REMARKS ON SOME WENDROFF TYPE INEQUALITIES

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Abstract. In this paper we give new proofs for some recent generalizations of Wendroff inequality (see [1]) and we obtain a representation for the best upper bound in the Wendroff inequality. Moreover we point out that the proofs of theorem 2.1 and 2.2 from [1] contain some errors, hence a new proof is necessary. Our method can be applied to a wide class of Gronwall type inequalities and gives elegant and powerful proofs for most of the known Gronwall type inequalities.

1. Introduction

The Wendroff inequality is a generalization of the Gronwall inequality for 2 independent variables, has its origin in the theory of partial differential equations and can be found in many monographs on inequalities ([4], [3], [9], [8]). Recently the authors in [1] gave a sharpened version for a Wendroff type inequality proved by Pachpatte (see [9]) but unfortunately their proof contains some errors. In this paper we prove the inequality given in [1] (theorem 2.2) and we use the abstract comparison Gronwall lemma to obtain new proofs for well known generalizations of the Wendroff inequality. Our method uses an operatorial point of view and can be used to simplify the proofs of many other Wendroff type inequalities.

1.1. Wendroff type inequalities.

In what follows we consider $D = [0, l] \times [0, l] \subset \mathbb{R}^2$. As a starting point we recall the following generalization of the original Wendroff inequality proved by B.G. Pachpatte:

Theorem 1.1. *([1],[9]) Let $u(x, y)$, $w(x, y)$ and $a(x, y)$ non-negative continuous functions defined for $(x, y) \in D$, and let $w(x, y)$ be non-decreasing in each variable $x, y \in D$. If*

$$u(x, y) \le w(x, y) + \int_0^x \int_0^y a(s, t)u(s, t)dtds, \ \forall (x, y) \in D \tag{1.1}$$

then

$$u(x, y) \le w(x, y) \exp\left(\int_0^x \int_0^y a(s, t)dtds\right), \ \forall (x, y) \in D. \tag{1.2}$$

This inequality was generalized in [1] as follows:

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Theorem 1.2. *([1] - Theorem 2.1) Suppose $u(x,y)$, $w(x,y)$ and $a(x,y)$ are non-negative continuous functions defined on a domain D and w is nondecreasing in the second variable. If inequality (1.1) is satisfied for all $(x,y)\in D$, then*

$$u(x,y) \le w(x,y) + G^{-1}\left(\int_0^x\int_0^y a(s,t)dtds\right),\ \forall (x,y)\in D, \tag{1.3}$$

where

$$G(r) = \int_{r_0}^{r}\frac{ds}{s+w}, r\ge r_0>0, \tag{1.4}$$

G^{-1} is the inverse function of G and $\int_0^x\int_0^y a(s,t)dtds \in Dom(G^{-1}),\ \forall (x,y)\in D$.

Remark 1.3. In [1] the last condition (w is nondecreasing) is omitted in the statement of the theorem but it is used in the proof (line 7 of the proof).

Theorem 1.4. *([1] - Theorem 2.2) Suppose $u(x,y)$, $w(x,y)$ and $a(x,y)$ are non-negative continuous functions defined on a domain D, and let $w(x,y)$ be nondecreasing in each variable $(x,y)\in D$. If u satisfies inequality (1.1), then $\forall (x,y)\in D$ we have*

$$u(x,y) \le w(x,y)\left[1+\int_0^x\int_0^y a(s,y)\exp\left(\int_s^x\int_t^y a(\xi,\eta)d\xi d\eta\right)dtds\right]. \tag{1.5}$$

Remark 1.5. The proof of theorem 2.2 in [1] contains 2 errors. The first error is on line 5-6 of the proof and can be corrected only by adding further assumptions on the functions w and a. This motivates the need of a new proof for this theorem. The second error is on page 611, line 11 but this error can be corrected only by replacing the right hand side of the inequality with an other expression and this weakens the inequality.

1.2. **Picard operators.** The Picard operator technique was developed by I.A. Rus (see [11] and the references therein) in order to handle some important problems in the theory of differential equations (existence, uniqueness, differentiability of the solutions, etc.) and can be applied also in the study of Gronwall type inequalities (see [10], [13], [14], [5], [11], [12], [2] and the references therein).

We recall some notations and some properties from [11].

Let $(X,\to)$ be an L-space, $A: X\to X$ an operator. We denote by F_A the fixed points of A. We also denote $A^0 := 1_X,\ A^1 := A, \dots, A^{n+1} := A^n\circ A,\ n\in\mathbb{N}$ the iterate operators of the operator A.

Definition 1.6. ([10], [11], [12]) A is a Picard operator (briefly PO), if there exists $x_A^*\in X$ such that:
(i) $F_A = \{x_A^*\}$;
(ii) $A^n(x)\to x_A^*$ as $n\to\infty, \forall x\in X$.

Remark 1.7. The Banach fixed point theorem guaranties that if A is a contraction, then A is a Picard operator.

The following two abstract Gronwall lemmas can be used in order to give simple and elegant proofs for most of the known Gronwall type inequalities.

Lemma 1.8 ([10], [11])**.** *(Abstract Gronwall lemma) Let $(X,\to,\le)$ be an orderd L-space and $A: X\to X$ an operator. We assume that:*
(i) A is PO;

(ii) A is increasing.
If we denote by x_A^* the unique fixed point of A, then:
(a) $x \leq A(x) \Rightarrow x \leq x_A^*$;
(b) $x \geq A(x) \Rightarrow x \geq x_A^*$.

Lemma 1.9 ([10], [11])**.** *(Abstract Gronwall-comparison lemma) Let $(X, \rightarrow, \leq)$ be an orderd L-space and $A_1, A_2 : X \rightarrow X$ be two operators. We assume that:*
(i) A_1 is increasing;
(ii) A_1 and A_2 are POs;
(iii) $A_1 \leq A_2$.
If we denote by x_2^* the unique fixed point of A_2, then

$$x \leq A_1(x) \Rightarrow x \leq x_2^*.$$

2. Main results

The inequality (1.1) is linear in u, so we can obtain a representation by applying the successive approximation method. This representation gives also the solution of the integral equation

$$u(x,y) = w(x,y) + \int_0^x \int_0^y a(s,t)u(s,t)dtds, \ \forall (x,y) \in D \tag{2.1}$$

starting from the $u_0 = w$, hence this is the maximal solution of the inequality (1.1). In [5] the authors proved that the right hand side of the classical Wendroff inequality is not the fixed point of the corresponding integral operator (it is not the solution of the associated integral equation). In what follows we prove that this is also valid for the Wendroff type inequalities proved in [1] and we construct the best possible estimation (see theorem 2.1). We use this representation to give a correct proof of theorem 2.2 from [1]. In the last section we use the abstract Gronwall comparison lemma (see [14]) for proving theorem 1.2 and 2.1. from [1].

2.1. The representation for the best estimation.

Theorem 2.1. *Suppose $u(x,y)$, $w(x,y)$ and $a(x,y)$ are non-negative continuous functions defined on a domain D. If u satisfies inequality* (1.1)*, then*

$$u(x,y) \leq w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t)H(x,y,s,t)dtds, \ \forall (x,y) \in D \tag{2.2}$$

where

$$H(x,y,s,t) = \sum_{j=0}^{\infty} K_j(x,y,s,t) \text{ and}$$

$$K_{j+1}(x,y,s,t) = \int_s^x \int_t^y a(\xi,\eta)K_j(x,y,\xi,\eta)d\eta d\xi, \ K_0 \equiv 1.$$

Proof. The integral operator $A : C(D) \rightarrow C(D)$ defined by

$$A(u)(x,y) = w(x,y) + \int_0^x \int_0^y a(s,t)u(s,t)dtds \tag{2.3}$$

is a Picard operator if we consider the Bielecki norm on the set $C(D)$:

$$\|u\| = \max_{(x,y)\in D} e^{-\tau(x+y)}|u(x,y)|.$$

Moreover the space $(C(D), \|\cdot\|)$ is an ordered Banach space with the natural ordering $u \leq v \Leftrightarrow u(x,y) \leq v(x,y)$, $\forall (x,y) \in D$ and the operator A is an increasing operator. These observations allow us to apply the abstract Gronwall lemma, so

$$u(x,y) \leq u^*(x,y),$$

where $u^*(x,y)$ is the solution of the integral equation (2.1). But this solution can be obtained as the limit of the successive approximation sequence starting from $u_0 = w$ and the terms of this sequence can be calculated as follows:

$$\begin{aligned} u_1(x,y) &= A(u)(x,y) \\ &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t)dtds \end{aligned}$$

$$\begin{aligned} u_2(x,y) &= A(u_1)(x,y) \\ &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t)dtds + \\ &+ \int_0^x \int_0^y a(s,t) \int_0^s \int_0^t a(\xi,\eta)w(\xi,\eta)d\eta d\xi dtds \end{aligned}$$

Changing the order of integration in the last integral we obtain

$$\begin{aligned} u_2(x,y) = w(x,y) &+ \int_0^x \int_0^y a(s,t)w(s,t)dtds + \\ &+ \int_0^x \int_0^y a(s,t)w(s,t) \int_s^x \int_t^y a(\xi,\eta)d\eta d\xi dtds \\ &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t)\left[K_0(x,y,s,t) + K_1(x,y,s,t)\right]dtds. \end{aligned}$$

Applying the operator A one more time we obtain

$$\begin{aligned} u_3(x,y) &= A(u_2)(x,y) \\ &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t) \sum_{j=0}^{2} K_j(x,y,s,t)dtds \end{aligned}$$

and by an inductive argument we deduce

$$\begin{aligned} u_{k+1}(x,y) &= A(u_k)(x,y) \\ &= w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t) \sum_{j=0}^{k} K_j(x,y,s,t)dtds. \end{aligned}$$

Hence the solution can be represented as

$$u^*(x,y) = w(x,y) + \int_0^x \int_0^y a(s,t)w(s,t)H(x,y,s,t)dtds,\ \forall (x,y) \in D,$$

where

$$H(x,y,s,t) = \sum_{j=0}^{\infty} K_j(x,y,s,t) \text{ and}$$

$$K_{j+1}(x,y,s,t) = \int_s^x \int_t^y a(\xi,\eta) K_j(x,y,\xi,\eta) d\eta d\xi, \ K_0 \equiv 1.$$

□

Theorem 2.2 (Corrected statement of theorem 2.2 from [1]). *Suppose $u(x,y)$, $w(x,y)$ and $a(x,y)$ are non-negative continuous functions defined on a domain D and w is nondecreasing in both variables. If u satisfies inequality* (1.1)*, then*

$$u(x,y) \le w(x,y) + \int_0^x \int_0^y a(s,t) w(s,t) \exp\left(\int_s^x \int_t^y a(\xi,\eta) d\eta d\xi \right) dt ds, \ \forall (x,y) \in D. \tag{2.4}$$

Proof. Denote by $\overline{u}(x,y)$ the right hand side of the inequality (2.4). Using the representation from theorem 2.1 it is sufficient to prove that

$$H(x,y,s,t) \le \exp\left(\int_s^x \int_t^y a(\xi,\eta) d\eta d\xi \right), \ \forall (x,y) \in D.$$

In order to prove this inequality we proceed by mathematical induction and we prove that for all $k \in \mathbb{N}$

$$\sum_{j=0}^{k} K_j(x,y,s,t) \le \exp\left(\int_s^x \int_t^y a(\xi,\eta) d\eta d\xi \right), \ \forall (x,y) \in D \text{ and } s \le x, t \le y.$$

This inequality is trivial for $k = 0$. For a fixed k by replacing s with ξ and t with η, multiplying with $a(\xi,\eta)$ and integrating from s to x and from t to y we obtain

$$\sum_{j=0}^{k} \int_s^x \int_t^y a(\xi,\eta) K_j(x,y,\xi,\eta) d\eta d\xi \le \int_s^x \int_t^y a(\xi,\eta) \exp\left(\int_\xi^x \int_\eta^y a(\alpha,\beta) d\beta d\alpha \right) d\eta d\xi \tag{2.5}$$

which implies

$$\sum_{j=0}^{k+1} K_j(x,y,s,t) \le 1 + \int_s^x \int_t^y a(\xi,\eta) \exp\left(\int_\xi^x \int_\eta^y a(\alpha,\beta) d\beta d\alpha \right) d\eta d\xi \tag{2.6}$$

In order to complete the inductive argument (and also the proof) it is sufficient to prove that

$$1 + \int_s^x \int_t^y a(\xi,\eta) \exp\left(\int_\xi^x \int_\eta^y a(\alpha,\beta) d\beta d\alpha \right) d\eta d\xi \le \exp\left(\int_s^x \int_t^y a(\xi,\eta) d\eta d\xi \right).$$

Consider the function

$$G(\xi,\eta) = \exp\left(\int_\xi^x \int_\eta^y a(\alpha,\beta) d\beta d\alpha \right). \tag{2.7}$$

For this function we have

$$\frac{\partial G}{\partial \xi}(\xi,\eta) = -\int_\eta^y a(x,\beta) d\beta \cdot G(x,y) \text{ and}$$

$$\frac{\partial^2 G}{\partial \xi \partial \eta}(\xi,\eta) = a(\xi,\eta) \cdot G(x,y) + \int_\xi^x a(\alpha,\eta) d\alpha \cdot \int_\eta^y a(\xi,\beta) d\beta \cdot G(\xi,\eta).$$

From this equality and the nonnegativity of a we obtain

$$a(\xi,\eta) G(\xi,\eta) \le \frac{\partial^2 G}{\partial x \partial y}(\xi,\eta),$$

hence

$$\int_s^x \int_t^y a(\xi,\eta)G(\xi,\eta))d\eta d\xi \le \int_s^x \int_t^y \frac{\partial^2 G}{\partial\xi\partial\eta}(\xi,\eta)d\eta d\xi.$$

But calculating the integrals from the right hand side expression we obtain $-1 + \exp\left(\int_s^x \int_t^y a(\xi,\eta)d\eta d\xi\right)$, so the proof is complete. □

2.2. **The abstract comparison lemma.** The proof of theorem 2.1 in [1] contains an error on line 7 (if $n(x,y) = \int\limits_0^x \int\limits_0^y a(s,t)u(s,t)dtds$, then the quantity $n_y(x,y) + w_y(x,y)$ is not necessary nonnegative. This error can be corrected if we assume that w is increasing in y.

In the following we use the abstract Gronwall-comparison lemma (Lemma 1.9) to prove theorem 1.2 (theorem 2.1 from [1]) and to prove a known version of a nonlinear Gronwall-Bihari inequaity.

Theorem 2.3 ([1],[5]). *Suppose $u(x,y)$, $w(x,y)$ and $a(x,y)$ are non-negative continuous functions defined on a domain D, w is nondecreasing in y. If inequality (1.1) is satisfied for all $(x,y) \in D$, then*

$$u(x,y) \le w(x,y) + G^{-1}\left(\int_0^x \int_0^y a(s,t)dtds\right),\ \forall (x,y) \in D, \tag{2.8}$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{s+w}, r \ge r_0 > 0, \tag{2.9}$$

G^{-1} is the inverse function of G and $\int_0^x \int_0^y a(s,t)dtds \in Dom(G^{-1})$, $\forall (x,y) \in D$.

Proof. We can see in [5], that if we take the integral operator $A_1 : C(D) \to C(D)$ defined by

$$A_1(u)(x,y) = w(x,y) + \int_0^x \int_0^y a(s,t)u(s,t)dtds \tag{2.10}$$

it's a PO, but its fixed point is not function

$$u(x,y) = w(x,y) + G^{-1}\left(\int_0^x \int_0^y a(s,t)dtds\right),\ \forall (x,y) \in D, \tag{2.11}$$

Now we have to find a PO $A_2 : C(D) \to C(D)$, with the property $A_1 \le A_2$ and with the fixed point defined by (2.11). Due to Lemma 1.9 if we construct A_2, the proof will be completed.

From (2.11) we get:

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial w}{\partial x}(x,y) + (G^{-1})'\left(\int_0^x \int_0^y a(s,t)dtds\right)\int_0^y a(x,t)dt, \forall (x,y) \in D.$$

But

$$(G^{-1})'\left(\int_0^x \int_0^y a(s,t)dtds\right) = u(x,y), \forall (x,y) \in D,$$

so we have

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial w}{\partial x}(x,y) + u(x,y)\int_0^y a(x,t)dt, \forall (x,y) \in D.$$

From the basic theorem of the calculus we have:

$$u(x,y) - u(0,y) = \int_0^x \frac{\partial u}{\partial x}(s,y)ds = \int_0^x \left(\frac{\partial w}{\partial x}(s,y) + u(s,y)\int_0^y a(s,t)dt\right)ds.$$

From here we get the PO $A_2 : C(D) \to C(D)$, defined by

$$A_2(u)(x,y) = w(x,y) + \int_0^x \int_0^y u(s,y)a(s,t)dtds,$$

with the fixed point satisfying (2.11). If we consider the set

$$X = \{u \in C(D) | u \text{ increasing in y and } u(0,y) = w(0,y),\ u(x,0) = w(x,0)\}$$

and the restrictions of A_1, A_2 to X, then

$$A_1, A_2 : X \to X,\ \ A_1(u) \leq A_2(u),\ \ \forall u \in X$$

because $u(s,t) \leq u(s,y)$ for $t \leq y$. On the other hand $A_1(u) \in X$, for all $u \in C(D)$, A_1 and A_2 are Picard operators on X, and X is a closed subset of $C(D)$, so using Lemma 1.9 the proof is complete. □

Theorem 2.4. *Let $u(x,y)$, $w(x,y)$ and $a(x,y)$ non-negative continuous functions defined on a domain D, w is non-decreasing in both variables and let $g : [0,\infty) \to (0,\infty)$ a continuous non-decreasing function. If*

$$u(x,y) \leq w(x,y) + \int_0^x \int_0^y a(s,t)g(u(s,t))dtds, \forall (x,y) \in D \tag{2.12}$$

then

$$u(x,y) \leq G^{-1}\left(G(w(x,y)) + \int_0^x \int_0^y a(s,t)dtds\right), \forall (x,y) \in D, \tag{2.13}$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{g(s)}, r \geq r_0 > 0 \tag{2.14}$$

G^{-1} is the inverse function of G and

$$G(w(x,y)) + \int_0^x \int_0^y a(s,t)dtds \in Dom(G^{-1}), \forall (x,y) \in D.$$

Proof. Let us consider the integral operator $A_1 : C(D) \to C(D)$ defined by the right side of the the inequality (2.12), namely

$$A_1(u)(x,y) = w(x,y) + \int_0^x \int_0^y a(s,t)g(u(s,t))dtds, \forall (x,y) \in D,$$

and the function

$$u(x,y) = G^{-1}\left(G(w(x,y)) + \int_0^x \int_0^y a(s,t)dtds\right), \forall (x,y) \in D. \tag{2.15}$$

We can easily check that A_1 is a PO, but its fixed point is not the function defined by (2.15), so we cannot use the first abstract Gronwall lemma, we have to deal with the abstract Gronwall-comparison lemma, like in the proof of previous theorem.

From the (2.15) we have:

$$\frac{\partial u}{\partial x}(x,y) = (G^{-1})'\left(G(w(x,y)) + \int_0^x \int_0^y a(s,t)dtds\right) \cdot$$
$$\cdot \left(G'(w(x,y))\frac{\partial w}{\partial x}(x,y) + \int_0^y a(x,t)dt\right)$$

But

$$(G^{-1})'\left(G(w(x,y)) + \int_0^x \int_0^y a(s,t)dtds\right) = g(u(x,y)),$$

so

$$\frac{\partial u}{\partial x}(x,y) = g(u(x,y))\left(\frac{1}{g(w(x,y))}\frac{\partial w}{\partial x}(x,y) + \int_0^y a(x,t)dt\right).$$

From the basic theorem of calculus we have:

$$\begin{aligned} u(x,y) - u(0,y) &= \int_0^x \frac{\partial u}{\partial x}(s,y)ds \\ &= \int_0^x \frac{g(u(s,y))}{g(w(s,y))}\frac{\partial w}{\partial x}(s,y)ds + \int_0^x \int_0^y a(s,t)g(u(s,y))dtds \end{aligned}$$

This relation shows that the function u defined by (2.15) is the fixed point of the integral operator $A_2 : C(D) \to C(D)$, defined by

$$A_2(u)(x,y) = w(0,y) + \int_0^x \frac{g(u(s,y))}{g(w(s,y))}\frac{\partial w}{\partial x}(s,y)ds + \int_0^x \int_0^y a(s,t)g(u(s,y))dtds,$$

$\forall (x,y) \in D$. If we want to apply the abstract Gronwall-comparison lemma, we need to consider the set

$$X = \{u \in C(D) | u \text{ increasing in y and } u(0,y) = w(0,y),\ u(x,0) = w(x,0)\}$$

and the restrictions of A_1, A_2 to X. This is necessary in order to obtain $A_1 u \le A_2 u$. But we do not know $A_2 u \in X$, and hence we can not apply the abstract Gronwall-comparison lemma as stated in [10] or [14]. This difficulty can be overcame if we observe that:

- from $u \le A_1 u$ we deduce $u \le u^*$, where u^* is the limit of successive approximation sequence for the operator A_1 starting from u;
- if $\overline{u}$ is the fixed point of A_2, then it is sufficient to have $A_1(\overline{u}) \le A_2(\overline{u})$.

Indeed if A_1 is a PO, then u^* is the limit of the successive approximation sequence starting from $\overline{u}$ and the second observation guaranties $A_1^k(\overline{u}) \le \overline{u}$, so $u^* \le \overline{u}$. Due to this observation it is sufficient to prove $A_1\overline{u} \le A_2\overline{u}$. But $\overline{u}$ is defined by (2.15), so $w(x,y) \le \overline{u}(x,y)$, hence

$$g(w(s,y)) \le g(\overline{u}(s,y)),\ \forall 0 \le s \le x,$$

so

$$w(x,y) \le w(0,y) + \int_0^x \frac{g(\overline{u}(s,y))}{g(w(s,y))}\frac{\partial w}{\partial x}(s,y)ds. \tag{2.16}$$

From (2.15) we can deduce that $\overline{u}$ is nondecreasing in the second variable, hence

$$\int_0^x \int_0^y a(s,t)g(\overline{u}(s,t))dtds \le \int_0^x \int_0^y a(s,t)g(\overline{u}(s,y))dtds,\ \forall (x,y) \in D. \tag{2.17}$$

From (2.16) and (2.17) we deduce $A_1(\overline{u}) \le A_2(\overline{u})$, so the proof is complete.

□

Remark 2.5. This inequality generalizes some results from [7].

3. Concluding remarks

Remark 3.1. The abstract Gronwall and the abstract Gronwall-comparison lemma enables us to rewrite the proofs of many Gronwall type inequalities in a unitary, structured, simplified way.

Remark 3.2. The proof of theorems 2.4 shows that some Gronwall type inequalities can not be proved by using the abstract Gronwall-comparison lemma, because the condition $A_1 \leq A_2$ is too strong.

Remark 3.3. The proof of theorems 1.2, 2.2 and 2.4 shows that the abstract Gronwall-comparison lemma needs a revision in order to cover a wider range of concrete Gronwall type inequalities.

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