39. Put $|f\rangle = \sum_{n=0}^{\infty} b_n |n\rangle \qquad \hat{a}^+|f\rangle = f|f\rangle$

so $|f\rangle = b_0|0\rangle + b_1|1\rangle + b_2|2\rangle + \cdots$

$\hat{a}^+|n\rangle = \sqrt{\hbar\omega}\,\sqrt{n}\,|n-1\rangle$

so $\hat{a}^+|f\rangle = \sqrt{\hbar\omega}\left[b_1|0\rangle + \sqrt{2}\,b_2|1\rangle + \sqrt{3}\,b_3|2\rangle + \cdots\right]$

i.e. $\sqrt{\hbar\omega}\sum_{i=0}^{\infty} b_{i+1}\sqrt{i+1}\,|i\rangle$.

Require that $|f\rangle$ is an eigenstate with f eigenvalue

i.e. $f\left[b_0|0\rangle + b_1|1\rangle + b_2|2\rangle + b_3|3\rangle + \cdots\right] = \sqrt{\hbar\omega}\left[b_1|0\rangle + \sqrt{2}\,b_2|1\rangle + \cdots\right]$

so we have

$f b_0 = \sqrt{\hbar\omega}\,b_1, \quad f b_1 = \sqrt{\hbar\omega}\,\sqrt{2}\,b_2, \ldots ; \quad f b_k = \sqrt{\hbar\omega}\,\sqrt{k+1}\,b_{k+1} \cdots$

i.e. $b_1 = \frac{f b_0}{\sqrt{\hbar\omega}}, \quad b_2 = \frac{f b_1}{\sqrt{\hbar\omega}\sqrt{2}} = \frac{f^2 b_0}{\hbar\omega\sqrt{2}}, \quad b_3 = \frac{f b_2}{\sqrt{\hbar\omega}\sqrt{3}} = \frac{f^3 b_0}{(\hbar\omega)^{3/2}\sqrt{2\cdot 6}}$

and in general

$$b_k = \frac{f^k\, b_0}{(\hbar\omega)^{k/2}\sqrt{k!}}.$$

Normalization $\langle f|f\rangle = b_0^2 + b_1^2 + b_2^2 + \cdots + b_k^2 + \cdots$

$$= \sum_{k=0}^{\infty} \frac{f^{2k}\, b_0^2}{(\hbar\omega)^k\, k!} = b_0^2 \exp\left(\frac{f^2}{\hbar\omega}\right).$$

So for normalization we choose $b_0 = \exp\left(\frac{-f^2}{2\hbar\omega}\right)$

this assumes f to be real, otherwise $|f|^2$ instead of f^2.