QM Problem Sheet – Solutions (30, 32, 38)

Finding simultaneous eigenstates of $\widehat{J}_x$, $\widehat{J}_y$ and $\widehat{J}_z$

Recall that to find simultaneous eigenstates of several operators, those operators must commute with each other. The main point of this question is that the operators are only required to commute when acting on those eigenstates (and so any linear combinations of them), not when acting on a general state in the Hilbert space. Now we know that $[\widehat{J}_x, \widehat{J}_y] = i\hbar\widehat{J}_z$ so a state can only be an eigenstate of both $\widehat{J}_x$ and $\widehat{J}_y$ if it is an eigenstate of $\widehat{J}_z$ with eigenvalue 0. Similarly we see that the only states which are simultaneous eigenstates of $\widehat{J}_x$, $\widehat{J}_y$ and $\widehat{J}_z$ have eigenvalue 0 for each operator. We have seen in the lectures that such states exist, they are the eigenstates of $\widehat{J}^2$ with eigenvalue 0 (they may also be labelled by eigenvalues of other compatible operators.)

30 The states $\{|j,m\rangle\}$ are orthonormal eigenstates of $\widehat{J}^2$ and $\widehat{J}_z$ so $\langle j,m|j',m'\rangle = \delta_{jj'}\delta_{mm'}$. So we can see immediately that

$$\langle j,m|\widehat{J}^2|j',m'\rangle = j'(j'+1)\hbar^2\delta_{jj'}\delta_{mm'}$$

For $j = j' = \frac{1}{2}$, m and m' can have values $-\frac{1}{2}$ or $\frac{1}{2}$ so we can express the four possible combinations as a 2×2 matrix

$$\frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly for $j = j' = 1$, m and m' can have values -1, 0 or 1 so we can express the nine possible combinations as a 3×3 matrix

$$2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can also immediately see that

$$\langle j,m|\widehat{J}_z|j',m'\rangle = m'\hbar\delta_{jj'}\delta_{mm'}$$

with corresponding matrices

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Recall that $\widehat{J}_\pm$ raise and lower m with the following normalisation

$$\widehat{J}_\pm|j',m'\rangle = \sqrt{(j'\mp m')(j'\pm m'+1)}\hbar|j',m'\pm 1\rangle$$

So we have

$$\langle j,m|\widehat{J}_\pm|j',m'\rangle = \sqrt{(j'\mp m')(j'\pm m'+1)}\hbar\delta_{jj'}\delta_{m,m'\pm 1}$$

with matrices

$$\hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

for $\widehat{J}_+$ and

$$\hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

for $\widehat{J}_-$.

Since $\widehat{J}_\pm = \widehat{J}_x \pm i\widehat{J}_y$ we can easily find the matrix elements for $\widehat{J}_x$ and $\widehat{J}_y$ by taking linear combinations of the above matrix elements of $\widehat{J}_\pm$. So for $\widehat{J}_x$ we have

$$\langle j,m|\widehat{J}_x|j',m'\rangle = \frac{\hbar}{2}\delta_{jj'}\left(\sqrt{(j'-m')(j'+m'+1)}\delta_{m,m'+1} + \sqrt{(j'+m')(j'-m'+1)}\delta_{m,m'-1}\right)$$

with matrices

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

while for $\widehat{J}_y$

$$\langle j,m|\widehat{J}_y|j',m'\rangle = \frac{\hbar}{2i}\delta_{jj'}\left(\sqrt{(j'-m')(j'+m'+1)}\delta_{m,m'+1} - \sqrt{(j'+m')(j'-m'+1)}\delta_{m,m'-1}\right)$$

with matrices

$$\frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \frac{\hbar}{\sqrt{2}i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

You can easily check that these matrices satisfy the angular momentum commutation relations.

32 An arbitrary state with $j = \frac{1}{2}$ is a linear combination of $|\frac{1}{2}, \frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle$. The two coefficients should have the same magnitude if the probabilities of measuring the z-component of spin as $\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$ are to be the same. So the most general such normalised state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{i\theta}|\frac{1}{2}, \frac{1}{2}\rangle + e^{i\phi}|\frac{1}{2}, -\frac{1}{2}\rangle\right)$$

where θ and ϕ are arbitrary (real) phases. Since the overall phase will not affect the expectation values we could set θ or ϕ to zero, or keeping both phases the expectation values can only depend on the difference $\theta - \phi$. Note that we cannot remove both phases by absorbing them into the definitions of $|\frac{1}{2}, \frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle$ since these states are already related by the action of $\widehat{J}_\pm$, fixing their relative phases. So using the results of question 30 we can easily calculate

$$\begin{aligned}
\langle J_x\rangle = \langle\psi|\widehat{J}_x|\psi\rangle &= \frac{1}{2}\left(e^{-i\theta}\langle\frac{1}{2}, \frac{1}{2}| + e^{-i\phi}\langle\frac{1}{2}, -\frac{1}{2}|\right)\widehat{J}_x\left(e^{i\theta}|\frac{1}{2}, \frac{1}{2}\rangle + e^{i\phi}|\frac{1}{2}, -\frac{1}{2}\rangle\right)\\
&= \frac{1}{2}\langle\frac{1}{2}, \frac{1}{2}|\widehat{J}_x|\frac{1}{2}, \frac{1}{2}\rangle + \frac{1}{2}e^{-i(\theta-\phi)}\langle\frac{1}{2}, \frac{1}{2}|\widehat{J}_x|\frac{1}{2}, -\frac{1}{2}\rangle +\\
&\quad \frac{1}{2}e^{i(\theta-\phi)}\langle\frac{1}{2}, -\frac{1}{2}|\widehat{J}_x|\frac{1}{2}, \frac{1}{2}\rangle + \frac{1}{2}\langle\frac{1}{2}, -\frac{1}{2}|\widehat{J}_x|\frac{1}{2}, -\frac{1}{2}\rangle\\
&= 0 + \frac{1}{2}e^{-i(\theta-\phi)} + \frac{1}{2}e^{i(\theta-\phi)} + 0 = \frac{\hbar}{2}\cos(\theta-\phi)
\end{aligned}$$

Alternatively we can calculate using the matrices from question 30 so that

$$\langle J_y\rangle = \frac{1}{2}\frac{\hbar}{2i}\begin{pmatrix} e^{-i\theta} & e^{-i\phi}\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} e^{i\theta}\\ e^{i\phi}\end{pmatrix} = \frac{\hbar}{4i}\left(e^{-i(\theta-\phi)} - e^{i(\theta-\phi)}\right) = -\frac{\hbar}{2}\sin(\theta-\phi)$$

Of course, we could have chosen to label states as eigenstates of $\widehat{J}^2$ together with $\widehat{J}_x$ or $\widehat{J}_y$ rather than $\widehat{J}_z$. This would simply correspond to a different choice of basis and these basis states would be linear combinations of the eigenstates of $\widehat{J}_z$. From the above calculations we can see explicitly that the states with $\theta - \phi = 0, \pi$ are eigenstates of $\widehat{J}_x$ while those with $\theta - \phi = \frac{\pi}{2}, \frac{3\pi}{2}$ are eigenstates of $\widehat{J}_y$.

38 Recall that the orthonormal energy eigenstates of a one-dimensional harmonic oscillator are $\{|n\rangle, n = 0, 1, 2, \ldots\}$ with

$$\widehat{H}|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle$$

The eigenstates are related by the creation and annihilation operators $\widehat{a} = \frac{1}{\sqrt{2m}}(\widehat{p} + im\omega\widehat{x})$ and $\widehat{a}^\dagger$:

$$\widehat{a}|n\rangle = \sqrt{(n+1)\hbar\omega}|n+1\rangle \qquad \widehat{a}^\dagger|n\rangle = \sqrt{n\hbar\omega}|n-1\rangle$$

So we can calculate

$$
\begin{aligned}
\widehat{p}|n\rangle &= \sqrt{\frac{m}{2}}\left(\widehat{a}+\widehat{a}^{\dagger}\right)|n\rangle = \sqrt{\frac{m}{2}}\left(\sqrt{(n+1)\hbar\omega}|n+1\rangle + \sqrt{n\hbar\omega}|n-1\rangle\right) \\
\widehat{x}|n\rangle &= \frac{-i}{\sqrt{2m\omega}}\left(\widehat{a}-\widehat{a}^{\dagger}\right)|n\rangle = \frac{-i}{\sqrt{2m\omega}}\left(\sqrt{(n+1)\hbar\omega}|n+1\rangle - \sqrt{n\hbar\omega}|n-1\rangle\right)
\end{aligned}
$$

Now we want to calculate $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$ and $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ for the state $|n\rangle$. It is easy to see that

$$
\langle p\rangle = \langle n|\widehat{p}|n\rangle = 0 \quad \text{and} \quad \langle x\rangle = 0
$$

Since $\widehat{p}$ and $\widehat{x}$ are self-adjoint we have

$$
\begin{aligned}
\langle p^2\rangle &= \langle n|\widehat{p}^2|n\rangle = \|\widehat{p}|n\rangle\|^2 = \frac{m}{2}\left((n+1)\hbar\omega + n\hbar\omega\right) = (n+\frac{1}{2})\hbar\omega m \\
\langle x^2\rangle &= \langle n|\widehat{x}^2|n\rangle = \|\widehat{x}|n\rangle\|^2 = \frac{1}{2m\omega^2}\left((n+1)\hbar\omega + n\hbar\omega\right) = (n+\frac{1}{2})\hbar\omega\frac{1}{m\omega^2}
\end{aligned}
$$

So we see that the uncertainties in p and x are

$$
\Delta p = \sqrt{(n+\frac{1}{2})\hbar\omega m} \qquad \Delta x = \sqrt{\frac{(n+\frac{1}{2})\hbar}{\omega m}}
$$

and so the uncertainty principle is satisfied since

$$
\Delta p\Delta x = (n+\frac{1}{2})\hbar \geq \frac{1}{2}\hbar
$$

For a classical oscillator we have

$$
x = A\cos(\omega t+\delta), \qquad p = m\frac{\mathrm{d}x}{\mathrm{d}t} = -Am\omega\sin(\omega t+\delta)
$$

We define the potential energy to be zero at $x=0$ so the total energy is simply the kinetic energy at $x=0$, giving $E = \frac{1}{2}mA^2\omega^2$. We define the average $\langle\cdots\rangle$ by time-averaging over one period of oscillation $T = \frac{2\pi}{\omega}$. So

$$
\begin{aligned}
\langle p\rangle &= \frac{1}{T}\int_{t_0}^{t_0+T}\mathrm{d}t(-Am\omega)\sin(\omega t+\delta) = 0 \\
\langle x\rangle &= \frac{1}{T}\int_{t_0}^{t_0+T}\mathrm{d}tA\cos(\omega t+\delta) = 0 \\
\langle p^2\rangle &= \frac{1}{T}\int_{t_0}^{t_0+T}\mathrm{d}t(-Am\omega)^2\sin^2(\omega t+\delta) = \frac{1}{2}A^2m^2\omega^2 = Em \\
\langle x^2\rangle &= \frac{1}{T}\int_{t_0}^{t_0+T}\mathrm{d}tA^2\cos^2(\omega t+\delta) = \frac{1}{2}A^2 = \frac{E}{m\omega^2}
\end{aligned}
$$

So we see that for quantum and classical simple harmonic oscillators of the same energy, we get the same average results for these measurements.