\mathcal{N}}^{sp}\langle j\rangle$ be the k-submodule in $\widetilde{\mathcal{N}}^{sp}$ generated by all $t^{-\iota}l\in\widetilde{\mathcal{N}}^{sp}$ such that for some $m\geqslant 0$, $t^{-\iota}l\in\widetilde{\mathcal{N}}^{sp}[m]$ and $\mathrm{ch}(\iota)\geqslant m+j$;

b) $\widetilde{\mathcal{N}}^{sp}(s,j\rangle$ be the submodule in $\widetilde{\mathcal{N}}^{sp}\langle j\rangle$ generated by $t^{-\iota}l$ (in the above notation) such that $l\in C_s(\bar{\mathcal{L}}^{\dagger}_k)$.

Note that:

— $\widetilde{\mathcal{N}}^{sp}\langle 0\rangle=\widetilde{\mathcal{N}}^{sp}(1,0\rangle=\widetilde{\mathcal{N}}^{sp}$;

— all $\widetilde{\mathcal{N}}^{sp}\langle j\rangle$ and $\widetilde{\mathcal{N}}^{sp}(s,j\rangle$ are ideals in $\widetilde{\mathcal{N}}^{sp}$;

— for all j_1,j_2 and s_1,s_2, $[\widetilde{\mathcal{N}}^{sp}\langle j_1\rangle,\widetilde{\mathcal{N}}^{sp}\langle j_2\rangle]\subset\widetilde{\mathcal{N}}^{sp}\langle j_1+j_2\rangle$ and $[\widetilde{\mathcal{N}}^{sp}(s_1,j_1\rangle,\widetilde{\mathcal{N}}^{sp}(s_2,j_2\rangle]\subset\widetilde{\mathcal{N}}^{sp}(s_1+s_2,j_1+j_2\rangle$;

— $\widetilde{\mathcal{N}}^{sp}\langle p-1\rangle=0$.

— for any $\iota\in\mathfrak{A}^0(p)$, $t^{-\iota}D^{\dagger}_{\iota 0}\in\widetilde{\mathcal{N}}^{sp}$.

2.5. **The action Ω_γ.** Suppose $\gamma\in\mathbb{Z}/p$.

If $\iota=p^n(q\alpha-(q-1)\beta)\in\mathfrak{A}^0$ and $t^{-\iota}l\in\widetilde{\mathcal{N}}^{sp}$, where $l\in\bar{\mathcal{L}}^{\dagger}_k$, then by Lemma 2.9

$$\Omega_\gamma(t^{-\iota}l):=t^{-\iota}\widetilde{\exp}(\gamma(p^n\alpha)t^{b^*})l\ \in\widetilde{\mathcal{N}}^{sp}\,.$$

If $w\in\widetilde{\mathcal{N}}^{sp}$ then there is a unique presentation $w=\sum\limits_{\iota\in\mathfrak{A}^0}t^{-\iota}l_\iota$, where all $t^{-\iota}l_\iota\in\widetilde{\mathcal{N}}^{sp}$, and we set

$$\Omega_\gamma(w)=\sum_{\iota\in\mathfrak{A}^0}\Omega_\gamma(t^{-\iota}l_\iota)\,.$$

The correspondence $w\mapsto\Omega_\gamma(w)$ is a well-defined action of the elements γ of the (additive) group $\mathbb{Z}/p$ on the Lie algebra $\widetilde{\mathcal{N}}^{sp}$. This action is unipotent because for any $n\in\widetilde{\mathcal{N}}^{sp}\langle j\rangle$, $\Omega_\gamma(n)\equiv n\,\mathrm{mod}\,\widetilde{\mathcal{N}}^{sp}\langle j+1\rangle$.

Choose $\bar{e}^{sp}\in\bar{\mathcal{N}}^{\dagger}$ satisfying the following two conditions:

$$\bar{e}^{sp}\equiv\sum_{\iota\in\mathfrak{A}^0(p)}t^{-\iota}D^{\dagger}_{\iota 0}\,\mathrm{mod}\,C_2(\bar{\mathcal{L}}_{\mathcal{K}})\,; \tag{2.1}$$

$$\widetilde{e}^{sp}:=\bar{e}^{sp}\,\mathrm{mod}\,t^{(p-1)b^*}\bar{\mathcal{N}}^{\dagger}\in\widetilde{\mathcal{N}}^{sp}\,. \tag{2.2}$$

A choice of $\bar{e}^{\,sp}$ allows us to associate to the above defined action Ω_γ the "conjugated" action of $A^{\dagger}_\gamma$ on $\bar{\mathcal{L}}^{\dagger}$ as follows.

Proposition 2.10. *For any* $\gamma \in \mathbb{Z}/p$*, there are unique* $\widetilde{c}_\gamma \in \widetilde{\mathcal{N}}^{sp}\langle 1\rangle$ *and* $A^\dagger_\gamma \in \mathrm{Aut}_{\mathrm{Lie}}\bar{\mathcal{L}}^\dagger$ *such that*

a) $\sigma\tilde{c}_\gamma \in \widetilde{\mathcal{N}}^{sp}\langle 1\rangle$ *and* $\Omega_\gamma(\widetilde{e}^{\,sp}) = (\sigma\widetilde{c}_\gamma)\circ(A^\dagger_\gamma\otimes\mathrm{id}_{\mathcal{K}})\widetilde{e}^{\,sp}\circ(-\widetilde{c}_\gamma)$*;*

b) *for any* $\iota\in\mathfrak{A}^0(p)$*,* $A^\dagger_\gamma(D^\dagger_{\iota 0}) - D^\dagger_{\iota 0} \in \oplus_{m<\mathrm{ch}(\iota)}\bar{\mathcal{L}}^\dagger[m]_k$*.*

Proof. We need the following lemma.

Lemma 2.11. *Suppose* $j, s \geqslant 1$ *and* $n \in \widetilde{\mathcal{N}}^{sp}(s,j\rangle$*. Then there are unique* $\mathcal{S}(n), \mathcal{R}(n) \in \widetilde{\mathcal{N}}^{sp}(s,j\rangle$ *such that*

a) $\mathcal{R}(n) = \sum\limits_{\iota\in\mathfrak{A}^+(p)} t^{-\iota}l_\iota$ *with all* $l_\iota \in C_s(\bar{\mathcal{L}}^\dagger)_k$ *(if* $\mathrm{ch}(\iota)<j$ *then* $l_\iota = 0$*);*

b) $n = \mathcal{R}(n) + (\sigma - \mathrm{id})\mathcal{S}(n)$*.*

Proof of lemma. Note that any $n\in\widetilde{\mathcal{N}}^{sp}(s,j\rangle$ appears as a sum of elements of the form $t^{-\iota}l$, where for some m^0 and s^0, it holds $l\in\bar{\mathcal{L}}^\dagger(s^0)[m^0]_k\cap C_s(\bar{\mathcal{L}}^\dagger)_k$, $\iota+\mathrm{ch}(\iota)b^* < (s^0+m^0)b^*$, $\mathrm{ch}(\iota)\geqslant m^0+j$ and $\kappa(\iota)\leqslant(p-2)m^0+s^0$. When proving the existence of $\mathcal{S}(n)$ and $\mathcal{R}(n)$ we can assume that $n = t^{-\iota}l$.

— Let $\iota<0$.

Set $\mathcal{R}(n)=0$ and $\mathcal{S}(n) = -\sum_{m\geqslant 0}t^{-\iota p^m}\sigma^m l$.
If $-\iota p^m \geqslant b^*(p-1)$ then $t^{-\iota p^m}\sigma^m l\in t^{b^*(p-1)}\widetilde{\mathcal{N}}^\dagger = 0$.
If $-\iota p^m < b^*(p-1)$ then:

— $\iota p^m + \mathrm{ch}(\iota p^m)b^* \leqslant \iota + \mathrm{ch}(\iota)b^* < (s^0+m^0)b^*$;

— $m^0 = \mathrm{ch}(l) = \mathrm{ch}(\sigma^m l)$ and $\mathrm{ch}(\iota p^m) = \mathrm{ch}(\iota)\geqslant m^0+j$;

— $\kappa(\iota p^m) = \kappa(\iota)$.

Therefore, if $\iota<0$ then both $\mathcal{R}(n),\mathcal{S}(n)\in\widetilde{\mathcal{N}}^{sp}(s,j\rangle$.

— Let $\iota>0$.

Suppose $p^{m(\iota)}$ is the maximal power of p such that $\iota = p^{m(\iota)}\iota_1$ and $\iota_1\in\mathfrak{A}^0$. Then $\iota_1\in\mathfrak{A}^+(p)$: it will be sufficient to verify just the last inequality for $\kappa(\iota_1)=\kappa(\iota)$ from the definition of $\mathfrak{A}^+(p)$ in Sect.2.3. Using that $t^{-\iota}l\in\widetilde{\mathcal{N}}^{sp}$, $w^0(\iota)\geqslant 1$ and $\mathrm{ch}(\iota)\geqslant m^0+1$ we obtain that

$$(p-2)\mathrm{ch}(\iota) + w^0(\iota) \geqslant (p-2)m^0 + p - 1 \geqslant (p-2)m^0 + s^0 \geqslant \kappa(\iota_1)\,.$$

Then we set

$$\mathcal{R}(n) = t^{-\iota_1}\sigma^{-m(\iota)}l,\quad \mathcal{S}(n) = \sum_{0\leqslant m<m(\iota)}\sigma^m(\mathcal{R}(n))\,.$$

Finally, if $0\leqslant m\leqslant m(\iota)$ then $\sigma^m\mathcal{R}(n)\in\widetilde{\mathcal{N}}^{sp}\langle j\rangle$. Indeed,

— $\iota_1 p^m + \mathrm{ch}(\iota_1 p^m)b^* \leqslant \iota + \mathrm{ch}(\iota)b^* < (s^0+m^0)b^*$;

— $\mathrm{ch}(\iota_1 p^m) = \mathrm{ch}(\iota)\geqslant m^0+j$, $\sigma^{-m(\iota)+m}l\in\bar{\mathcal{L}}^\dagger(s^0)[m^0]_k\cap C_s(\bar{\mathcal{L}}^\dagger)_k$;

— $\kappa(\iota_1 p^m)=\kappa(\iota)$.

So, we proved the existence of $\mathcal{R}(n)$ and $\mathcal{S}(n)$.

The uniqueness follows from the fact that for $j\geqslant 1$, $\widetilde{\mathcal{N}}^{sp}\langle j\rangle|_{\sigma=\mathrm{id}}=0$ and the appropriate $t^{-\iota}$ are independent modulo $(\sigma-\mathrm{id})\mathcal{K}$, cf. Prop.2.8. The lemma is proved. $\square$

Continue the proof of Prop.2.10.

Use induction on $i\geqslant 1$ to prove the proposition modulo $\widetilde{\mathcal{N}}^{sp}(i,i\rangle$.

— If $i=1$ take $\widetilde{c}_\gamma=0$, $A^\dagger_\gamma=\mathrm{id}$ and use $\Omega_\gamma(\widetilde{e}^{sp})-\widetilde{e}^{sp}\in\widetilde{\mathcal{N}}^{sp}(1,1\rangle$.

— Assume $1\leqslant i<p$ and for $\widetilde{c}_\gamma\in\widetilde{\mathcal{N}}^{sp}(1,1\rangle$ and $A^\dagger_\gamma\in\mathrm{Aut}_{\mathrm{Lie}}(\bar{\mathcal{L}}^\dagger)$,

$$H=\Omega_\gamma\widetilde{e}^{sp}-(\sigma\widetilde{c}_\gamma)\circ(A^\dagger_\gamma\otimes\mathrm{id}_{\mathcal{K}})\widetilde{e}^{sp}\circ(-\widetilde{c}_\gamma)\in\widetilde{\mathcal{N}}^{sp}(i,i\rangle\,.$$

Then $\mathcal{R}(H),\mathcal{S}(H)\in\widetilde{\mathcal{N}}^{sp}(i,i\rangle$. Set $\mathcal{R}(H)=\sum\limits_{\mathrm{ch}(\iota)\geqslant i+m}t^{-\iota}H_{\iota m}$, where all $H_{\iota m}\in\widetilde{\mathcal{L}}^\dagger[m]_k\cap C_i(\bar{\mathcal{L}}^\dagger)_k$. Introduce $A^{\dagger\prime}_\gamma\in\mathrm{Aut}_{\mathrm{Lie}}(\bar{\mathcal{L}}^\dagger)$ by setting for all involved ι and m, $A^{\dagger\prime}_\gamma(D^\dagger_{\iota 0})=A^\dagger_\gamma(D^\dagger_{\iota 0})-\sum_m H_{\iota m}$. Set also $\widetilde{c}^{\,\prime}_\gamma=\widetilde{c}_\gamma-\mathcal{S}(H)$. Then

$$\Omega_\gamma\widetilde{e}^{sp}\equiv(\sigma\widetilde{c}^{\,\prime}_{\gamma 1})\circ(A^{\dagger\prime}_\gamma\otimes\mathrm{id}_{\mathcal{K}})\widetilde{e}^{sp}\circ(-\widetilde{c}^{\,\prime}_\gamma)\,\mathrm{mod}\,\widetilde{\mathcal{N}}^{sp}(i+1,i+1\rangle\,.$$

The uniqueness follows similarly by induction on i and the uniqueness part of Lemma 2.11.

The proposition is proved. $\square$

We have obviously the following properties.

Corollary 2.12. *For any $\gamma,\gamma_1\in\mathbb{Z}/p$,*

a) $A^\dagger_{\gamma+\gamma_1}=A^\dagger_\gamma A^\dagger_{\gamma_1}$;

b) $\Omega_\gamma(\widetilde{c}_{\gamma_1})\circ(A^\dagger_{\gamma_1}\otimes\mathrm{id}_{\mathcal{K}})\widetilde{c}_\gamma=\widetilde{c}_{\gamma+\gamma_1}$;

c) *if $l\in\bar{\mathcal{L}}^\dagger[m]$ then $A^\dagger_\gamma(l)-l\in\oplus_{m'<m}\bar{\mathcal{L}}^\dagger[m']$, e.g. $A^\dagger_\gamma|_{\bar{\mathcal{L}}^{\dagger 0}}=\mathrm{id}$.*

2.6. **The action Ω_U.** Let $A^\dagger:=A^\dagger_\gamma|_{\gamma=1}$. Then for any $\gamma=n\,\mathrm{mod}\,p$, $A^\dagger_\gamma=A^{\dagger n}$, in particular, $A^{\dagger p}=\mathrm{id}_{\bar{\mathcal{L}}^\dagger}$. By part c) of the above corollary, for all $m\geqslant 0$,

$$(A^\dagger_\gamma-\mathrm{id}_{\bar{\mathcal{L}}^\dagger})\left(\oplus_{m'\leqslant m}\bar{\mathcal{L}}^\dagger[m']\right)\subset\oplus_{m'<m}\bar{\mathcal{L}}^\dagger[m']\,.$$

Therefore, there is a differentiation $B^\dagger\in\mathrm{End}_{\mathrm{Lie}}\bar{\mathcal{L}}^\dagger$ such that for all $m\geqslant 0$, $B^\dagger(\bar{\mathcal{L}}^\dagger[m])\subset\oplus_{m'<m}\bar{\mathcal{L}}^\dagger[m']$ and for all $\gamma\in\mathbb{Z}/p$, $A^\dagger_\gamma=\widetilde{\exp}(\gamma B^\dagger)$.

Recover this derivation by applying the methods from [8], Sect.3.

Namely, define a coaction of the formal finite group scheme $\alpha_p=\mathrm{Spec}\mathbb{F}_p[U]$ on $\widetilde{\mathcal{N}}^{sp}$ as follows. (Here $U^p=0$ and the coaddition is such that $\Delta U=U\otimes 1+1\otimes U$.)

If $\iota=p^n(q\alpha-(q-1)\beta)\in\mathfrak{A}^0$ and $t^{-\iota}l\in\widetilde{\mathcal{N}}^{sp}$, where $l\in\bar{\mathcal{L}}^\dagger_k$, set

$$\Omega_U(t^{-\iota}l):=t^{-\iota}\widetilde{\exp}(U\otimes(p^n\alpha)t^{b^*})l\,\in\mathbb{F}_p[U]\otimes\widetilde{\mathcal{N}}^{sp}\,.$$

As a result,

$$\Omega_U(\widetilde{e}^{sp}) = \sigma(\widetilde{c}_U)\circ(A_U^{\dagger}\otimes\mathrm{id})\widetilde{e}^{sp}\circ(-\widetilde{c}_U)\,, \tag{2.3}$$

where for all $\gamma\in\mathbb{Z}/p$, $A_U^{\dagger}=\widetilde{\exp}(UB^{\dagger})$ and $\widetilde{c}_U|_{U=\gamma}=\widetilde{c}_{\gamma}$. In [?] we also established that

— $\widetilde{c}_U=\widetilde{c}^{(1)}U+\dots+\widetilde{c}^{(p-1)}U^{p-1}$, where all $c^{(j)},\sigma c^{(j)}\in\widetilde{\mathcal{N}}^{sp}\langle j\rangle$;

— the cocycle $\widetilde{c}_U$ is determined uniquely by its linear part $\widetilde{c}^{(1)}$;

— the action $\Omega_U=\sum_{0\leqslant i<p}\Omega^iU^i$ (here $\Omega^0=\mathrm{id}$) is recovered uniquely from its differential $d\,\Omega_U:=\Omega^1U$.

2.7. **Ideals $\bar{\mathcal{L}}^{\dagger}[v_0]$ and $\bar{\mathcal{L}}[v_0]$.**

Recall that $\bar{\mathcal{L}}^{\dagger}[0]$ is the minimal Lie subalgebra of $\bar{\mathcal{L}}^{\dagger}$ such that $\bar{\mathcal{L}}^{\dagger}[0]_k$ contains all $D^{\dagger}_{\iota n}$ with $\iota\in\mathfrak{A}^0_0(p)=\{\iota\in\mathfrak{A}^0(p)\mid \mathrm{ch}(\iota)=0\}$. Then $\bar{\mathcal{L}}^{\dagger}[0]$ has the induced filtration $\{\bar{\mathcal{L}}^{\dagger}(s)[0]\}_{s\geqslant 1}$ and there is epimorphism of filtered Lie algebras $\mathcal{V}^0:\bar{\mathcal{L}}^{\dagger}\longrightarrow\bar{\mathcal{L}}[0]$ such that $D^{\dagger}_{\iota n}\mapsto D^{\dagger}_{\iota n}$ if $\iota\in\mathfrak{A}^0_0(p)$ and $D^{\dagger}_{\iota n}\mapsto 0$, otherwise.

By Lemma 2.6, $\mathfrak{A}^0_0(p)=\{qa\mid a\in[0,(p-1)v_0)\cap\mathbb{Z}^+(p)\}$. By Remark from the proof of Proposition 2.8a), the correspondences $D^{\dagger}_{qa,n}\mapsto D_{an}$ establish isomorphism of filtered Lie algebras $j^0:\bar{\mathcal{L}}^{\dagger}[0]\longrightarrow\bar{\mathcal{L}}$.

Let $\mathcal{V}:=j^0\mathcal{V}^0:\bar{\mathcal{L}}^{\dagger}\longrightarrow\bar{\mathcal{L}}$.

Define the ideal $\bar{\mathcal{L}}^{\dagger}[v_0]$ as the minimal ideal in $\bar{\mathcal{L}}^{\dagger}$ containing all $A^{\dagger}_{\gamma}(\mathrm{Ker}\,\mathcal{V})$, $\gamma\in\mathbb{Z}/p$. Set $\bar{\mathcal{L}}[v_0]=\mathcal{V}(\bar{\mathcal{L}}^{\dagger}[v_0])$. Then $\bar{\mathcal{L}}[v_0]$ is the minimal ideal in $\bar{\mathcal{L}}$ such that $\mathcal{V}^{-1}(\bar{\mathcal{L}}[v_0])$ is invariant with respect to all $A^{\dagger}_{\gamma}$.

Proposition 2.13. *If $l\in\bar{\mathcal{L}}^{\dagger}$ and $\gamma\in\mathbb{Z}/p$ then*

$$\mathcal{V}(A^{\dagger}_{\gamma}l)\equiv\mathcal{V}(l)\,\mathrm{mod}\,\bar{\mathcal{L}}[v_0]\,.$$

Proof. a) Let $l'=\mathcal{V}^0(l)$. Then $l\in l'+\mathrm{Ker}\mathcal{V}$ and, therefore,

$$A^{\dagger}_{\gamma}(l)\in A^{\dagger}_{\gamma}(l')+A^{\dagger}_{\gamma}(\mathrm{Ker}\mathcal{V})\subset l'+\mathcal{V}^{-1}\bar{\mathcal{L}}[v_0]\,.$$

It remains to apply $\mathcal{V}$ to this embedding. (Use that $A^{\dagger}_{\gamma}|_{\mathrm{Im}\,\mathcal{V}^0}=\mathrm{id}$.) □

The ideal $\bar{\mathcal{L}}[v_0]$ can be also defined in terms related to the action Ω_U. If $B^{\dagger}$ is the differentiation from Sect.2.6 then $\bar{\mathcal{L}}[v_0]$ appears as the minimal ideal in $\bar{\mathcal{L}}$ such that $\bar{\mathcal{L}}[v_0]_k$ contains all the elements $\mathcal{V}B^{\dagger}_k(D^{\dagger}_{\iota 0})$, where $\iota\in\mathfrak{A}^0(p)$ and $\mathrm{ch}(\iota)\geqslant 1$ (if $\iota\in\mathfrak{A}^0_0(p)$ then $B^{\dagger}(D^{\dagger}_{\iota 0})=0$). This is implied by the following proposition.

Proposition 2.14. *Suppose $\mathcal{I}$ is an ideal in $\bar{\mathcal{L}}$. Then the following conditions are equivalent:*

a) *for any $\gamma\in\mathbb{Z}/p$, $A^{\dagger}_{\gamma}(\mathrm{Ker}\mathcal{V})\subset\mathcal{V}^{-1}(\mathcal{I})$;*

b) $B^{\dagger}(\mathrm{Ker}\mathcal{V})\subset\mathcal{V}^{-1}(\mathcal{I})$.

Proof. Part a) implies b) because for any $l \in \bar{\mathcal{L}}^{\dagger}$ we have a non-degenerate system of linear relations

$$(A^{\dagger}_{\gamma} - \mathrm{id}_{\bar{\mathcal{L}}^{\dagger}})l \equiv \sum_{1\leqslant s<p} \gamma^s B^{\dagger s}(l)/s! \,\mathrm{mod}\, \mathcal{I} \tag{2.4}$$

with $\gamma = 1, \dots, p-1$.

Vice versa, b) implies that for all $s \geqslant 1$, $B^{\dagger s}(\mathrm{Ker}\mathcal{V}) \subset B^{\dagger}(\mathrm{Ker}\,\mathcal{V})$. Indeed, $\bar{\mathcal{L}}^{\dagger} = \mathrm{Ker}\mathcal{V} \oplus \bar{\mathcal{L}}^{\dagger}[0]$ implies that $\mathcal{V}^{-1}(\mathcal{I}) = \mathrm{Ker}\mathcal{V} \oplus (j^0)^{-1}(\mathcal{I})$. Therefore, $B^{\dagger\, 2}(\mathrm{Ker}\,\mathcal{V}) \subset B^{\dagger}(\mathcal{V}^{-1}(\mathcal{I})) = B^{\dagger}(\mathrm{Ker}\mathcal{V})$ (use $B^{\dagger}|_{\bar{\mathcal{L}}^{\dagger}[0])} = 0$). It remains to use relations (2.4). Proposition is proved. □

2.8. **Lie algebras $\mathcal{N}^{(q)}$, $\bar{\mathcal{N}}^{(q)}$ and $\widetilde{\mathcal{N}}^{(q)}$.**

Introduce an analogue $\mathcal{N}^{(q)} \subset \mathcal{L}_{\mathcal{K}}$ of $\mathcal{N}$ as the k-module generated by all $t^{-a}l$, where for some $s \geqslant 1$, $l \in \mathcal{L}(s)_k$ and $a < sb^*$. It is a Lie k-algebra and $e^{(q)}$ together with all $\gamma * e^{(q)}$, $\gamma \in \mathbb{Z}/p$, cf. Sect.1.5, belong to $\mathcal{N}^{(q)}$.

Similarly, introduce the Lie algebras $\bar{\mathcal{N}}^{(q)}$ (use the algebra $\bar{\mathcal{L}}$ instead of $\mathcal{L}$) and $\widetilde{\mathcal{N}}^{(q)} = \bar{\mathcal{N}}^{(q)}/t^{(p-1)b^*}\bar{\mathcal{N}}^{(q)}$. These algebras are related to $\mathcal{N}^{(q)}$ via the natural projection $\bar{\mathrm{pr}}_{\mathcal{K}} : \mathcal{L}_{\mathcal{K}} \longrightarrow \bar{\mathcal{L}}_{\mathcal{K}}$. The appropriate images of $e^{(q)}$ in $\bar{\mathcal{N}}^{(q)}$ and $\widetilde{\mathcal{N}}^{(q)}$ will be denoted, resp., by $\bar{e}^{(q)}$ and $\widetilde{e}^{(q)}$. Note that there are natural identifications $\bar{\mathcal{N}}^{(q)} = \mathcal{V}_{\mathcal{K}}(\bar{\mathcal{N}}^{\dagger})$ and $\widetilde{\mathcal{N}}^{(q)} = \mathcal{V}_{\mathcal{K}}(\widetilde{\mathcal{N}}^{\dagger})$, where $\bar{\mathcal{N}}^{\dagger}$, $\widetilde{\mathcal{N}}^{\dagger}$ and $\mathcal{V}_{\mathcal{K}}$ were defined in Sect.2.4.

2.9. **Generators of $\bar{\mathcal{L}}[v_0]$.** Introduce the following condition of compatibility

$$\mathcal{V}_{\mathcal{K}}(\bar{e}^{sp}) = \bar{e}^{(q)}\,. \tag{2.5}$$

By Prop.2.14, $\bar{\mathcal{L}}[v_0]$ is the minimal ideal in $\bar{\mathcal{L}}$ such that for all $\iota \in \mathfrak{A}^0(p)$ with $\mathrm{ch}(\iota) \geqslant 1$, $\mathcal{V}_k B^{\dagger}_k(D^{\dagger}_{\iota 0}) \in \bar{\mathcal{L}}[v_0]_k$. (Note that this implies $\mathcal{V}B^{\dagger}(C_2(\bar{\mathcal{L}}^{\dagger})) \subset [\bar{\mathcal{L}}[v_0], \bar{\mathcal{L}}]$.)

Proposition 2.15. *If* $\mathrm{ch}(\iota) \geqslant 2$ *then* $\mathcal{V}_k B^{\dagger}_k(D^{\dagger}_{\iota 0}) \in [\bar{\mathcal{L}}[v_0], \bar{\mathcal{L}}]_k$.

Proof. Suppose $\tilde{e}^{\,(q)} = \sum_a t^{-qa} l^{(q)}_a$ and $\tilde{e}^{\,sp} = \sum_{\iota} t^{-\iota} l^{sp}_{\iota}$, where all $l^{(q)}_a \in \bar{\mathcal{L}}_k$ and $l^{sp}_{\iota} \in \bar{\mathcal{L}}^{\dagger}_k$. Note that:

— if $a \in \mathbb{Z}^0(p)$ then $l^{(q)}_a \equiv D_{a0} \,\mathrm{mod}\, C_2(\bar{\mathcal{L}})_k$;

— if $a \notin \mathbb{Z}^0(p)$ then $l^{(q)}_a \in C_2(\bar{\mathcal{L}})_k$;

— if $\iota = qa$ with $a \in \mathbb{Z}$, then $\mathcal{V}_k(l^{sp}_{\iota}) = l^{(q)}_a$, otherwise $\mathcal{V}_k(l^{sp}_{\iota}) = 0$;

— if $\iota \in \mathfrak{A}^0(p)$ then $l^{sp}_{\iota} \equiv D^{\dagger}_{\iota 0} \,\mathrm{mod}\, C_2(\bar{\mathcal{L}}^{\dagger})_k$, otherwise, $l^{sp}_{\iota} \in C_2(\bar{\mathcal{L}}^{\dagger})_k$.

Applying formalism from Sect.2.6 we obtain

$$\Omega_U(\widetilde{e}^{\,sp}) \equiv (U\sigma\tilde{c}^1) \circ (\widetilde{e}^{\,sp} + U(B^{\dagger} \otimes \mathrm{id}_{\mathcal{K}})\widetilde{e}^{\,sp}) \circ (-U\tilde{c}^1) \,\mathrm{mod}\, U^2 \widetilde{\mathcal{N}}^{\dagger}\,, \tag{2.6}$$

where $\tilde{c}^1, \sigma\tilde{c}^1 \in \widetilde{\mathcal{N}}^{sp}\langle 1\rangle$. Note that

$$\Omega_U(\widetilde{e}^{sp}) \equiv \widetilde{e}^{sp} + U\sum_\iota (p^m\alpha)_\iota t^{-\iota+b^*} l_\iota^{sp} \operatorname{mod} U^2\,,$$

where $\iota = q(p^m\alpha)_\iota - (q-1)(p^m\beta)_\iota$.

Applying $\mathcal{V}_{\mathcal{K}}$ to relation (2.6) and setting $\tilde{x} := \mathcal{V}_{\mathcal{K}}\tilde{c}^1$ we obtain

$$\tilde{e}^{(q)} + U\sum_a at^{-qa+b^*} l_a^{(q)} \equiv \tag{2.7}$$

$$(U\sigma\tilde{x})\circ\left(\tilde{e}^{(q)} + U\sum_\iota t^{-\iota}\mathcal{V}_k B_k^\dagger(l_\iota^{sp})\right)\circ(-U\tilde{x}) \operatorname{mod} U^2\tilde{\mathcal{N}}^{(q)}\,.$$

Let $\tilde{f}_1, \tilde{f}_2 \in \widetilde{\mathcal{N}}^{(q)}$ be such that

$$(U\sigma\tilde{x})\circ\tilde{e}^{(q)} \equiv \tilde{e}^{(q)} + U(\sigma\tilde{x} + \tilde{f}_1) \operatorname{mod} U^2$$

$$\tilde{e}^{(q)}\circ(-U\tilde{x}) \equiv \tilde{e}^{(q)} + U(-\tilde{x} + \tilde{f}_2) \operatorname{mod} U^2\,.$$

There are explicit formulas for $\tilde{f}_1$ and $\tilde{f}_2$, cf. e.g. Sect.3.2 of [8], but we need only that they are just $\mathbb{F}_p$-linear combinations of the commutators $[\dots[\sigma\tilde{x}, \tilde{e}^{(q)}], \dots, \tilde{e}^{(q)}]$ and, resp., $[\dots[\tilde{x}, \tilde{e}^{(q)}], \dots, \tilde{e}^{(q)}]$.

Comparing the coefficients for U in (2.7) we obtain

$$\sum_a at^{-qa+b^*} l_a^{(q)} = \sigma\tilde{x} - \tilde{x} + \sum_\iota t^{-\iota}\mathcal{V}_k B_k^\dagger(l_\iota^{sp}) + \tilde{f}_1 + \tilde{f}_2\,. \tag{2.8}$$

Note that:

a) $\tilde{c}^1, \sigma\tilde{c}^1 \in \widetilde{\mathcal{N}}^{sp}\langle 1\rangle$ implies that $\tilde{x}, \sigma\tilde{x} \in \sum_{\mathrm{ch}(\iota)\geqslant 1} t^{-\iota}\bar{\mathcal{L}}_k$;

b) $\{\, y \in \sum_{\mathrm{ch}(\iota)\geqslant 1} t^{-\iota}\bar{\mathcal{L}}_k \mid \sigma y = y\} = 0$;

c) if $\iota \notin \mathfrak{A}^0(p)$ then $\mathcal{V}_k B_k^\dagger(l_\iota^{sp}) \in [\bar{\mathcal{L}}[v_0], \bar{\mathcal{L}}]_k$;

d) if $\iota \in \mathfrak{A}^0(p)$ then $\mathcal{V}_k B_k^\dagger(l_\iota^{sp}) \equiv \mathcal{V}_k B_k^\dagger(D_{\iota 0}^\dagger) \operatorname{mod} [\bar{\mathcal{L}}[v_0], \bar{\mathcal{L}}]_k$.

Let $\tilde{x} = \sum_\iota t^{-\iota}x_\iota$ and $\tilde{f}_1 + \tilde{f}_2 = \sum_\iota t^{-\iota} f_\iota$, where $x_\iota, f_\iota \in \bar{\mathcal{L}}_k$ and the both sums are taken for $\iota \in \mathfrak{A}^0$ such that $\mathrm{ch}(\iota) \geqslant 1$. Let $\tilde{x}[m]$ be a part of the first sum containing all the summands $t^{-\iota}x_\iota$ with $\mathrm{ch}(\iota) = m$. Similarly, define a part $\tilde{f}[m]$ of the second sum. Note that $\tilde{f}[m]$ is a linear combination of the commutators $[\dots[\tilde{x}[m], \tilde{e}^{(q)}], \dots, \tilde{e}^{(q)}]$ and $[\dots[\sigma(\tilde{x}[m]), \tilde{e}^{(q)}], \dots, \tilde{e}^{(q)}]$.

Then (2.8) and above congruence d) imply that for any $m \geqslant 2$,

$$\mathcal{H}[m] \equiv -\tilde{f}[m] \in \operatorname{mod} [\bar{\mathcal{L}}[v_0], \bar{\mathcal{L}}]_{\mathcal{K}}\,,$$

where $\mathcal{H}[m] := \sigma(\tilde{x}[m]) - \tilde{x}[m] + \sum_{\mathrm{ch}(\iota)=m} t^{-\iota}\mathcal{V}_k B_k^\dagger(D_{\iota 0}^\dagger)$.

Let $\bar{\mathcal{D}}(s) := [\bar{\mathcal{L}}[v_0], \bar{\mathcal{L}}] + \bar{\mathcal{L}}(s)$.

Prove by induction on $s\geqslant 1$ that $\tilde{x}[m]\in\bar{\mathcal{D}}(s)_{\mathcal{K}}$ and $\mathcal{V}_kB_k^{\dagger}(D_{\iota 0}^{\dagger})\in\bar{\mathcal{D}}(s)_k$ (here $\mathrm{ch}(\iota)=m$).

If $s=1$ there is nothing to prove.

Suppose it is proved for $s<p$.

Then $\tilde{f}[m]\in\bar{\mathcal{D}}(s+1)_{\mathcal{K}}$ and, therefore, $\mathcal{H}[m]\in\bar{\mathcal{D}}(s+1)_{\mathcal{K}}$. Then analog of Lemma 2.11 implies that $\tilde{x}[m]$ and $\sum\limits_{\mathrm{ch}(\iota)=m}t^{-\iota}\mathcal{V}_kB_k^{\dagger}(D_{\iota 0}^{\dagger})$ belong to $\bar{\mathcal{D}}(s+1)_{\mathcal{K}}$. In particular, all $\mathcal{V}_kB_k^{\dagger}(D_{\iota 0})\in\bar{\mathcal{D}}(s+1)_k$.

The proposition is proved because $\bar{\mathcal{D}}(p)=[\bar{\mathcal{L}}[v_0],\bar{\mathcal{L}}]$. □

Corollary 2.16. *$\bar{\mathcal{L}}[v_0]$ is the minimal ideal in $\bar{\mathcal{L}}$ such that for all $\iota\in\mathfrak{A}_1^0(p):=\{\iota\in\mathfrak{A}^0(p)\mid \mathrm{ch}(\iota)=1\}$, $\mathcal{V}_kB_k^{\dagger}(D_{\iota 0}^{\dagger})\in\bar{\mathcal{L}}[v_0]_k$.*

3. Application to the ramification filtration

3.1. **Statement of the main result.** Recall that in Sect.1 we fixed an element $e\in\mathcal{L}_{\mathcal{K}}$ satisfying conditions (1.1) and (1.3). We also fixed $f\in\mathcal{L}_{sep}$ such that $\sigma f=e\circ f$ and introduced epimoirphism $\eta_e=\pi_f(e):\mathcal{G}\longrightarrow G(\mathcal{L})$ which induces identification $\mathcal{G}_{<p}\simeq G(\mathcal{L})$. Conditions (1.1) and (1.3) mean that η_e is a "sufficiently good" lift of the reciprocity map of class field theory. We are going to describe the ideal $\mathcal{L}^{(v_0)}$ of $\mathcal{L}$ such that $\eta_e(\mathcal{G}^{(v_0)})=\mathcal{L}^{(v_0)}$ via the ideal $\bar{\mathcal{L}}[v_0]$ introduced in Sect. 2. In the next section this result will be related to the explicit description of $\mathcal{L}^{(v_0)}$ from [1].

Consider the parameters δ_0,r^*,N^*,q from Sect.2 (they depend just on the original $v_0>0$). Note that if $e^{(q)}=\sigma^{N^*}(e)$ and $f^{(q)}=\sigma^{N^*}(f)$ then the appropriate morphism $\pi_{f^{(q)}}(e^{(q)})$ coincides with η_e.

The ideal $\bar{\mathcal{L}}[v_0]\subset\bar{\mathcal{L}}$ was defined in the terms of action of the formal group α_p on $\widetilde{e}^{sp}=\bar{e}^{sp}\,\mathrm{mod}\,t^{(p-1)b^*}\bar{\mathcal{N}}^{\dagger}\in\widetilde{\mathcal{N}}^{sp}\subset\widetilde{\mathcal{N}}^{\dagger}$ which satisfies assumption (2.1). In Sect. 2.8 we introduced compatibility condition (2.5) relating the elements e and $\bar{e}^{sp}$. This condition can be definitely satisfied if e.g. $e=\sum\limits_{a\geqslant 0}t^{-a}l_a$, where all $l_a\in\mathcal{L}_k$.

Theorem 3.1. *Under condition* (2.5), $\mathcal{L}^{(v_0)}=\bar{\mathrm{pr}}^{-1}\bar{\mathcal{L}}[v_0]$.

3.2. **Inductive assumption.** Prove theorem by induction on $s_0\geqslant 1$ in the folowing form (the statement of theorem appears with $s_0=p$)

$$\mathcal{L}^{(v_0)}+C_{s_0}(\mathcal{L})=\bar{\mathrm{pr}}^{-1}(\bar{\mathcal{L}}[v_0])+\mathcal{L}(s_0)\,. \tag{3.1}$$

It is obviously true for $s_0=1$ because $C_1(\mathcal{L})=\mathcal{L}(1)=\mathcal{L}$.

Suppose (3.1) holds for some $1\leqslant s_0<p$.

Note that for any $\gamma\in\mathbb{Z}/p$, the image $\gamma*\widetilde{e}^{(q)}$ of $\gamma*\bar{e}^{(q)}$ in $\widetilde{\mathcal{N}}^{(q)}$ coincides with $\mathcal{V}_{\mathcal{K}}\Omega_{\gamma}(\widetilde{e}^{sp})$.

Proposition 3.2. *There is $x_\gamma\in t^{b^*}\sum\limits_{1\leqslant s<s_0}t^{-sb^*}\mathcal{L}(s)_{\mathrm{m}}\subset\mathcal{N}^{(q)}$ such that*

$$\gamma*e^{(q)}\equiv(\sigma x_\gamma)\circ e^{(q)}\circ(-x_\gamma)\,\mathrm{mod}\,(\mathcal{L}^{(v_0)}+C_{s_0}(\mathcal{L}))_{\mathcal{K}}\,.$$

Proof. From Prop.2.10 a) it follows that

$$\gamma * \widetilde{e}^{(q)} = (\sigma\widetilde{x}_\gamma)\circ(\mathcal{V}A^\dagger_\gamma\otimes \mathrm{id}_\mathcal{K})\widetilde{e}^{sp}\circ(-\widetilde{x}_\gamma)\,,$$

where $\widetilde{x}_\gamma = \mathcal{V}_\mathcal{K}(\widetilde{c}_\gamma)\in\mathcal{V}_\mathcal{K}(\widetilde{\mathcal{N}}^{sp}\langle 1\rangle)\subset\mathcal{V}_\mathcal{K}(t^{b^*}\widetilde{\mathcal{N}}^\dagger) = t^{b^*}\widetilde{\mathcal{N}}^{(q)}$ (use Remark before Lemma 2.9).

Recall that $\bar{e}^{sp}\in\bar{\mathcal{N}}^\dagger$ is a lift of $\widetilde{e}^{sp}\in\widetilde{\mathcal{N}}^{sp}\subset\widetilde{\mathcal{N}}^\dagger$ such that $\mathcal{V}_\mathcal{K}(\bar{e}^{sp}) = \bar{e}^{(q)}$ and σ is nilpotent on the kernel of the projection $\bar{\mathcal{N}}^\dagger\longrightarrow\widetilde{\mathcal{N}}^\dagger$. Therefore, proceeding simialrly to the proof of Prop.1.4 we can establish the existence of a unique lift $\bar{x}_\gamma\in t^{b^*}\bar{\mathcal{N}}^{(q)}$ of $\tilde{x}_\gamma$ such that

$$\gamma * \bar{e}^{(q)} = (\sigma\bar{x}_\gamma)\circ(\mathcal{V}A^\dagger_\gamma\otimes\mathrm{id}_\mathcal{K})\bar{e}^\dagger\circ(-\bar{x}_\gamma)\,. \tag{3.2}$$

Prop.2.13 implies that $(\mathcal{V}A^\dagger_\gamma\otimes\mathrm{id}_\mathcal{K})\bar{e}^\dagger\equiv\bar{e}^{(q)}\operatorname{mod}\bar{\mathcal{L}}[v_0]_\mathcal{K}$ and we obtain the following congruence

$$\gamma * e^{(q)}\equiv(\sigma x_\gamma)\circ e^{(q)}\circ(-x_\gamma)\operatorname{mod}\bar{\mathrm{pr}}^{-1}\bar{\mathcal{L}}[v_0]_\mathcal{K}\,, \tag{3.3}$$

where $x_\gamma\in\mathcal{L}_\mathcal{K}$ is any lift of $\bar{x}_\gamma$.

We can choose $x_\gamma\in t^{b^*}\sum_{1\leqslant s<s_0}t^{-sb^*}\mathcal{L}(s)_\mathrm{m}$ when taking this congruence modulo the ideal $(\bar{\mathrm{pr}}^{-1}\bar{\mathcal{L}}[v_0]+\mathcal{L}(s_0))_\mathcal{K}$. It remains to use the inductive assumption. The proposition is proved. □

Remark. a)Due to the criterion from Sect.1.5 congruence (3.3) already implies that $\mathcal{L}^{(v_0)}\subset\bar{\mathrm{pr}}^{-1}\bar{\mathcal{L}}[v_0]$ (use that all x_γ are defined over $\mathcal{K}$).

b) In the above proof we have automatically that $\sigma\bar{x}_\gamma\in t^{b^*}\bar{\mathcal{N}}^{(q)}$ and $\sigma x_\gamma\in t^{b^*}\sum_{1\leqslant s<s_0}t^{-sb^*}\mathcal{L}(s)_\mathrm{m}$.

For (non-commuting) variables U and V from some Lie $\mathbb{F}_p$-algebra L of nilpotent class $<p$, let $\delta^0(U,V) := U\circ V-(U+V)$. Note, if U and V are well-defined modulo $C_{s_0}(L)$ then $\delta^0(U,V)$ is well-defined modulo $C_{s_0+1}(L)$.

Let $y_\gamma = \gamma * e^{(q)} - e^{(q)} + \delta^0(\gamma * e^{(q)}, x_\gamma) - \delta^0(\sigma x_\gamma, e^{(q)})$.

Lemma 3.3. *For any $\gamma\in\mathbb{Z}/p$, there is $X_\gamma\in\mathcal{L}_{sep}$ such that*

a) $\gamma * e^{(q)}\equiv(\sigma X_\gamma)\circ e^{(q)}\circ(-X_\gamma)\operatorname{mod}([\mathcal{L}^{(v_0)},\mathcal{L}]+C_{s_0+1}(\mathcal{L}))_{sep}$;

b) $X_\gamma\equiv x_\gamma\operatorname{mod}(\mathcal{L}^{(v_0)}+C_{s_0}(\mathcal{L}))_{sep}$.

Proof of lemma. Prop.3.2 implies that

$$y_\gamma\equiv\sigma x_\gamma - x_\gamma\operatorname{mod}(\mathcal{L}^{(v_0)}+C_{s_0}(\mathcal{L}))_\mathcal{K}\,.$$

Therefore, there is $X_\gamma\in\mathcal{L}_{sep}$ such that $\sigma X_\gamma - X_\gamma = y_\gamma$ and X_γ satisfies the congruence b).

It remains to note that a) is equivalent to the following congruence

$$\sigma X_\gamma - X_\gamma\equiv\gamma * e^{(q)} - e^{(q)} + \delta^0(\gamma * e^{(q)}, X_\gamma) - \delta^0(\sigma X_\gamma, e^{(q)})$$

modulo $([\mathcal{L}^{(v_0)},\mathcal{L}]+C_{s_0+1}(\mathcal{L}))_{sep}$ and by the same modulo we have $\delta^0(\gamma * e^{(q)}, X_\gamma)\equiv\delta^0(\gamma * e^{(q)}, x_\gamma)$ and $\delta^0(\sigma X_\gamma, e^{(q)})\equiv\delta^0(\sigma x_\gamma, e^{(q)})$. □

The element y_γ can be uniquely written as

$$y_\gamma = \sum_{\substack{m\geqslant 0\\ a\in\mathbb{Z}^+(p)}} t^{-ap^m} l_{am} + l_O\,,$$

where all $l_{am}\in\mathcal{L}_k$ and $l_O\in\mathcal{L}_O$ (and $O=k[[t]]\subset\mathcal{K}$). By Prop.1.3 the ideal $\mathcal{L}^{(v_0)}+C_{s_0+1}(\mathcal{L})$ appears as the minimal ideal in the set of all ideals $\mathcal{I}$ such that:

— $\mathcal{I}\supset[\mathcal{L}^{(v_0)},\mathcal{L}]+C_{s_0+1}(\mathcal{L})$;

— if $a\in\mathbb{Z}^+(p)$ and $a\geqslant qv_0-b^*$ then $l^{(a)}:=\sum_{m\geqslant 0}\sigma^{-m}l_{am}\in\mathcal{I}_k$.

Proposition 3.4. *$\mathcal{L}(s_0+1)\subset\mathcal{L}^{(v_0)}+C_{s_0+1}(\mathcal{L})$, or (equivalently) if $a\geqslant s_0v_0$ then all $D_{an}\in\mathcal{L}_k^{(v_0)}+C_{s_0+1}(\mathcal{L})_k$.*

Proof. We have $e^{(q)},\gamma * e^{(q)}\in\mathcal{N}^{(q)}$ and $\gamma * e^{(q)}-e^{(q)}\in t^{b^*}\mathcal{N}^{(q)}$. Then from Prop.3.2 we obtain that $y_\gamma\equiv\gamma * e^{(q)}-e^{(q)}$ modulo

$$[t^{b^*}\mathcal{N},\mathcal{N}]\subset(t^{b^*}\mathcal{N})\cap C_2(\mathcal{L})_{\mathcal{K}}\subset$$

$$\subset\sum_{2\leqslant s_1+s_2\leqslant s_0}[\mathcal{L}(s_1),\mathcal{L}(s_2)]_{\mathrm{m}}t^{-(s_1+s_2-1)b^*}+\mathcal{L}(s_0+1)_{\mathcal{K}}\cap C_2(\mathcal{L})_{\mathcal{K}}$$

$$\subset t^{-(s_0-1)b^*}\mathcal{L}_{\mathrm{m}}+\mathcal{L}(s_0+1)_{\mathcal{K}}\cap C_2(\mathcal{L})_{\mathcal{K}}.$$

Lemma 3.5. *$\mathcal{L}(s_0+1)\cap C_2(\mathcal{L})\subset\mathcal{L}^{(v_0)}\cap C_2(\mathcal{L})+C_{s_0+1}(\mathcal{L})$.*

Proof of lemma. From the definition of $\mathcal{L}(s_0+1)$ it follows that the k-module $\mathcal{L}(s_0+1)_k\cap C_2(\mathcal{L})_k$ is generated by the commutators

$$[\dots[D_{a_1n_1},D_{a_2n_2}],\dots,D_{a_rn_r}]$$

such that $r\geqslant 2$ and $\mathrm{wt}(D_{a_1n_1})+\dots+\mathrm{wt}(D_{a_rn_r})\geqslant s_0+1$.

Here for $1\leqslant i\leqslant r$, $\mathrm{wt}(D_{a_in_i})=s_i$, where $(s_i-1)v_0\leqslant a_i<s_iv_0$. Hence, if $s'_i:=\min\{s_i,s_0\}$ then $\sum_i s'_i\geqslant s_0+1$ (use that $r\geqslant 2$). By inductive assumption all $D_{a_in_i}\in\mathcal{L}(s'_i)_k\subset\mathcal{L}_k^{(v_0)}+C_{s'_i}(\mathcal{L})_k$ and, therefore, our commutator belongs to $\mathcal{L}_k^{(v_0)}+C_{s_0+1}(\mathcal{L})_k$. It remains to note that $(\mathcal{L}^{(v_0)}+C_{s_0+1}(\mathcal{L}))\cap C_2(\mathcal{L})=\mathcal{L}^{(v_0)}\cap C_2(\mathcal{L})+C_{s_0+1}(\mathcal{L})$.

The lemma is proved. □

Lemma 3.5 implies that for $a\geqslant(s_0-1)b^*$, all $l^{(a)}$ modulo the ideal $\mathcal{L}_k^{(v_0)}\cap C_2(\mathcal{L})_k+C_{s_0+1}(\mathcal{L})_k$ appear as linear combinations of the linear terms of $\gamma * e^{(q)}-e^{(q)}$. More precisely, this can be stated as follows.

Let

$$\sum_{a\in\mathbb{Z}^+(p)}t^{-aq}(E(at^{b^*})-1)D_{a0}=\sum_{a',u}\alpha(a',u)t^{-qa'+ub^*}D_{a'0}\,,$$

where a' and u run over $\mathbb{Z}^+(p)$ and $\mathbb{N}$, resp., and all $\alpha(a',u)\in\mathbb{F}_p$ (note that $\alpha(a',1)=a'$).

Lemma 3.6. *If $a\geqslant (s_0-1)b^*$ then*

$$l^{(a)}\equiv \sum_{\substack{m\geqslant 0\\ qa'-ub^*=ap^m}}\alpha(a',u)D_{a',-m}\,\mathrm{mod}\,(\mathcal{L}_k^{(v_0)}\cap C_2(\mathcal{L})_k+C_{s_0+1}(\mathcal{L})_k)\,.$$

Proof of Lemma. Suppose $a_0\in\mathbb{Z}^0(p)$ satisfies the following inequality $a_0\geqslant s_0v_0$. Then $a=qa_0-b^*\geqslant (s_0-1)b^*$ and $l^{(a)}$ is congruent to

$$a_0D_{a_00}+\{k\text{-(pro)linear combination of } D_{a'm'} \text{ with } a'>a_0\}\,.$$

Since all such $l^{(a)}$ should belong to $\mathcal{L}^{(v_0)}+C_{s_0+1}(\mathcal{L})$, this implies that all D_{a_00} with $a_0\geqslant s_0v_0$ (or, equivalently, with the weight $\geqslant s_0+1$) must belong to $\mathcal{L}_k^{(v_0)}+C_{s_0+1}(\mathcal{L})_k$. □

Proposition is proved. □

3.3. **Interpretation in $\bar{\mathcal{L}}^{\dagger}$.** It remains to prove that in $\bar{\mathcal{L}}$ we have $\bar{\mathcal{L}}^{(v_0)}+C_{s_0+1}(\bar{\mathcal{L}})=\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0+1)$. By Prop.3.4 and Remark from Sect.3.2 it will be sufficient to establish that

$$\bar{\mathcal{L}}^{(v_0)}+\bar{\mathcal{L}}(s_0+1)\supset\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0+1)\,.$$

We can use the inductive assumption in the following form

$$\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0)=\bar{\mathcal{L}}^{(v_0)}+\bar{\mathcal{L}}(s_0)\,.$$

By the definition of $\bar{\mathcal{L}}[v_0]$ and Prop.2.15 the ideal $\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0+1)$ appears as the minimal ideal in the set of all ideals $\mathcal{I}$ of $\bar{\mathcal{L}}$ such that :

- $\mathcal{I}\supset\bar{\mathcal{D}}(s_0+1):=[\bar{\mathcal{L}}[v_0],\bar{\mathcal{L}}]+\bar{\mathcal{L}}(s_0+1)=[\bar{\mathcal{L}}^{(v_0)},\bar{\mathcal{L}}]+\bar{\mathcal{L}}(s_0+1)$;
- if $\iota\in\mathfrak{A}_1^0(p)$ then $\mathcal{V}_kB_k^{\dagger}(D_{\iota0}^{\dagger})\in\mathcal{I}_k$.

We must prove that for any $\iota\in\mathfrak{A}_1^0(p)$, $\mathcal{V}_kB_k^{\dagger}(D_{\iota0}^{\dagger})\in\bar{\mathcal{L}}_k^{(v_0)}+\bar{\mathcal{L}}(s_0+1)_k$ or, equivalently, for any $\gamma\neq 0$,

$$\mathcal{V}_k(A_{\gamma}^{\dagger}-\mathrm{id}_{\bar{\mathcal{L}}_k})(D_{\iota0}^{\dagger})\in\bar{\mathcal{L}}_k^{(v_0)}+\bar{\mathcal{L}}(s_0+1)_k\,.$$

Fix $\gamma\neq 0$ and consider equality (3.2).

By Prop.2.13, $(\mathcal{V}A_{\gamma}^{\dagger}\otimes\mathrm{id}_{\mathcal{K}})\bar{e}^{\dagger}\equiv\bar{e}^{(q)}\,\mathrm{mod}\,(\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0))_{\mathcal{K}}$. Hence, there is $Z_{\gamma}\in(\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0))_{sep}$ such that

$$\sigma Z_{\gamma}-Z_{\gamma}=(\mathcal{V}(A_{\gamma}^{\dagger}-\mathrm{id}_{\bar{\mathcal{L}}^{\dagger}})\otimes\mathrm{id}_{\mathcal{K}})\bar{e}^{\dagger}\,,$$

and we obtain (use that $(\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0+1))\,\mathrm{mod}\,\bar{\mathcal{D}}(s_0+1)$ is abelian)

$$\gamma*\bar{e}^{(q)}=\sigma(\bar{x}_{\gamma}\circ Z_{\gamma})\circ\bar{e}^{(q)}\circ(-(\bar{x}_{\gamma}\circ Z_{\gamma}))\,\mathrm{mod}\,\bar{\mathcal{D}}(s_0+1)_{sep}.$$

Therefore, the ideal $\bar{\mathcal{L}}^{(v_0)}+\bar{\mathcal{L}}(s_0+1)\supset\bar{\mathcal{D}}(s_0+1)$ is the minimal in the family of all ideals $\mathcal{I}$ such that

- $\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0+1)\supset\mathcal{I}\supset\bar{\mathcal{D}}(s_0+1)$;
- $v(Z_{\gamma}\,\mathrm{mod}\,\mathcal{I}_{sep}/\mathcal{K})<qv_0-b^*$ (use that $\bar{x}_{\gamma}$ is defined over $\mathcal{K}$).

By Prop.2.15 we have the following congruence modulo $\bar{\mathcal{D}}(s_0+1)_{\mathcal{K}}$

$$(\mathcal{V}(A_\gamma^\dagger - \mathrm{id}_{\bar{\mathcal{L}}^\dagger})\otimes \mathrm{id}_{\mathcal{K}})\bar{e}^\dagger \equiv \sum_{\mathrm{ch}(\iota)=1} t^{-\iota}\mathcal{V}_k(A_\gamma^\dagger - \mathrm{id}_{\bar{\mathcal{L}}^\dagger})_k D_{\iota 0}^\dagger\,.$$

For any $\iota\in\mathfrak{A}_1^0(p)$, consider $\bar{W}_{\gamma\iota} := \mathcal{V}_k(A_\gamma^\dagger - \mathrm{id}_{\bar{\mathcal{L}}^\dagger})_k D_{\iota 0}^\dagger \in \bar{\mathcal{L}}_k$.

Recall that $\bar{\mathcal{L}}[v_0]_k+\bar{\mathcal{L}}(s_0+1)_k$ is generated by all $\bar{W}_{\gamma\iota}$ and the elements of $\bar{\mathcal{D}}(s_0+1)_k$. Then

$$Z_\gamma \equiv \sum_{\mathrm{ch}(\iota)=1} Z_{\gamma\iota} \,\mathrm{mod}\, \bar{\mathcal{D}}(s_0+1)_{sep}\,,$$

where $Z_{\gamma\iota}\in(\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0+1)/\bar{\mathcal{D}}(s_0+1))_{sep}$, $\sigma Z_{\gamma\iota}-Z_{\gamma\iota}=t^{-\iota}W_{\gamma\iota}$ and $W_{\gamma\iota}:=\bar{W}_{\gamma\iota}\,\mathrm{mod}\,\bar{\mathcal{D}}(s_0+1)_k \in ((\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0+1)/\bar{\mathcal{D}}(s_0+1))_k$.

All above $Z_{\gamma\iota}$ come from elementary Artin-Schreier equations. Indeed, suppose $\{\omega_j\}$ is a (finite) $\mathbb{F}_p$-basis of $(\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0+1))/\bar{\mathcal{D}}(s_0+1)$. Then for some $w_{\gamma\iota j}\in k$, $W_{\gamma\iota}=\sum\limits_j w_{\gamma\iota j}\omega_j$ and $Z_{\gamma\iota}=\sum\limits_j z_{\gamma\iota j}\omega_j$, where $z_{\gamma\iota j}^p - z_{\gamma\iota j} = w_{\gamma\iota j}t^{-\iota}$. In particular, for any fixed ι (and γ), $\mathcal{K}(Z_{\gamma\iota})$ is a composit of all $\mathcal{K}(z_{\gamma\iota j})$. Therefore, $\mathcal{K}(Z_{\gamma\iota})/\mathcal{K}$ is an elementary abelian p-extension, which is either trivial or has only one (upper) ramification number $\iota p^{-v_p(\iota)}$.

This implies that

— $\mathcal{K}(Z_\gamma\,\mathrm{mod}\,\mathcal{I}_{sep})/\mathcal{K}$ coincides with the composite of all $\mathcal{K}(Z_{\gamma\iota}\,\mathrm{mod}\,(\mathcal{I}/\bar{\mathcal{D}}(s_0+1))_{sep})/\mathcal{K}$ (use that for different ι these extensions are linearly disjoint because by Prop. 2.8a) their ramification numbers are different).

In particular,

— if $W_{\gamma\iota}\notin\mathcal{I}_k$ then the field $\mathcal{K}(Z_{\gamma\iota}\,\mathrm{mod}\,(\bar{\mathcal{I}}/\bar{\mathcal{D}}(s_0+1))_{sep})/\mathcal{K}$ is a finite abelian p-extension with only one ramification number $\iota p^{-v_p(\iota)}$;

— by Prop.2.8a), the ramification numbers of different non-trivial extensions $\mathcal{K}(Z_{\gamma\iota}\,\mathrm{mod}\,(\mathcal{I}/\bar{\mathcal{L}}^*(s_0+1))_{sep})/\mathcal{K}$ are different.

As a result, the biggest upper ramification number of the field extension $\mathcal{K}(Z_\gamma\,\mathrm{mod}\,\mathcal{I}_{sep})/\mathcal{K}$ coincides with $\max\{\iota p^{-v_p(\iota)}\mid W_{\gamma\iota}\notin\mathcal{I}_k\}$.

By Prop. 2.8b), if $\iota\in\mathfrak{A}_1^0(p)$ then $\iota p^{-v_p(\iota)}\geqslant qv_0-b^*$. This implies that the biggest upper ramification number $v(\mathcal{K}(Z_\gamma\,\mathrm{mod}\,\mathcal{I}_{sep})/\mathcal{K})<qv_0-b^*$ if and only if all $W_{\gamma\iota}\in\mathcal{I}_k$, i.e. $\mathcal{I}=\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s_0+1)$.

Theorem 3.1 is completely proved.

4. Construction of explicit generators of $\bar{\mathcal{L}}[v_0]$

4.1. **Choice of** $e\in\mathcal{L}_{\mathcal{K}}$**.** In [1, 2, 3] we fixed the group isomorphism $\mathcal{G}_{<p}\simeq G(\mathcal{L})$ induced by the epimorphism $\eta_e=\pi_f(e):\mathcal{G}\longrightarrow G(\mathcal{L})$ via

a special choice of $e\in\mathcal{L}_{\mathcal{K}}$. In this paper we use more general element e by assuming that

$$\widetilde{\exp}\, e\equiv 1+\sum_{1\leqslant s<p}\eta(a_1,\dots,a_s)t^{-(a_1+\dots+a_s)}D_{a_10}\dots D_{a_s0}\operatorname{mod}\mathcal{J}_{\mathcal{K}}^p. \tag{4.1}$$

Here $\mathcal{J}$ is the augmentation ideal in the enveloping algebra $\mathcal{A}$ of $\mathcal{L}$. In the above sum the indices $a_1,\dots,a_s$ run over $\mathbb{Z}^0(p)$ and the "structural constants" $\eta(a_1,\dots,a_s)\in k$ satisfy the following identities:

$\mathrm{I_e}$) $\eta(a_1)=1$;

II_e) if $0\leqslant s_1\leqslant s<p$ then

$$\eta(a_1,\dots,a_{s_1})\eta(a_{s_1+1},\dots,a_s)=\sum_{\pi\in I_{s_1s}}\eta(a_{\pi(1)},\dots,a_{\pi(s)}),$$

where I_{s_1s} consists of all permutations π of order s such that the sequences $\pi^{-1}(1),\dots,\pi^{-1}(s_1)$ and $\pi^{-1}(s_1+1),\dots,\pi^{-1}(s)$ are increasing (i.e. I_{s_1s} is the set of all "insertions" of the ordered set $\{1,\dots,s_1\}$ into the ordered set $\{s_1+1,\dots,s\}$).

Assumption $\mathrm{I_e}$) means that e satisfies (1.1) from Sect.1.

Assumption II_e) means that

$$\Delta(\widetilde{\exp}(e))\equiv\widetilde{\exp}(e)\otimes\widetilde{\exp}(e)\operatorname{mod}(\mathcal{J}_{\mathcal{K}}\hat{\otimes}1+1\hat{\otimes}\mathcal{J}_{\mathcal{K}})^p,$$

i.e. $\widetilde{\exp}(e)$ is diagonal modulo degree p. This means that e is a k-linear combination of the commutators $t^{-(a_1+\dots+a_r)}[\dots[D_{a_10},\dots],D_{a_s0}]$. In particular, e satisfies the assumption (1.3) from Sect.1 and the compatibility (2.5) can be easily satisfied. Therefore, we can use Theorem 3.1 to obtain generators of the ramification ideal $\mathcal{L}^{(v_0)}$. Note that in most applications of the results from [1, 2, 3] we used the simplest choice $e=\sum_{a\in\mathbb{Z}^0(p)}t^{-a}D_{a0}$, where all $\eta(a_1,\dots,a_s)=1/s!$

4.2. **Statement of the main result.** For $\bar a=(a_1,\dots,a_s)$ with all $a_i\in\mathbb{Z}^0(p)$, set $\eta(\bar a)=\eta(a_1,\dots,a_s)$.

Definition. Let $\bar n=(n_1,\dots,n_s)$ with $s\geqslant 1$. Suppose there is a partition $0=i_0<i_1<\dots<i_r=s$ such that if $i_j<u\leqslant i_{j+1}$ then $n_u=m_{j+1}$ and $m_1>m_2>\dots>m_r$. Then set

$$\eta(\bar a,\bar n)_s=\sigma^{m_1}\eta(\bar a^{(1)})\dots\sigma^{m_r}\eta(\bar a^{(r)}),$$

where $\bar a^{(j)}=(a_{i_{j-1}+1},\dots,a_{i_j})$. If such a partition does not exist we set $\eta(\bar a,\bar n)_s=0$. (If there is no risk of confusion we just write $\eta(\bar a,\bar n)$ instead of $\eta(\bar a,\bar n)_s$.)

If $s=0$ we set $\eta(\bar a,\bar n)_s=1$.

For $\bar a=(a_1,\dots,a_s)$, $\bar n=(n_1,\dots,n_s)$, set $D_{(\bar a,\bar n)}=D_{a_1n_1}\dots D_{a_sn_s}$.

Note, if $e_{(N^*,0]}:=\sigma^{N^*-1}(e)\circ\sigma^{N^*-2}(e)\circ\dots\sigma(e)\circ e$ then

$$\widetilde{\exp}\, e_{(N^*,0]}\equiv\sum_{\bar a,\bar n}\eta(\bar a,\bar n)_sD_{(\bar a,\bar n)}\operatorname{mod}\mathcal{J}_{\mathcal{K}}^p.$$

For $\alpha \geqslant 0$ and $N \in \mathbb{Z}_{\geqslant 0}$, introduce $\mathcal{F}^0_{\alpha,-N} \in \mathcal{L}_k$ such that

$$\mathcal{F}^0_{\alpha,-N} = \sum_{\substack{1\leqslant s<p\\ a_i,n_i}} a_1\eta(\bar{a},\bar{n})[\dots[D_{a_1n_1}, D_{a_2n_2}],\dots,D_{a_sn_s}]\,.$$

Here:

— $\bar{a} = (a_1,\dots,a_s)$, $n_1 = 0$ and all $n_i \geqslant -N$;

— $\alpha = \gamma(\bar{a},\bar{n}) = a_1p^{n_1} + a_2p^{n_2} + \dots + a_sp^{n_s}$.

Note that non-zero terms in the above expression for $\mathcal{F}^0_{\alpha,-N}$ can appear only if $0 = n_1 \geqslant n_2 \geqslant \dots \geqslant n_s$ and $\alpha \in A[p-1, N]$.

Our result about explicit generators of $\bar{\mathcal{L}}[v_0]$ can be stated in the following form.

Let $\bar{\mathcal{F}}^0_{\alpha,-N}$ be the image of $\mathcal{F}^0_{\alpha,-N}$ in $\bar{\mathcal{L}}_k$.

If $\iota = qp^m\alpha - p^mb^* \in \mathfrak{A}^0_1$ is the standard presentation from Sect.2.2 we indicate the dependance of α and m on ι by setting $\alpha = \alpha[\iota]$ and $m = m[\iota]$. Recall that $\alpha[\iota] \in A[p-1, m[\iota]]$ and $m[\iota] < N^*$.

Let $m(\iota)$ be the maximal non-negative integer such that $\iota p^{m(\iota)} \leqslant (p-1)b^*$. For any $\iota \in \mathfrak{A}^0_1$, fix a choice of $m_\iota \geqslant m(\iota)$.

Theorem 4.1. *$\bar{\mathcal{L}}[v_0]$ is the minimal ideal in $\bar{\mathcal{L}}$ such that for all $\iota \in \mathfrak{A}^0_1$ with $\alpha[\iota] \geqslant 0$, $\bar{\mathcal{F}}^0_{\alpha[\iota],-(m[\iota]+m_\iota)} \in \bar{\mathcal{L}}[v_0]_k$.*

The proof is given in Sect. 4.3-4.6 below.

4.3. **Recurrent relation.** We are going to carry out computations in the enveloping algebra $\bar{\mathcal{A}}$ of the Lie algebra $\bar{\mathcal{L}}$. Note that the natural embedding $\bar{\mathcal{L}}_{\mathcal{K}} \subset \bar{\mathcal{A}}_{\mathcal{K}}$ remains still injective when taken modulo $\bar{\mathcal{J}}^p_{\mathcal{K}}$. This can be establshed similarly to the corresponding property for Lie $\mathbb{F}_p$-algebras from Sect.1.2.

Using universal properties of enveloping algebras obtain the following lemma. (We are going to use these properties slightly later.)

Lemma 4.2. *Suppose I is an ideal in the Lie algebra L of nilpotence class $< p$. Let A be an enveloping algebra of L with augmentation ideal J and $J_I := IA$ – the corresponding (two-sided) ideal in A. Then:*

a) $(J_I + J^p) \cap L = I$;

b) $(JJ_I + J_IJ + J^p) \cap L = [I, L]$.

Consider relation (2.5) and choose $\widetilde{e}^{\,sp} = \sum_\iota t^{-\iota}l^{sp}_\iota$ such that for all $\iota \in \mathfrak{A}^0(p)$ with $\mathrm{ch}(\iota) \geqslant 1$, $l^{sp}_\iota = D^\dagger_{\iota 0}$ if $\iota \in \mathfrak{A}^0(p)$ and $l^{sp}_\iota = 0$, otherwise. In other words, the part of $\widetilde{e}^{\,sp}$ which "disappears under $\mathcal{V}_{\mathcal{K}}$" coincides with $\sum\limits_{\mathrm{ch}(\iota)\geqslant 1} t^{-\iota}D^\dagger_{\iota 0}$.

Note that $\widetilde{\exp}\,(U * \bar{e}^{\,(q)}) \equiv \widetilde{\exp}\,\bar{e}^{\,(q)} + \bar{\mathcal{E}}U \bmod \bar{\mathcal{A}}_{\mathcal{K}}U^2$, where

$$\bar{\mathcal{E}} = \sum_{\substack{s\geqslant 1\\ a_i\in\mathbb{Z}^0(p)}} \eta(a_1,\dots,a_s)t^{-(a_1+\dots+a_s)q+b^*}(a_1+\dots+a_s)D_{a_10}\dots D_{a_s0}\,.$$

Apply $\widetilde{\exp}$ to (2.7) and find a lift $\bar{x}$ of $\tilde{x}$ to $\bar{\mathcal{N}}^{(q)}$ such that

$$\widetilde{\exp}\,\bar{e}^{(q)}+\bar{\mathcal{E}}U\equiv(1+U\sigma\bar{x})\left(\widetilde{\exp}\,\bar{e}^{(q)}+U\sum_{\mathrm{ch}(\iota)\geqslant 1}t^{-\iota}\mathcal{V}_kB_k^{\dagger}(D_{\iota 0}^{\dagger})\right)(1-U\bar{x})$$

modulo $\bar{\mathcal{J}}_{\mathcal{K}}^pU+\bar{\mathcal{A}}_{\mathcal{K}}U^2$ (proceed similarly to the proof of Prop.1.4). Comparing the coefficients for U and setting $\mathcal{V}_kB_k^{\dagger}(D_{\iota 0}^{\dagger})=V_{\iota 0}$ we obtain in $\bar{\mathcal{A}}_{\mathcal{K}}$ the following congruence modulo $\bar{\mathcal{J}}_{\mathcal{K}}^p$

$$\sigma\bar{x}-\bar{x}+\sum_{\iota}t^{-\iota}V_{\iota 0}\equiv\bar{\mathcal{E}}+(\widetilde{\exp}\,\bar{e}^{(q)}-1)\cdot\bar{x}-\sigma\bar{x}\cdot(\widetilde{\exp}\,\bar{e}^{(q)}-1)\,. \tag{4.2}$$

This equality gives a recurrent procedure to determine uniquely the elements $\bar{x}\in\sum\limits_{\mathrm{ch}(\iota)\geqslant 1}t^{-\iota}\bar{\mathcal{L}}_k+t^{(p-1)b^*}\bar{\mathcal{N}}^{(q)}$ and $V_{\iota 0}\in\bar{\mathcal{L}}_k$.

4.4. **Some combinatorial identities.** Let

$$-e_{[0,N^*)}:=(-e)\circ(-\sigma e)\circ\ldots\circ(-\sigma^{N^*-1}e)$$

and introduce the constants $\eta^o(\bar{a},\bar{n})\in k$ by the following congruence

$$\widetilde{\exp}(-e_{[0,N^*)})\equiv\sum\eta^o(\bar{a},\bar{n})D_{(\bar{a},\bar{n})}\,\mathrm{mod}\,\mathcal{J}_{\mathcal{K}}^p.$$

Set $\eta^o(\bar{a}):=\eta^o(\bar{a},\bar{0})$.

It can be easily seen that if there is a partition from the definition of η-constants in Sect.4.2 such that $m_1<m_2<\cdots<m_r$ then

$$\eta^o(\bar{a},\bar{n})_s=\sigma^{m_1}\eta^o(\bar{a}^{(1)})\cdot\sigma^{m_2}\eta^o(\bar{a}^{(2)})\cdot\ldots\cdot\sigma^{m_r}\eta^o(\bar{a}^{(r)})\,.$$

Otherwise, $\eta^o(\bar{a},\bar{n})_s=0$.

If there is no risk of confusion we just write $\eta(1,\ldots,s)$ instead of $\eta(\bar{a},\bar{n})_s$ and use the similar agreement for η^o. E.g. the equalities

$$e_{(N^*,0]}\circ(-e_{[0,N^*)})=e_{[0,N^*)}\circ(-e_{(N^*,0]})=0$$

can be written as the following identities

$$\sum_{0\leqslant s_1\leqslant s}\eta(1,\ldots,s_1)\eta^o(s_1+1,\ldots,s)=\sum_{0\leqslant s_1\leqslant s}\eta^o(1,\ldots,s_1)\eta(s_1+1,\ldots,s)=\delta_{0s} \tag{4.3}$$

(here δ_{0s} is the Kronecker symbol).

For $1\leqslant s_1\leqslant s<p$, consider the subset Φ_{ss_1} of permutations π of order s such that $\pi(1)=s_1$ and for any $1\leqslant l\leqslant s$, the subset $\{\pi(1),\ldots,\pi(l)\}$ of the segment $[1,s]$ is "connected", i.e. there exists $n(l)\in\mathbb{N}$ such that

$$\{\pi(1),\ldots,\pi(l)\}=\{n(l),n(l)+1,\ldots,n(l)+l-1\}.$$

By definition, we set $\Phi_{s0}=\Phi_{s,s+1}=\emptyset$.

Set $B_{s_1}(1,\ldots,s)=\sum_{\pi\in\Phi_{ss_1}}\eta(\pi(1),\ldots,\pi(s))$.

Note that:
— $B_0(1,\dots,s)=B_{s+1}(1,\dots,s)=0$;
— $B_1(1,\dots,s)=\eta(1,2,\dots,s)$;
— $B_s(1,\dots,s)=\eta(s,s-1,\dots,1)$.

Lemma 4.3. *Suppose* $0\leqslant s_1\leqslant s<p$. *Then:*
a) $B_{s_1}(1,\dots,s)+B_{s_1+1}(1,\dots,s)=\eta(s_1,\dots,1)\eta(s_1+1,\dots,s)$*;*
b) $\eta^o(1,\dots,s)=(-1)^s\eta(s,s-1,\dots,1)$*;*
c) *for indeterminates* $X_1,\dots,X_s$*,*

$$\sum_{\substack{1\leqslant s_1\leqslant s\\ \pi\in\Phi_{ss_1}}}(-1)^{s_1-1}X_{\pi^{-1}(1)}\dots X_{\pi^{-1}(s)}=[\dots[X_1,X_2],\dots,X_s].$$

Proof. a) Use that all insertions of $(s_1,...,1)$ into $(s_1+1,\dots s)$ are “connected” and start either with s_1 or s_1+1.

b) Clearly, part a) implies that

$$\sum_{0\leqslant s_1\leqslant s}(-1)^{s_1}\eta(s_1,\dots,1)\eta(s_1+1,\dots,s)=\delta_{0s}\,.$$

Then our statement follows from above relation (4.3).

c) Use that the right-hand side is a linear combintion of the monomials $X_{i_1}\dots X_{i_s}$ such that for any $l\geqslant 1$, $\{j\mid i_j\in[1,l]\}$ is a “connected” segment of consecutive l integers. ∎

4.5. **Lie elements $\bar{\mathcal{F}}[\iota]$ and $\bar{\mathcal{F}}[\iota]_0$.** Introduce the following notation:

— $\bar n=(n_1,\dots,n_s)\geqslant M$ means that all $n_i\geqslant M$. Similarly, we interpret $\bar n>M$, $\bar n\leqslant M$ and $\bar n<M$.

— $\gamma(\bar a,\bar n)=a_1p^{n_1}+\dots+a_sp^{n_s}$.

For $1\leqslant s_1\leqslant s$, let $\gamma^*_{[s_1,s]}(\bar a,\bar n)=\sum\limits_{s_1\leqslant u\leqslant s}a_up^{n_u}$ where $n_u^*=0$ if $n_u=M(\bar n):=\max\{n_1,\dots,n_s\}$ and $n_u^*=-\infty$ (i.e. $p^{n_u^*}=0$), otherwise.

For $\iota\in\mathfrak{A}_1^0$, introduce

$$\bar{\mathcal{F}}[\iota]=\sum_{(\bar a,\bar n)}\sum_{1\leqslant s_1\leqslant s}\eta^o(1,\dots,s_1-1)\eta(s_1,\dots,s)\gamma^*_{[s_1,s]}(\bar a,\bar n)D_{(\bar a,\bar n)}\in\bar{\mathcal{A}}_k\,.$$

Here the first sum is taken over all $(\bar a,\bar n)$ of lengths $1\leqslant s<p$ such that $\bar n\geqslant 0$ and $\gamma(\bar a,\bar n)-p^{M(\bar n)}b^*=\iota$. Note that $M(\bar n)$ depends only on ι and, therefore, all non-zero summands in $\bar{\mathcal{F}}[\iota]$ depend on $(\bar a,\bar n)$ with the same $M(\bar n)$.

Let $\bar{\mathcal{F}}[\iota]_0$ be a part of the above sum taken under the condition $m(\bar n):=\min\{n_1,\dots,n_s\}=0$. Then for any $\iota\in\mathfrak{A}_1^0$ and $m\geqslant 0$,

$$\sigma^m\bar{\mathcal{F}}[\iota]_0+\sigma^{m-1}\bar{\mathcal{F}}[\iota p]_0+\dots+\bar{\mathcal{F}}[\iota p^m]_0=\bar{\mathcal{F}}[\iota p^m]\,.$$

In particular, $\bar{\mathcal{F}}[\iota]=\sum_{\iota',m}\sigma^m\bar{\mathcal{F}}[\iota']_0$ where the sum is taken over all $\iota'\in\mathfrak{A}_1^0$ and $m\geqslant 0$ such that $\iota'p^m=\iota$.

Proposition 4.4. *If $\iota = qp^m\alpha - p^m b^* \in \mathfrak{A}_1^0$ (standard notation) then $\bar{\mathcal{F}}[\iota p^n] = \sigma^{m+n}\bar{\mathcal{F}}^0_{\alpha,-(m+n)}$.*

Proof. We have

$$\sigma^{-(m+n)}\bar{\mathcal{F}}[\iota p^n] = \sum_{\substack{1\leqslant s_1\leqslant s<p\\ (\bar a,\bar n)}} \eta^o(1,\dots,s_1-1)\eta(s_1,\dots,s)\gamma^*_{[s_1,s]}(\bar a,\bar n)D_{(\bar a,\bar n)}\,,$$

where the sum is taken for $(\bar a,\bar n)$ with $M(\bar n)=0$, $\bar n\geqslant -(m+n)$ and $\gamma(\bar a,\bar n)=\alpha$. By Lemma 4.3, $\eta^o(1,\dots,s_1-1)=(-1)^{s_1-1}\eta(s_1-1,\dots,1)$, and we obtain

$$\sum_{\substack{1\leqslant s_1\leqslant s<p\\ (\bar a,\bar n)}} (-1)^{s_1-1}(B_{s_1-1}(1,\dots,s)+B_{s_1}(1,\dots,s))\gamma^*_{[s_1,s]}(\bar a,\bar n)D_{(\bar a\bar n)} =$$

$$\sum_{\substack{1\leqslant s_1\leqslant s<p\\ (\bar a,\bar n)}} (-1)^{s_1-1}B_{s_1}(1,\dots,s)(\gamma^*_{[s_1,s]}(\bar a,\bar n)-\gamma^*_{[s_1+1,s]}(\bar a,\bar n))D_{(\bar a,\bar n)} =$$

$$\sum_{\substack{1\leqslant s<p\\ (\bar a,\bar n)}}\sum_{1\leqslant s_1\leqslant s} (-1)^{s_1-1}B_{s_1}(1,\dots,s)a_{s_1}p^{n^*_{s_1}}D_{(\bar a\bar n)} =$$

$$\sum_{\substack{1\leqslant s<p\\ (\bar a,\bar n)}}\sum_{\substack{1\leqslant s_1\leqslant s\\ \pi\in\Phi_{ss_1}}} (-1)^{s_1-1}\eta(\pi(1),\dots,\pi(s))a_{s_1}p^{n^*_{s_1}}D_{(\bar a,\bar n)} =$$

$$\sum_{\substack{1\leqslant s<p\\ (\bar a,\bar n)}}\eta(1,\dots,s)a_1\sum_{\substack{1\leqslant s_1\leqslant s\\ \pi\in\Phi_{ss_1}}} (-1)^{s_1-1}D_{a_{\pi^{-1}(1)}n_{\pi^{-1}(1)}}\dots D_{a_{\pi^{-1}(s)}n_{\pi^{-1}(s)}} =$$

$$\sum_{\substack{1\leqslant s<p\\ (\bar a,\bar n)}}\eta(1,\dots,s)a_1[\dots[D_{a_1n_1},D_{a_2n_2}],\dots,D_{a_sn_s}] = \bar{\mathcal{F}}^0_{\alpha,-(m+n)}\,.$$

The proposition is proved. □

Corollary 4.5. *All $\bar{\mathcal{F}}[\iota p^m]$ and $\bar{\mathcal{F}}[\iota p^m]_0$ belong to $\bar{\mathcal{L}}_k$.*

4.6. **Solving recurrent relation** (4.2)**.** For $\iota\in\mathfrak{A}^0$, let

— $m(\iota):=\max\{m\mid \iota p^m\in\mathfrak{A}^0\}(=\max\{m\mid |\iota p^m|\leqslant(p-1)b^*\})$.

— $\mathfrak{A}_1^{\mathrm{prim}}=\mathfrak{A}_1^0\setminus p\mathfrak{A}_1^0$ (note that $\mathfrak{A}_1^0(p)=\{\iota\in\mathfrak{A}_1^{\mathrm{prim}}\mid \iota>0\}$).

As earlier, set $\bar{\mathcal{D}}:=[\bar{\mathcal{L}}[v_0],\bar{\mathcal{L}}]$, $\bar{\mathcal{L}}[v_0](s):=\bar{\mathcal{L}}[v_0]+\bar{\mathcal{L}}(s)$ and $\bar{\mathcal{D}}(s):=\bar{\mathcal{D}}+\bar{\mathcal{L}}(s)$. Clearly, $\bar{\mathcal{L}}[v_0]=\bar{\mathcal{L}}[v_0](p)$ and $\bar{\mathcal{D}}=\bar{\mathcal{D}}(p)$.

Proposition 4.6. a) $\bar x\equiv\sum\limits_{\iota,m}\bar{\mathcal{F}}[\iota p^m]t^{-\iota p^m}\bmod\bar{\mathcal{L}}[v_0]_{\mathcal{K}}$, *where the sum is taken over all $i\in\mathfrak{A}_1^{\mathrm{prim}}$ and $m\geqslant 0$;*

b) *if $\iota\in\mathfrak{A}_1^0(p)$, then $V_{\iota 0}\equiv-\sigma^{-m(\iota)}\bar{\mathcal{F}}[\iota p^{m(\iota)}]\bmod\bar{\mathcal{D}}_k$.*

Proof. Apply induction on $1 \leqslant s_0 < p$ by assuming that a) holds modulo $\bar{\mathcal{L}}[v_0](s_0)_{\mathcal{K}}$ and deducing from this that a) and b) hold modulo the ideals $\bar{\mathcal{L}}[v_0](s_0+1)_{\mathcal{K}}$ and, resp., $\bar{\mathcal{D}}(s_0+1)_k$.

Clearly, a) holds modulo $\bar{\mathcal{L}}[v_0](1)_{\mathcal{K}} = \bar{\mathcal{L}}_{\mathcal{K}}$.

Suppose $1 \leqslant s_0 < p$ and part a) holds modulo $\bar{\mathcal{L}}[v_0](s_0)_{\mathcal{K}}$. Applying this assumption to the right-hand side of (4.2) we obtain (use (4.3)) that

$$\sigma\bar{x} - \bar{x} + \sum_{\iota} t^{-\iota} V_{\iota 0} \equiv -\sum_{\iota,m} \bar{\mathcal{F}}[\iota p^m]_0 t^{-\iota p^m} \tag{4.4}$$

modulo $(\bar{\mathcal{J}}\bar{\mathcal{J}}_{\bar{\mathcal{L}}[v_0](s_0)} + \bar{\mathcal{J}}_{\bar{\mathcal{L}}[v_0](s_0)}\bar{\mathcal{J}} + \bar{\mathcal{J}}^p)_{\mathcal{K}}$, cf. notation from Lemma 4.2. (Here the right-hand sum is taken over all $\iota \in \mathfrak{A}_1^{\mathrm{prim}}$ and $m \geqslant 0$.)

Since the both parts of congruence (4.4) belong to $\bar{\mathcal{L}}_{\mathcal{K}}$, part b) of Lemma 4.2 implies that (4.4) holds modulo $[\bar{\mathcal{L}}[v_0](s_0), \bar{\mathcal{L}}]_{\mathcal{K}} = \bar{\mathcal{D}}(s_0+1)_{\mathcal{K}}$.

Remark. Since $\bar{x}, \sigma\bar{x} \in \bar{\mathcal{N}}^{(q)}$ relation (4.4) implies that $\bar{\mathcal{F}}[\iota p^m]_0 \in \bar{\mathcal{D}}(s_0+1)_k = \bar{\mathcal{D}}_k + \bar{\mathcal{L}}(s_0+1)_k$ if $\iota p^m > s_0 b^*$.

Apply the operators $\mathcal{S}$ and $\mathcal{R}$ from Lemma 2.11 to recover the elements $\sum_{i\in\mathfrak{A}^0(p)} t^{-\iota} V_{\iota 0}$ and $\bar{x}$ modulo $\bar{\mathcal{D}}(s_0+1)_{\mathcal{K}}$ as follows.

Let $\bar{x} = \bar{x}^+ + \bar{x}^-$, where $\bar{x}^+$ (resp., $\bar{x}^-$) is the linear combination of elements of $\bar{\mathcal{L}}_k$ with positive (resp., negative) powers of t.

If $\iota < 0$ then $\mathcal{S}(\bar{\mathcal{F}}[\iota p^m]_0 t^{-\iota p^m}) = -\sum\limits_{n\geqslant 0} \sigma^n \bar{\mathcal{F}}[p^m \iota]_0 t^{-\iota p^{n+m}}$ and, therefore, $\bar{x}^+ \equiv \sum\limits_{\iota,m} \bar{\mathcal{F}}[\iota p^m] t^{-\iota p^m} \operatorname{mod} \bar{\mathcal{D}}(s_0+1)_{\mathcal{K}}$, where the sum is taken over all $\iota \in \mathfrak{A}_1^{\mathrm{prim}} \setminus \mathfrak{A}_1^0(p)$ and $m \geqslant 0$. This gives part a) modulo $\bar{\mathcal{L}}[v_0](s_0+1)_{\mathcal{K}} \supset \bar{\mathcal{D}}(s_0+1)_{\mathcal{K}}$ at the level of positive powers of t.

Let $\iota \in \mathfrak{A}_1^0(p)$. Then $\mathcal{R}(t^{-\iota p^m} \bar{\mathcal{F}}[\iota p^m]_0) = t^{-\iota}\sigma^{-m}\bar{\mathcal{F}}[\iota p^m]_0$ and

$$V_{\iota 0} t^{-\iota} \equiv -t^{-\iota} \sum_{0\leqslant m\leqslant m(\iota)} \sigma^{-m} \bar{\mathcal{F}}[\iota p^m]_0 \equiv -t^{-\iota}\sigma^{-m(\iota)} \bar{\mathcal{F}}[\iota p^{m(\iota)}]$$

modulo $\bar{\mathcal{D}}(s_0+1)_{\mathcal{K}}$. This gives part b).

As a result, we have the following congruences modulo $\bar{\mathcal{L}}[v_0](s_0+1)_{\mathcal{K}}$:

$$\mathcal{S}(\bar{x}^-) \equiv -\sum_{\iota,m} \mathcal{S}(t^{-\iota p^m} \bar{\mathcal{F}}[\iota p^m]_0) \equiv -\sum_{\iota,m} \sum_{0\leqslant m_1 < m} t^{-\iota p^{m_1}} \sigma^{-m+m_1} \bar{\mathcal{F}}[\iota p^m]_0$$

$$\equiv -\sum_{\iota,m} t^{-\iota p^m} \sum_{m_1 > m} \sigma^{-m_1+m} \bar{\mathcal{F}}[\iota p^{m_1}]_0 \equiv -\sum_{\iota,m} t^{-\iota p^m} \sum_{\substack{m_1+m_2=m\\ m_2<0}} \sigma^{m_2} \bar{\mathcal{F}}[\iota p^{m_1}]_0$$

$$\equiv \sum_{\iota,m} t^{-\iota p^m} \sum_{\substack{m_1+m_2=m\\ m_1,m_2\geqslant 0}} \sigma^{m_2} \bar{\mathcal{F}}[\iota p^{m_1}]_0 = \sum_{\iota,m} \bar{\mathcal{F}}[\iota p^m] t^{-\iota p^m}$$

(here ι and m run over $\mathfrak{A}_1^0(p)$ and, resp., $\mathbb{Z}_{\geqslant 0}$) because

$$\sum_{m_2+m_1=m} \sigma^{m_2}\bar{\mathcal{F}}[\iota p^{m_1}]_0 = \bar{\mathcal{F}}[\iota p^m] \equiv \sigma^{m-m(\iota)}V_\iota \operatorname{mod} \bar{\mathcal{L}}[v_0](s_0+1)_k\,.$$

This completes the induction step for part a). □

Corollary 4.7. *$\bar{\mathcal{L}}[v_0]$ is the minimal ideal in $\bar{\mathcal{L}}$ such that $\bar{\mathcal{L}}[v_0]_k$ contains all $\bar{\mathcal{F}}[\iota p^{m_\iota}]$.*

Proof. If $m > m(\iota)$ then $\iota p^m > (p-1)b^*$ and by remark in the proof of Prop. 4.6, $\bar{\mathcal{F}}[\iota p^m]_0 \in \bar{\mathcal{D}}_k$. Therefore, $\bar{\mathcal{F}}[\iota p^{m_\iota}] \equiv \bar{\mathcal{F}}[\iota p^{m(\iota)}] \operatorname{mod} \bar{\mathcal{D}}_k$. □

Theorem 4.1 is completely proved.

5. Effective choice of parameters

Theorem 4.1 gives explicit description of the ramification ideal $\mathcal{L}^{(v_0)}$ but this description depends on a choice of parameters δ_0, b^* and $q = p^{N^*}$ involved into the construction of the module of auxilliary coefficients $\mathfrak{A}^0$. More precisely, the corresponding generating elements $\mathcal{F}^0_{\alpha[\iota],-(m[\iota]+m(\iota))}$ of the ideal $\mathcal{L}_k^{(v_0)}$ depend on the rational numbers $\alpha[\iota] \geqslant v_0$ and the integers $m[\iota]$ coming from the appropriate $\iota \in \mathfrak{A}_1^0(p)$. The values of δ_0, b^* and N^* can be specified in a sufficiently constructive way directly from their definitions but it is highly unlikely that this could be done in a more or less optimal way. The reason is that a choice of δ_0, b^* and N^* depends on the whole set $\mathfrak{A}^0$ but the construction of generators uses only the subset $\mathfrak{A}_1^0(p) \subset \mathfrak{A}^0$.

As we have noticed in the Introduction, an analogue of Theor. 4.1 was obtained in [1] by different methods, and was stated in the terms of generators $\mathcal{F}^0_{\alpha,-N}$ with arbitrary rationals $\alpha \geqslant v_0$ and a boundary value $\widetilde{N}(v_0)$ such that $N \geqslant \widetilde{N}(v_0)$. In Sect. 5.1 we deduce this analogue from Theor. 4.1 and prove that it holds with the boundary value $\widetilde{N}(v_0) = N^* - 1$.

It should be noticed that if $\widetilde{N}(v_0)$ is (unreasonably) growing then the number of dependent generators among $\mathcal{F}^0_{\alpha,-N}$, $\alpha \geqslant v_0$, also grows. As a result, the description of the ideal $\mathcal{L}^{(v_0)}$ is getting more and more complicated. In Sect. 5.2 we use the left-continuity property of ramification filtration to provide “flexible” boundaries $\widetilde{N}(v_0, \alpha)$ depending on the parameter α. This allows us to obtain more effective description of the whole filtration $\{\mathcal{L}^{(v)}\}_{v\geqslant 1}$ under the condition that the set of its jumps is known.

5.1. **Relation to the main result of** [1]**.**

In [1] (cf. also [3]) we proved the following theorem.

Theorem 5.1. *There is $\widetilde{N}(v_0) \in \mathbb{N}$ such that if $N \geqslant \widetilde{N}(v_0)$ is fixed then $\mathcal{L}^{(v_0)}$ is the minimal ideal in $\mathcal{L}$ such that for all $\alpha \geqslant v_0$, $\mathcal{F}^0_{\alpha,-N} \in \mathcal{L}_k^{(v_0)}$.*

Proposition 5.2. *Theorem 5.1 holds with* $\widetilde{N}(v_0) = N^* - 1$.

Proof. Let $\mathcal{L}^{\star}_N[v_0]$ be the minimal ideal in $\mathcal{L}$ such that for all $\alpha \geqslant v_0$, $\mathcal{F}^0_{\alpha,-N} \in \mathcal{L}^{\star}_N[v_0]_k$. We should prove that for $N \geqslant N^*-1$, $\mathcal{L}^{\star}_N[v_0] = \mathcal{L}^{(v_0)}$.

Use induction on $s \geqslant 1$ to deduce (use that $\mathcal{F}^0_{a,-N} \in \mathcal{L}^{\star}_N[v_0]_k$) for $a \geqslant sv_0$ that $D_{a0} \in \mathcal{L}^{\star}_N[v_0]_k + \mathcal{L}(s+1)_k$, e.g. cf. Lemma 3.5 or Lemma 2.3 from [9]. This implies that $\mathcal{L}(p) \subset \mathcal{L}^{\star}_N[v_0]$.

Denote by $\bar{\mathcal{L}}^{\star}_N[v_0]$ the image of $\mathcal{L}^{\star}_N[v_0]$ in $\bar{\mathcal{L}} = \mathcal{L}/\mathcal{L}(p)$.

It follows from Theor. 4.1 that $\bar{\mathcal{L}}[v_0]$ is already the minimal ideal in $\bar{\mathcal{L}}$ such that $\{\mathcal{F}^0_{\alpha[\iota],-N} \mid \iota \in \mathfrak{A}^0_1(p)\} \subset \bar{\mathcal{L}}[v_0]_k$ (use that m_ι could be chosen such that $m[\iota] + m(\iota) = N \geqslant N^* - 1$). Therefore, it remains to show that for any $\alpha \geqslant v_0$, it holds $\bar{\mathcal{F}}^0_{\alpha,-N} \in \bar{\mathcal{L}}[v_0]$.

Note that $\bar{\mathcal{F}}^0_{\alpha,-N} \neq 0$ implies that $\alpha \in A[p-1, N]$. Then by Prop. 2.5, $p^{N+1}(q\alpha - b^*) \geqslant q(q\alpha - (q-1)r^*) > (p-1)b^*$. Therefore, our proposition is implied by the following lemma. ∎

Lemma 5.3. *Suppose* $M \geqslant 0$ *and* $p^{M+1}(q\alpha - b^*) > (p-1)b^*$. *Then*

a) $\bar{\mathcal{F}}^0_{\alpha,-M} \in \bar{\mathcal{L}}[v_0]_k$;

b) *if in addition* $p^M(q\alpha - b^*) > (p-1)b^*$ *then*

$$\bar{\mathcal{F}}^0_{\alpha,-(M-1)} \equiv \bar{\mathcal{F}}^0_{\alpha,-M} \bmod [\bar{\mathcal{L}}[v_0], \bar{\mathcal{L}}]_k\,.$$

Proof of lemma. Apply induction on $M \geqslant 0$.

Let $M = 0$, i.e. $p(q\alpha - b^*) > (p-1)b^*$. Here $\bar{\mathcal{F}}^0_{\alpha,-M} = \bar{\mathcal{F}}^0_{\alpha,0}$ is a k-linear combination of the commutators

$$[\dots[D_{a'_1 0}, D_{a'_2 0}], \dots, D_{a'_r 0}]\,,$$

where $a'_1 + \dots + a'_r = \alpha$. Consider two cases:

(i) If $q\alpha - b^* > (p-1)b^*$ then $\alpha > pb^*/q > (p-1)(v_0 - \delta_0)$ and this implies $\alpha \geqslant (p-1)v_0$, cf. Lemma 2.6 a). Therefore, the above commutators belong to $\bar{\mathcal{L}}(p) = 0$. (Indeed, if $(s_i - 1)v_0 \leqslant a'_i < s_i v_0$ then $\sum_i \mathrm{wt}(D_{a'_i 0}) = \sum_i s_i > \alpha/v_0 \geqslant p-1$.)

(ii) If $q\alpha - b^* \leqslant (p-1)b^*$ then $\iota := q\alpha - b^* \in \mathfrak{A}^0_1(p)$ and $m(\iota) = 0$. Then Theor. 4.1 implies $\bar{\mathcal{F}}^0_{\alpha,0} \in \bar{\mathcal{L}}[v_0]_k$.

Suppose $M \geqslant 1$. We have the following two cases:

(i) If $qp^M\alpha - p^M b^* \leqslant (p-1)b^*$ then there is $\iota \in \mathfrak{A}^0_1(p)$ and $n \geqslant 0$ such that $\iota p^n = qp^M\alpha - p^M b^* \in \mathfrak{A}^0_1$ and, therefore, $m(\iota) = n \leqslant M$. Then by Theorem 4.1 with $m_\iota = M - n$ we obtain $\bar{\mathcal{F}}^0_{\alpha,-M} = \sigma^{-M}\bar{\mathcal{F}}[\iota p^n] \in \bar{\mathcal{L}}[v_0]_k$.

(ii) Suppose now that $p^M(q\alpha - b^*) > (p-1)b^*$. Prove simultaneously the remaining case of a) and the statement b).

By the induction assumption $\bar{\mathcal{F}}^0_{\alpha,-(M-1)} \in \bar{\mathcal{L}}[v_0]_k$.

Note that $\bar{\mathcal{F}}^0_{\alpha,-M} - \bar{\mathcal{F}}^0_{\alpha,-(M-1)}$ is a linear combination of the terms of the form

$$[\dots[\bar{\mathcal{F}}^0_{\alpha',-(M-1)}, D_{a'_1,-M}], \dots, D_{a'_r,-M}]\,, \tag{5.1}$$

where $\alpha=\alpha'+(a'_1+\dots+a'_r)/p^M$, $r\geqslant 1$ and $\alpha'\in A[p-1,M-1]$.

It remains to prove that (5.1) belongs to $[\bar{\mathcal{L}}[v_0],\bar{\mathcal{L}}]_k$.

Let $s\in\mathbb{N}$ be such that $sb^*/q>a'_1+\ldots+a'_r\geqslant (s-1)b^*/q$.

Then $a'_1+\ldots+a'_r\geqslant (s-1)v_0$, cf. Sect.2.1, and $\sum\limits_i \mathrm{wt}(D_{a'_i,-M})\geqslant s$. (If $(s_i-1)v_0\leqslant a'_i<s_iv_0$ then $\sum\limits_i s_i>(a'_1+\ldots+a'_r)/v_0\geqslant s-1$.) We can assume that $s\leqslant p-2$ because, otherwise, (5.1) belongs to $\bar{\mathcal{L}}(p)_k=0$. Now the inequality $a'_1+\ldots+a'_r<sb^*/q$ implies

$$(p-1)b^*/p^M<q\alpha-b^*\leqslant q\alpha'-b^*+sb^*/p^M$$

and, therefore,

$$p^M(q\alpha'-b^*)>(p-1-s)b^*\,. \tag{5.2}$$

If $p^M(q\alpha'-b^*)>(p-1)b^*$ then by the induction assumption we have $\bar{\mathcal{F}}^0_{\alpha',-(M-1)}\in\bar{\mathcal{L}}[v_0]_k$ and (5.1) belongs to $[\bar{\mathcal{L}}[v_0],\bar{\mathcal{L}}]_k$.

If $p^M(q\alpha'-b^*)\leqslant(p-1)b^*$ then $\iota':=p^M(q\alpha'-b^*)\in\mathfrak{A}^0_1$, $m(\iota')=0$ (use that $\iota'>b^*$) and, therefore, $\bar{\mathcal{F}}[\iota']\in\bar{\mathcal{L}}[v_0]_k$.
Then from inequality (5.2) and Remark from Sect. 4.6 it follows that

$$\bar{\mathcal{F}}[\iota']_0\in[\bar{\mathcal{L}}[v_0],\bar{\mathcal{L}}]_k+\bar{\mathcal{L}}(p-s)_k\,.$$

Note that $\alpha'\in A[p-1,M-1]$ implies that $p^M\alpha'\equiv 0\operatorname{mod}p$ and, therefore, $\iota'/p\in\mathfrak{A}^0_1$. Now from the the identity $\bar{\mathcal{F}}[\iota']=\bar{\mathcal{F}}[\iota']_0+\sigma\bar{\mathcal{F}}[\iota'/p]$ we deduce that

$$\bar{\mathcal{F}}[\iota'/p]\in\bar{\mathcal{L}}[v_0]_k+\bar{\mathcal{L}}(p-s)_k\,,$$

and from $\sigma^{M-1}\bar{\mathcal{F}}^0_{\alpha',-(M-1)}=\bar{\mathcal{F}}[\iota'/p]$ it follows that

$$\bar{\mathcal{F}}^0_{\alpha',-(M-1)}\in\bar{\mathcal{L}}[v_0]_k+\bar{\mathcal{L}}(p-s)_k\,.$$

Finally, the commutator $[\ldots[\bar{\mathcal{L}}(p-s)_k,D_{a'_1,-M}],\ldots,D_{a'_r,-M}]\subset\bar{\mathcal{L}}(p)=0$ because $\sum\limits_i\mathrm{wt}(D_{a'_i,-M})\geqslant s$. As a result, (5.1) belongs to $[\bar{\mathcal{L}}[v_0],\bar{\mathcal{L}}]_k$.

The lemma is proved. □

Remark. If in the above notation $p^M(q\alpha-b^*)>(p-1)b^*$, then

$$\mathcal{F}^0_{\alpha,-(M-1)}\equiv\mathcal{F}^0_{\alpha,-M}\operatorname{mod}[\mathcal{L}[v_0],\mathcal{L}]_k\,.$$

Indeed, since $\mathcal{F}^0_{\alpha,-M}$ and $\mathcal{F}^0_{\alpha,-M+1}$ have the same linear term the part b) of the above lemma implies that

$$\mathcal{F}^0_{\alpha,-(M-1)}-\mathcal{F}^0_{\alpha,-M}\in\mathcal{L}(p)_k\cap C_2(\mathcal{L}_k)+[\mathcal{L}[v_0],\mathcal{L}]_k\,.$$

It remains to note that Lemma 3.5 implies $\mathcal{L}(p)\cap C_2(\mathcal{L})\subset[\mathcal{L}^{(v_0)},\mathcal{L}]$.

5.2. **Flexible boundaries.** Suppose $v \geqslant 1$. Introduce the weight function wt_v on $\mathcal{L}_k$ such that $\mathrm{wt}_v(D_{an}) = s \in \mathbb{N}$ iff $(s-1)v \leqslant v < sv$. Denote by $\mathcal{L}_v(p)$ the ideal of elements with wt_v-weight $\geqslant p$. Note that in notation from Sect.1.6 $\mathrm{wt} = \mathrm{wt}_{v_0}$.

Introduce another weight function wt_v^+ on $\mathcal{L}_k$ such that $\mathrm{wt}^+(D_0) = 1$ and for $a \in \mathbb{Z}^+(p)$, $\mathrm{wt}_v^+(D_{an} = s$ iff $(s-1)v < a \leqslant sv$. Denote by $\mathcal{L}_v^+(p)$ the ideal of elements with wt_v^+-weight $\geqslant p$.

Clearly, we have the following property:

Proposition 5.4. $\mathcal{L}_v^+(p) = \cup_{v'>v}\mathcal{L}_{v'}(p)$.

Suppose $v > 1$ and $v^\flat \in [1, v)$ is such that for any $v' \in (v^\flat, v]$, we have $\mathcal{G}_{<p}^{(v')} = \mathcal{G}_{<p}^{(v)}$. The existence of $v^\flat$ follows from the left-continuity property of ramification filtration.

Remark. There is the following upper estimate for $v^\flat$. Let $\mathfrak{B}$ be the set of all $a_1 + a_2p^{-n_2} + \dots a_{p-1}p^{-n_{p-1}} < v$ with $a_i \in \mathbb{Z}^+(p) \cap [1, (p-1)v)$ and $n_i \geqslant 0$. Let $\delta_0(1) = \min\{v - b \mid b \in \mathfrak{B}\}$, cf. Sect. 2.1. Then $v^\flat \leqslant v - \delta_0(1)$. This follows easily from Theorem 5.1 because if $\alpha \notin \mathfrak{B}$ and $\alpha < v$ then $\mathcal{F}^0_{\alpha,-N} = 0$ and, therefore, the set $\mathfrak{B}$ contains all possible breaks of the filtration $\{\mathcal{G}_{<p}^{(v')}\}_{1 \leqslant v' < v}$.

For any $\alpha > v^\flat$ choose $N_\alpha \geqslant 0$ such that

$$p^{N_\alpha+1}(\alpha - v^\flat) > (p-1)v^\flat\,.$$

There is the following more effective version of Theor. 5.1.

Theorem 5.5. *$\mathcal{L}^{(v)}$ is the minimal ideal in $\mathcal{L}$ such that for all $\alpha \geqslant v$, $\mathcal{F}^0_{\alpha,-N_\alpha} \in \mathcal{L}_k^{(v)}$.*

Proof. Apply Theorem 5.1 with $v_0 = v$ by choosing $N \geqslant \widetilde{N}(v)$ such that for all $\alpha \geqslant v$, $p^{N+1}(\alpha - v^\flat) > (p-1)v^\flat$. Then $\mathcal{L}^{(v)}$ is the minimal ideal in $\mathcal{L}$ such that $\mathcal{L}_k^{(v)}$ contains all $\mathcal{F}^0_{\alpha,-N}$ with $\alpha \geqslant v$.

Fix $\alpha \geqslant v$ and choose $v_\alpha \in (v^\flat, v)$ such that we still have the inequalities $p^{N_\alpha+1}(\alpha - v_\alpha) > (p-1)v_\alpha$ and $p^{N+1}(\alpha - v_\alpha) > (p-1)v_\alpha$

Let b_α^* and q_α be analogs of the parameters b^* and q chosen in Sect.2.1 when v_0 is replaced by v_α.

Then the inequality $p^{M+1}(q_\alpha\alpha - b_\alpha^*) > (p-1)b_\alpha^*$ from Lemma 5.3 holds with $M = N$ and $M = N_\alpha$ (use that $b_\alpha^*/q_\alpha < v_\alpha$). Therefore, by remark from the end of Sect. 5.1 we have

$$\mathcal{F}^0_{\alpha,-N} \equiv \mathcal{F}^0_{\alpha,-N_\alpha} \operatorname{mod} [\mathcal{L}^{(v)}, \mathcal{L}]_k\,.$$

This means that the conditions $\mathcal{F}^0_{\alpha,-N} \in \mathcal{L}_k^{(v)}$ and $\mathcal{F}^0_{\alpha,-N_\alpha} \in \mathcal{L}_k^{(v)}$ are equivalent. Theorem is proved. □

5.3. **The whole filtration $\{\mathcal{G}_{<p}^{(v)}\mid v\geqslant 1\}$.**

Suppose $1=v_1<v_2<\ldots<v_r<\ldots$ are all jumps of the ramification filtration $\{\mathcal{G}_{<p}^{(v)}\}_{v\geqslant 1}$. (This set is obviously discrete.) In other words,

– $\mathcal{G}_{<p}^{(v_1)}\supsetneqq\ldots\supsetneqq\mathcal{G}_{<p}^{(v_r)}\supsetneqq\ldots$;

– $\mathcal{G}_{<p}^{(1)}$ is the ramification subgroup in $\mathcal{G}_{<p}$, $(\mathcal{G}_{<p}:\mathcal{G}_{<p}^{(1)})=p$;

– if $r\geqslant 2$ and $v_{r-1}<v\leqslant v_r$ then $\mathcal{G}_{<p}^{(v)}=\mathcal{G}_{<p}^{(v_r)}$.

Use the identification $\mathcal{G}_{<p}\simeq G(\mathcal{L})$ from Sect. 1. Then the ramification filtration appears as ideals $\mathcal{L}^{(v_1)}\supsetneqq\ldots\supsetneqq\mathcal{L}^{(v_r)}\supsetneqq\ldots$ in $\mathcal{L}$, where $\mathcal{L}_k^{(1)}$ is generated by all D_{an}, $a\in\mathbb{Z}^+(p)$.

Suppose $u\geqslant 2$.

Introduce the weight function wt_u on $\mathcal{L}_k$ such that $\mathrm{wt}_u(D_0)=1$ and if $s\in\mathbb{N}$ is such that $(s-1)v_{u-1}<a\leqslant sv_{u-1}$ then $\mathrm{wt}_u(D_{an})=s$.

Introduce also the elements $\mathcal{F}^*[u]\in\mathcal{L}_k$ obtained from the elements $\mathcal{F}^0_{v_u,-M_u}$, cf. Sect. 4.2, where

$$p^{M_u+1}(v_u-v_{u-1})>(p-1)v_{u-1}$$

by imposing additional restriction $\mathrm{wt}_u(D_{a_1n_1})+\ldots+\mathrm{wt}_u(D_{a_sn_s})\leqslant p-1$ if $s\geqslant 2$. Clearly,

$$\mathcal{F}^*[u]\equiv\mathcal{F}^0_{v_u,-M_u}\,\mathrm{mod}\,[\mathcal{L}^{(v_u)},\mathcal{L}]_k \tag{5.3}$$

Theorem 5.6. *For $r\geqslant 2$, $\mathcal{L}^{(v_r)}$ is the minimal ideal in $\mathcal{L}$ such that $\mathcal{L}_k^{(v_r)}$ contains all $\mathcal{F}^*[u]$ with $u\geqslant r$.*

Proof. Consider $\alpha>1$ and let $u_\alpha\geqslant 2$ be such that $v_{u_\alpha-1}<\alpha\leqslant v_{u_\alpha}$. Let $N_\alpha\geqslant 0$ be such that $p^{N_\alpha+1}(\alpha-v_{u_\alpha-1})>(p-1)v_{u_\alpha-1}$. This implies that $\mathcal{F}_{\alpha,-N_\alpha}\in\mathcal{L}_k^{(v_{u_\alpha})}$.

Suppose $\alpha>v_{r-1}$. Then $v_{u_\alpha-1}\geqslant v_{r-1}$ and

$$p^{N_\alpha+1}(\alpha-v_{r-1})\geqslant p^{N_\alpha+1}(\alpha-v_{u_\alpha-1})>(p-1)v_{u_\alpha-1}\geqslant(p-1)v_{r-1}\,.$$

In particular, Theor. 5.5 implies that $\mathcal{L}^{(v_r)}$ is the minimal ideal in $\mathcal{L}$ such that for all $\alpha\geqslant v_r$, $\mathcal{F}^0_{\alpha,-N_\alpha}\in\mathcal{L}_k^{(v_r)}$.

If $\alpha>v_r$ then $u_\alpha\geqslant v_{r+1}$ and $\mathcal{F}^0_{\alpha,-N_\alpha}\in\mathcal{L}_k^{(v_{r+1})}$.

If $\alpha=v_r$ then $u_\alpha=r$ and $p^{N_\alpha+1}(v_r-v_{r-1})=p^{N_\alpha+1}(\alpha-v_{u_\alpha-1})>(p-1)v_{u_\alpha-1}=(p-1)v_{r-1}$. Set $M_r=N_\alpha$. Then congruence (5.3) implies that

$$\mathcal{F}^*[r]\equiv\mathcal{F}^0_{v_r,-M_r}\equiv\mathcal{F}^0_{\alpha,-N_\alpha}\,\mathrm{mod}\,[\mathcal{L}^{(v_r)},\mathcal{L}]_k\,.$$

As a result, $\mathcal{L}^{(v_r)}$ is the minimal ideal in $\mathcal{L}$ such that $\mathcal{F}^*[r]\in\mathcal{L}_k^{(v_r)}$ and $\mathcal{L}^{(v_{r+1})}\subset\mathcal{L}^{(v_r)}$.

By iterating this procedure we obtain the statement of our theorem. □

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