QM Homework Question 3 – Solutions

(a) To normalise the state we want:

$$1 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}x |\psi(x)|^2 = |N|^2 \int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{(x-y)^2}{a^2}}$$

Using the standard result for a Gaussian integral we can evaluate this via the substitution $u = x - y$ (or directly $u = (x-y)/a$)

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{(x-y)^2}{a^2}} = \int_{-\infty}^{\infty} \mathrm{d}u e^{-\frac{u^2}{a^2}} = a \int_{-\infty}^{\infty} \mathrm{d}\left(\frac{u}{a}\right) e^{-\frac{u^2}{a^2}} = a\sqrt{\pi}$$

So we can choose:

$$N = (a^2\pi)^{-\frac{1}{4}}$$

to normalise the wavefunction. Note that we can only determine $|N|$ so we could equally well choose $N = (a^2\pi)^{-1/4}e^{i\theta}$ for any real constant θ. The point is that whenever we calculate any expectation values $\langle\psi|\hat{A}|\psi\rangle$ the result will be independent of θ so it does not matter what we choose for θ.

We can now calculate expectation values of operators $\hat{A}(\hat{x}, \hat{p})$ using:

$$\langle A\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\psi^*(x)A(x, -i\hbar\frac{\mathrm{d}}{\mathrm{d}x})\psi(x) = |N|^2 \int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{(x-y)^2}{2a^2}} A(x, -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}) e^{-\frac{(x-y)^2}{2a^2}}$$

So we have:

$$\frac{1}{|N|^2}\langle x\rangle = \int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{(x-y)^2}{2a^2}} x e^{-\frac{(x-y)^2}{2a^2}} = \int_{-\infty}^{\infty} \mathrm{d}u(u+y)e^{-\frac{u^2}{a^2}} = \int_{-\infty}^{\infty} \mathrm{d}u u e^{-\frac{u^2}{a^2}} + y\int_{-\infty}^{\infty} \mathrm{d}u e^{-\frac{u^2}{a^2}}$$

So we finally get:

$$\langle x\rangle = |N|^2(0 + ya\sqrt{\pi}) = y$$

We can also calculate:

$$\frac{1}{|N|^2}\langle p\rangle = \int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{(x-y)^2}{2a^2}} (-i\hbar)\frac{\mathrm{d}}{\mathrm{d}x} e^{-\frac{(x-y)^2}{2a^2}} = \frac{i\hbar}{a^2} \int_{-\infty}^{\infty} \mathrm{d}x(x-y)e^{-\frac{(x-y)^2}{a^2}} = 0$$

Recalling the definitions:

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2, \qquad (\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2$$

we also calculate:

$$\frac{1}{|N|^2}\langle x^2\rangle = \int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{(x-y)^2}{2a^2}} x^2 e^{-\frac{(x-y)^2}{2a^2}} = \int_{-\infty}^{\infty} \mathrm{d}u(u+y)^2 e^{-\frac{u^2}{a^2}}$$

with the final result:

$$\langle x^2 \rangle = \frac{1}{2}a^2 + y^2$$

Similarly:

$$\frac{1}{|N|^2}\langle p^2 \rangle = \int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{(x-y)^2}{2a^2}} (-i\hbar)^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} e^{-\frac{(x-y)^2}{2a^2}} = -\hbar^2 \int_{-\infty}^{\infty} \mathrm{d}x \left(-\frac{1}{a^2} + \frac{(x-y)^2}{a^4}\right) e^{-\frac{(x-y)^2}{a^2}}$$

with the final result:

$$\langle p^2 \rangle = \frac{\hbar^2}{2a^2}$$

So we can easily see that:

$$\Delta x = \frac{a}{\sqrt{2}}, \qquad \Delta p = \frac{\hbar}{\sqrt{2}a}$$

(b) To find the momentum representation wavefunction we have to perform the Fourier transform of the position representation wavefunction:

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}x e^{-\frac{i}{\hbar}px} \psi(x)$$

We can perform this integral by completing the square in the exponent to write the integrand in a standard Gaussian form:

$$\begin{aligned} \tilde{\psi}(p) &= (4\pi^3 a^2 \hbar^2)^{-\frac{1}{4}} \int_{-\infty}^{\infty} \mathrm{d}x \exp\left(-\frac{1}{2a^2}(x - y + \frac{i}{\hbar}pa^2)^2 - \frac{i}{\hbar}py - \frac{p^2 a^2}{2\hbar^2}\right) \\ &= \left(\frac{a^2}{\pi\hbar^2}\right)^{\frac{1}{4}} \exp\left(-\frac{i}{\hbar}py - \frac{p^2a^2}{2\hbar^2}\right) \end{aligned}$$

Now we can calculate the expectation value of $\hat{A}(\hat{x}, \hat{p})$ using:

$$\langle A \rangle = \int_{-\infty}^{\infty} \mathrm{d}p \tilde{\psi}^*(p) A(i\hbar \frac{\mathrm{d}}{\mathrm{d}p}, p) \tilde{\psi}(p)$$

The calculations are similar to those in the position representation because the momentum representation wavefunction is also a Gaussian distribution. Taking care of the various factors will produce the same results as calculated in the position representation. In particular

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} \mathrm{d}p \tilde{\psi}^*(p) i\hbar \frac{\mathrm{d}}{\mathrm{d}p} \tilde{\psi}(p) \\ &= \frac{a}{\sqrt{\pi}\hbar} \int_{-\infty}^{\infty} \mathrm{d}p \left(y - i\frac{a^2}{\hbar}p\right) e^{-\frac{p^2a^2}{\hbar^2}} = y \end{aligned}$$

noting that the integral with the factor $i\frac{a^2}{\hbar}p$ vanishes since the integrand is an odd function of p. Note how the average position $\langle x \rangle = y$ is encoded by the momentum-dependent phase factor in the momentum representation wavefunction.

(c) The position-momentum uncertainty principle is:

$$\Delta x \Delta p \geq \frac{1}{2}\hbar$$

and it is clear from the results of part (a) that this bound is satisfied by the state $|\psi\rangle$. In fact we see that the lowest possible value is attained.

We have seen that the uncertainty in position is proportional to a while the uncertainty in momentum is inversely proportional to a. So if we take $a \to 0$ the state will have a well-defined position but a completely undefined momentum. Similarly taking $a \to \infty$ will produce a state with definite momentum but undefined position.

We can consider the form of the wavefunctions in these limits. For example if we take $a \to 0$ then for any $x \neq y$ we see that $\psi(x) \to 0$ since the exponential factor is much more important than the fixed power of a in the normalisation constant. However, for $x = y$ the exponential factor is equal to 1 so $\psi(y) \to \infty$. So as $a \to 0$ the state becomes more like a particle with a definite position at $x = y$. However, at the same time the momentum is becoming more uncertain. The momentum representation wavefunction becomes a pure phase factor times a normalisation constant. So the momentum is equally likely to be any value.

Note that the normalisation constant for the momentum wavefunction actually vanishes in this limit. Similarly it can be seen that the position wavefunction does not have the correct normalisation to be a delta-function. However, this is expected since we have chosen the state to have norm 1 whereas a position eigenstate should really have delta-function norm.