Quantum Mechanics — Solutions

Q4 $\quad f_1 \cdot f_2 \equiv \int_{-1}^{1} f_1^* f_2 \, dx \qquad 1 \geq x \geq -1$

Need to show that

a) $f_2 \cdot f_1 = (f_1 \cdot f_2)^*$

$$f_2 \cdot f_1 = \int_{-1}^{1} f_2^* f_1 \, dx = \left(\int_{-1}^{1} f_2 f_1^* \, dx\right)^* = (f_1 \cdot f_2)^*$$

b) $$f_1 \cdot (\alpha f_2 + \beta f_3) = \int_{-1}^{1} f_1^* (\alpha f_2 + \beta f_3) \, dx = \alpha \int_{-1}^{1} f_1^* f_2 \, dx + \beta \int_{-1}^{1} f_1^* f_3 \, dx$$

$$= \alpha (f_1 \cdot f_2) + \beta (f_1 \cdot f_3)$$

c) $f_1 \cdot f_1 \geq 0$ with equality only if $f_1 \equiv 0$

But $\quad f_1 \cdot f_1 = \int_{-1}^{1} f_1^* f_1 \, dx = \int_{-1}^{1} |f_1|^2 \, dx \geq 0 \quad$ clearly 0 iff $f_1 = 0$

as integrand ≥ 0.

We exclude functions differing from 0 only on sets of measure zero.

Examples

1) Fourier series $\qquad f_r = A_r \sin(r\pi x) \quad r = 1, 2 \ldots$

2) Legendre polynomials. $\quad f_n = A_n P_n(x)$

$(f_r \cdot f_k) = \delta_{rk}$ for orthonormality. $\qquad$ due to orthogonality

$$(f_r, f_k) = \int_{-1}^{1} A_r A_k P_r(x) P_k(x) = A_r A_k \int_{-1}^{1} P_r P_k \, dx = \frac{2}{2r+1} A_r^2 \delta_{rk}$$

$$\therefore A_r = \sqrt{\frac{2r+1}{2}}$$

$$(f_r, f_k) = \int_{-1}^{1} A_r A_k \sin(r\pi x) \sin(k\pi x) = A_r A_k \int_{-1}^{1} \sin r\pi x \sin k\pi x \, dx =$$

$$= A_r^2 \delta_{rk} \qquad \therefore A_r = 1.$$