Quantum Mechanics – Solutions

Q28. $[[\hat{x}, \hat{H}]\hat{x}] = ?$

Calculate this commutator

$[\hat{x}, \hat{H}] = [\hat{x}, \frac{\hat{p}^2}{2M} + V(\hat{x})] = [\hat{x}, \frac{\hat{p}^2}{2M}] = \frac{1}{2M}[\hat{x}, \hat{p}^2] =$

$= \frac{1}{2M}\{\hat{p}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{p}\} = \frac{1}{2M}(2i\hbar\hat{p}).$

So $[[\hat{x}, \hat{H}]\hat{x}] = \frac{i\hbar}{M}[\hat{p}, \hat{x}] = +\frac{\hbar^2}{M}\mathbb{1}.$

Take $\langle m|[[\hat{x}, \hat{H}], \hat{x}]|m\rangle$ where $\hat{H}|m\rangle = E_m|m\rangle$

Then $\langle m|[[\hat{x}, \hat{H}], \hat{x}]|m\rangle = \langle m|\frac{\hbar}{M}\mathbb{1}|m\rangle = \frac{\hbar}{M}$ and

$= \langle m|2\hat{x}\hat{H}\hat{x} - \hat{x}^2\hat{H} - \hat{H}\hat{x}^2|m\rangle = 2\langle m|\hat{x}\hat{H}\hat{x}|m\rangle - \langle m|\hat{x}^2\hat{H}|m\rangle$

$- \langle m|\hat{H}\hat{x}^2|m\rangle$

To calculate these matrix elements put

$\hat{H} = \sum_n |n\rangle E_n\langle n|$ in the first term.

$\sum_n 2\langle m|\hat{x}|n\rangle E_n\langle n|\hat{x}|m\rangle = 2E_n|\langle m|\hat{x}|n\rangle|^2$

On the other hand $\langle m|\hat{x}^2\hat{H}|m\rangle = E_m\langle m|\hat{x}^2|m\rangle =$

$= \langle m|\hat{H}\hat{x}^2|m\rangle.$

But put $\hat{x}^2 = \hat{x}\sum_n |n\rangle\langle n|\hat{x}$ as $\sum_n |n\rangle\langle n| = \hat{\mathbb{1}}$

and obtain $\langle m|\hat{x}^2\hat{H}|m\rangle = \langle m|\hat{H}\hat{x}^2|m\rangle = E_m\sum_{n=0}^{\infty}|\langle m|\hat{x}|n\rangle|^2$

So altogether

$$\frac{\hbar^2}{2M} = \sum_n (E_n - E_m)|\langle m|\hat{x}|n\rangle|^2 \quad \text{as required.}$$