# Ulam-Hyers stability of dynamic equations on time scales via Picard operators

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## Abstract

In this paper we study the Ulam-Hyers stability of some linear and nonlinear dynamic equations and integral equations on time scales. We use both direct and operatorial methods and we propose a unified approach to Ulam-Hyers stability based on the theory of Picard operators (see [29] and[34]). Our results extend some recent results from [25],[26], [8], [14], [13] to dynamic equations and are more general than the results from [1].

The operatorial point of view, based on the theory of Picard operators, allows to discuss the Ulam-Hyers stability of many types of differential- and integral equations on time scales and also to obtain simple and structured proofs to the existing results, but as we point out at our final remarks there are also a few disadvantages.

*Keywords:* Ulam-Hyers, stability, Picard operators, time scales, differential equations, integral equations
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## 1. Introduction

*1.1. The Ulam-Hyers stability*

In 1940, S.M. Ulam gave a wide range of talks at the Mathematics Club of the University of Wisconsin, in which discussed a number of important unsolved problems. These problems were also discussed in ([38]). Among those was the question concerning the stability of group homomorphisms, namely:

Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$ does there exists a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the

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inequality

$$d(h(xy), h(x)h(y)) < \delta, \ \forall x, y \in G_1,$$

then there exists a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon, \ \forall x \in G_1?$$

The case of approximately additive functions was solved in the next year by D.H. Hyers ([18]) under the assumption that $G_1$ and $G_2$ are Banach spaces. He proved that each solution of the inequality

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon, \ \forall x, y \in G_1,$$

can be approximated by an exact solution, an additive function. In this case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to have the Ulam-Hyers stability.

Since then, the stability of many algebraic, functional, differential, integral, operatorial equations have been extensively investigated (see [10]-[14],[25],[21],[32] and the references therein).

In the near past many research papers have been published about the Ulam-Hyers stability of functional, differential and difference equations. The main tool used by the authors for obtaining stability results was the direct method (see [19]). Recently I. A. Rus developed a unified approach based on Gronwall type inequalities and Picard operators (see [32], [35]). This approach can be applied to a wide range of problems (see [15], [16], [24]).

We recall the following definitions:

**Definition 1.1** ([32]). Let $(X, d)$ be a metric space and $A : X \to X$ be an operator. By definition, the fixed point equation

$$x = A(x) \tag{1.1}$$

is said to be Ulam-Hyers stable if there exists a real number $c_A > 0$ such that: for each $\varepsilon > 0$ real number and each solution $y^*$ of the equation

$$d(y, A(y)) \leq \varepsilon,$$

there exists a solution $x^*$ of the equation (1.1) such that

$$d(y^*, x^*) \leq c_A \cdot \varepsilon.$$

**Definition 1.2** ([35]). The equation

$$x'(t) = f(t, x(t)), \forall t \in [a, b] \tag{1.2}$$

is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1([a, b])$ of the inequation

$$|y'(t) - f(t, y(t))| < \varepsilon, \ \forall t \in [a, b]$$

there exists a solution $x \in C^1([a, b])$ of the equation 1.2 with the property

$$|y(t) - x(t)| \leq c_f \varepsilon, \forall t \in [a, b].$$

*Remark* 1.3. These definitions show that on a bounded interval these two notions are the same if $X$ is the space of continuous functions and $d$ is the Chebyshev metric while on an unbounded interval the Ulam-Hyers stability of the differential equation is not equivalent with the Ulam-Hyers stability of the corresponding integral equation because the Chebyshev functional cannot be defined on $C[0, \infty)$ (only on $CB[0, \infty)$).

*1.2. Time scale analysis*

The time scale calculus was founded by Stefan Hilger in his PhD thesis (see [17]) as a unification of the classical real analysis, the $q$-calculus and the theory of difference equations. Since then this theory has been extensively studied in order to obtain a better understanding and a unified viewpoint of mathematical phenomenons occurring in the theory of difference equations and in the theory of differential equations. For an excellent introduction to the calculus on time scales and to the theory of dynamic equations on time scales we recommend the books [3] and [4] by M. Bohner and A. Peterson. Throughout in this paper we use the basic notations from these books. For the sake of coherency we recall a few basic definitions, notations and theorems from [3].

**Definition 1.4.** A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

**Definition 1.5.** We define the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{R}$ by the relations

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

Using these operators we can classify the points of time scale $\mathbb{T}$ as left dense, left scattered, right dense and right scattered according to whether $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$ and $\sigma(t) > t$ respectively.

**Definition 1.6.** A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right dense point in $\mathbb{T}$. The set of all rd-continuous functions is denoted by $C_{rd}$. If $\mathbb{T}$ has left scattered maximum $m$, then

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus \{m\} & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \end{cases} \tag{1.3}$$

We define the graininess function $\mu : \mathbb{T}^\kappa \to \mathbb{R}$ by the relation

$$\mu(t) = \sigma(t) - t.$$

We also define for $f$ the function $f^\sigma : \mathbb{T}^\kappa \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)), \ \forall t \in \mathbb{T}.$$

**Definition 1.7.** Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided if exists) with the property that given any

$\varepsilon > 0$, there is a neighborhood $U$ of $t$ (i.e. $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| [f^\sigma(t) - f(s)] - f^\Delta(t) [\sigma(t) - s] \right| \leq |\sigma(t) - s|, \; \forall s \in U.$$

We call $f^\Delta(t)$ the delta (or Hilger) derivative of $f$ at t.

**Theorem 1.8.** *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:*
   *(i) If $f$ is differentiable at $t$, then $f$ is continuous at $t$.*
   *(ii) If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is differentiable at $t$ with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

   *(iii) If $t$ is right-dense, then $f$ is differentiable at $t$ if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

*exists and is a finite number. In this case*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

   *(iv) If $f$ is differentiable at $t$, then*

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

**Definition 1.9.** A function $F : \mathbb{T} \to \mathbb{R}$ is said to be an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. We define the integral of $f$ by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s), \tag{1.4}$$

where $s, t \in \mathbb{T}$.

**Definition 1.10.** The function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$, for all $t \in \mathbb{T}^\kappa$. We denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$ the set of all regressive and rd-continuous functions and define

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}. \tag{1.5}$$

**Definition 1.11.** For $p \in \mathcal{R}$ we define (see [3]) the exponential function $e_p(\cdot, t_0)$ on the time scale $\mathbb{T}$ as the unique solution to the scalar initial value problem

$$x^\Delta(t) = p(t)x(t), \quad x(t_0) = 1. \tag{1.6}$$

If $p \in \mathcal{R}^+$, then $e_p(t, t_0) > 0$, for all $t \in \mathbb{T}$. We note that, if $\mathbb{T} = \mathbb{R}$, the exponential function is given by

$$e_p(t, s) = \exp\left(\int_s^t p(\tau)d\tau\right), \quad e_\alpha(t, s) = \exp(\alpha(t - s)), \quad e_\alpha(t, 0) = \exp(\alpha t), \tag{1.7}$$

for $s, t \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a constant and $p : \mathbb{R} \to \mathbb{R}$ is a continuous function. To compare with the discrete case, if $\mathbb{T} = \mathbb{Z}$ (the set of integers), the exponential function is given by

$$e_p(t, s) = \prod_{\tau=s}^{t-1} [1 + p(\tau)], \quad e_\alpha(t, s) = (1 + \alpha)^{t-s}, \quad e_\alpha(t, 0) = (1 + \alpha)^t, \quad (1.8)$$

for $s, t \in \mathbb{Z}$ with $s < t$, where $\alpha \neq -1$ is a constant and $p : \mathbb{Z} \to \mathbb{R}$ is a sequence satisfying $p(t) \neq -1$ for all $t \in \mathbb{Z}$.

**Theorem 1.12** (Properties of the exponential function). *If $p, q \in \mathcal{R}$, then*
*(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;*
*(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;*
*(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;*
*(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;*
*(v) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$;*
*(vi) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p \ominus q}(t, s)$;*
*(vii) $\left( \frac{1}{e_p(\cdot,s)} \right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$,*
*where for all $p, q \in \mathcal{R}$ we define*

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t),$$

*and*

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)},$$

*for all $t \in \mathbb{T}^\kappa$.*
*We remark, that $(\mathcal{R}, \oplus)$ is an Abelian group, called the regressive group.*

For recent developments regarding the exponential functions on timescales we recommend [7].

*1.3. Picard operators and applications*

The Picard operator technique was applied by many authors to study some functional nonlinear integral equations, see for example [9], [28]-[34], [36]. In what follows we use the terminology and notations from [28], [29], [30].

Let $(X, \to)$ be an L-space ([29]), $A : X \to X$ an operator. We denote by $F_A$ the fixed points of $A$. We also denote $A^0 := 1_X$, $A^1 := A, \dots, A^{n+1} := A^n \circ A$, $n \in \mathbb{N}$ the iterate operators of the operator $A$. We also define $A^\infty(x) := \lim_{n \to \infty} A^n(x)$ if the limit exists.

**Definition 1.13** ([32]). By definition $A : X \to X$ is weakly Picard operator if the sequence of successive approximations, $A^n(x)$, converges for all $x \in X$ and the limit (which may be dependent on $x$) is a fixed point of $A$.

**Definition 1.14** ([28], [29], [30]). $A$ is a Picard operator (briefly PO), if there exists $x_A^* \in X$ such that:

(i) $F_A = \{x_A^*\}$;

(ii) $A^n(x) \to x_A^*$ as $n \to \infty, \forall x \in X$.

Equivalently we can say, that if for a weakly Picard operator $A : X \to X$ $F_A = \{x_A^*\}$, then $A$ is a PO.

The following class of weakly Picard operators is very important in our consideration. Let $(X, d)$ be a metric space.

**Definition 1.15** ([32]). Let $A : X \to X$ be a weakly Picard operator and $c > 0$ a real number. By definition the operator $A$ is c-weakly Picard operator if

$$d(x, A^\infty(x)) \le c \cdot d(x, A(x)), \ \forall x \in X.$$

We present two examples for c-weakly Picard operators from [32].

**Example 1.16.** Let $(X, d)$ be a complete metric space and $A : X \to X$ an operator with closed graphic. We suppose that $A$ is graphic $\alpha$-contraction, i.e.

$$d(A^2(x), A(x)) \le \alpha \cdot d(x, A(x)), \ \forall x \in X.$$

Then A is a c-weakly Picard operator, with $c = (1 - \alpha)^{-1}$.

**Example 1.17.** Let $(X, d)$ be a complete metric space, $\varphi : X \to \mathbb{R}_+$ a function and $A : X \to X$ an operator with closed graph. We suppose that:

(i) A is a $\varphi$-Caristi operator, i.e.

$$d(x, A(x)) \le \varphi(x) - \varphi(A(x)), \ \forall x \in X;$$

(ii) there exists $c > 0$ such that

$$\varphi(x) \le c \cdot d(x, A(x)), \ \forall x \in X.$$

Then A is a c-weakly Picard operator.

An interesting remark on Ulam-Hyers stability for c-weakly Picard operators is given bellow.

**Theorem 1.18** ([32]-Remark 2.1). *Let (X,d) be a metric space. If $A : X \to X$ is a c-weakly Picard operator, then the fixed point equation* (1.1) *is Ulam-Hyers stable.*

## 2. Linear dynamic equations with constant coefficients

First we recall some examples regarding Ulam-Hyers stability of differential, difference and dynamic equations.

**Example 2.1.** ([8] and [22]) The differential equation $y' = ay, a \in \mathbb{R}$ is Ulam-Hyers stable if and only if $a \neq 0$. Moreover, the equation

$$y^{(n)} - \sum_{k=1}^{n} a_k y^{(n-k)} = 0,$$

where $a_j \in \mathbb{R}, 1 \leq j \leq n$ is Ulam-Hyers stable if and only if the characteristic equation has no pure imaginary roots.

**Example 2.2.** ([26]) The difference equation $y_{n+1} = ay_n, a \in \mathbb{R}$ is Ulam-Hyers stable if and only if $|a| \neq 1$. Moreover, the equation

$$y_{n+p} = \sum_{k=0}^{p-1} a_k y_{n+k},$$

where $a_k \in \mathbb{R}, 0 \leq k \leq p-1$ is Ulam-Hyers stable if and only if the modulus of the roots of the corresponding characteristic equation are different from 1.

**Example 2.3.** If $\mathbb{T} = \{t_0, t_1, \ldots, t_n, \ldots\}$ with $t_0 = 1$, $\mu(t_{2j}) = \frac{1}{(2j+2)^2}$ and $\mu(t_{2j-1}) = 2 - \frac{1}{(2j+1)^2}$, then

$$|e_{-1}(t_j, t_0)| = \prod_{k=2}^{j+1} \left(1 - \frac{1}{k^2}\right) = \frac{1}{2} \cdot \frac{j+2}{j+1},$$

hence $|e_{-1}(t, t_0)|$ is bounded on $[t_0, \infty)_{\mathbb{T}}$ and $\lim_{t \to \infty} \int_{t_0}^{t} |e_{-1}(s, t_0)| \Delta s = \infty$. This property and the representation 2.5 imply that the equation $y^{\Delta} = -y$ is not Ulam-Hyers stable.

These (and also other) examples shows that the Ulam-Hyers stability of linear dynamical equations with constant coefficients on a timescale $\mathbb{T}$ is closely related to the behavior of the exponential functions defined on $\mathbb{T}$. Moreover, this behavior is connected also with the inner structure of the timescale, not only with the constants (or functions) defining the exponential functions. For this reason we formulate our results in terms of the asymptotical behavior of the exponential functions and by some remarks we emphasize special classes of constants, for which we have Ulam-Hyers stability. Let $a \in \mathbb{C}$ be a complex number and $\mathbb{T}$ a time scale. Consider the following conditions:

**S1** $|e_a(t, t_0)|$ and $\int_{t_0}^{t} |e_a(t, \sigma(s))| \Delta s$ are bounded on $[t_0, \infty)_{\mathbb{T}}$;

**S2** $\lim_{t \to \infty} |e_a(t, t_0)| = \infty$ and $\int_{t}^{\infty} |e_a(t_0, \sigma(s))| \Delta s < \infty$, for all $t \in [t_0, \infty)_{\mathbb{T}}$;

**S3** $|e_a(t, t_0)|$ is bounded on $[t_0, \infty)_{\mathbb{T}}$ and $\lim_{t \to \infty} \int_{t_0}^{t} |e_a(s, t_0)| \Delta s = \infty$.

*Remark 2.4.* **a)** If $\mathbb{T} = \mathbb{R}$, and $|a| \neq 0$, one of the conditions S1 and S2 holds.

**b)** If $\mathbb{T} = \mathbb{Z}$ and $a \notin \{-2, 0\}$, one of the conditions S1 and S2 holds.

**c)** If $t_0 = 1$, $\mu(t_{2j}) = \frac{1}{(2j+2)^2}$ and $\mu(t_{2j-1}) = 2 - \frac{1}{(2j+1)^2}$, then the exponential function $e_{-1}(t, t_0)$ changes sign on each interval $[t_j, t_{j+1}]$ and condition S3 holds.

**d)** There are timescales $\mathbb{T}$ for which the modulus of some exponential functions has arbitrary large and arbitrary small values on each interval $[t, \infty)$. In such a case none of the previous conditions hold.

**Theorem 2.5.** *Consider the following dynamic equation:*

$$\{ \; y^{\Delta}(t) = ay(t) \quad , \tag{2.1}$$

*where $a \in \mathbb{C}$ is a complex number. If S1 or S2 holds, then the above equation is Ulam-Hyers stable on $[t_0, +\infty)_{\mathbb{T}}$. The same property is valid also for the inhomogeneous equation.*

**Theorem 2.6.** *Consider the following $n^{th}$ order dynamic equation:*

$$\left\{ \; y^{\Delta(n)} - \sum_{k=1}^{n} a_k y^{\Delta(k)} = 0 \tag{2.2} \right.$$

*Denote by $\lambda_1, \lambda_2, \ldots \lambda_n$ the roots of the characteristic equation*

$$r^n - \sum_{k=1}^{n} a_k r^{n-k} = 0.$$

*If $1 + \mu(t)\lambda_j \neq 0, \forall t \in \mathbb{T}$ for all $1 \leq j \leq n$, and for each $\lambda_j$ S1 or S2 is verified, then equation (2.2) is Ulam-Hyers stable on $[t_0, +\infty)_{\mathbb{T}}$.*

*Remark* 2.7. If $a > 0$, the exponential function $e_a(t, t_0)$ is positive, hence the integrals in S1, S2 and S3 can be calculated effectively. In this case equation 2.1 is always Ulam-Hyers stable. If $a = 0$, the equation $y^{\Delta}(t) = 0$ has only constant solution, while the perturbed equation $y^{\Delta}(t) = \varepsilon$ has the solution $y(t) = y(t_0) + \varepsilon(t - t_0)$, hence the equation 2.1 is not Ulam-Hyers stable. The same example shows that if at least one of the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ is pure imaginary, then the equation 2.2 is not Ulam-Hyers stable. The condition $1 + \mu(t)a \neq 0$ (and $1 + \mu(t)\lambda_j \neq 0$, $1 \leq j \leq n$) is necessary for the existence of the corresponding exponential function(s).

*Remark* 2.8. The previous theorems are extensions of the results from [25], [8], [13] and [26]. In [26] the critical value seems to be 1, but this is only because the equation is in the form $y_{n+1} = ay_n$ which is $y^{\Delta} = (a-1)y$.

*Proof of theorem 2.5.* The solution of the equation 2.1 is

$$y(t) = y_0 e_a(t, t_0). \tag{2.3}$$

The solution of the perturbed equation

$$z^{\Delta}(t) = az(t) + h(t), \tag{2.4}$$

can be represented as

$$z(t) = z_0 e_a(t, t_0) + \int_{t_0}^{t} h(s) e_a(t, \sigma(s)) \Delta s. \tag{2.5}$$

We need to estimate the difference between $y(t)$ and $z(t)$, if $|h(t)| < \varepsilon$, $\forall t \in \mathbb{T}$.

**Case 1.** If S1 holds, we have

$$|z(t) - y(t)| = \left|(z_0 - y_0)e_a(t, t_0) + \int_{t_0}^{t} h(s)e_a(t, \sigma(s))\Delta s\right| \leq$$

$$\leq |(z_0 - y_0)| \cdot |e_a(t, t_0)| + \varepsilon \int_{t_0}^{t} |e_a(t, \sigma(s))|\Delta s.$$

These inequalities imply that if $M_1 > 0$ is an upper bound for $|e_a(t, t_0)|$ and $M_2 > 0$ an upper bound for $\int_{t_0}^{t} |e_a(t, \sigma(s))|\Delta s$, then by choosing $y_0$ such that $|y_0 - z_0| < \varepsilon$, we have $|y(t) - z(t)| < \varepsilon(M_1 + M_2)$, for all $t \in [t_0, \infty)_{\mathbb{T}}$.

*Remark* 2.9. If $\mathbb{T} = \mathbb{R}$ and $a < 0$, this situation occurs.

**Case 2.** If S2 holds $a \neq 0$ and we have

$$|z(t) - y(t)| = \left|(z_0 - y_0)e_a(t, t_0) + \int_{t_0}^{t} h(s)e_a(t, \sigma(s))\Delta s\right| \leq$$

$$\leq |e_a(t, t_0)| \left|z_0 - y_0 + \int_{t_0}^{t} h(s)e_a(t_0, \sigma(s))\Delta s\right|$$

From condition S2 we deduce that the improper integral $\int_{t_0}^{\infty} h(s)e_a(t_0, \sigma(s))\Delta s$ is absolutely convergent, hence it is also convergent. If we choose

$$y_0 = z_0 + \int_{t_0}^{\infty} h(s)e_a(t_0, \sigma(s))\Delta s,$$

we have

$$|z(t) - y(t)| \leq \varepsilon \left|e_a(t, t_0) \cdot \int_{t}^{\infty} e_a(t_0, \sigma(s))\Delta s\right|.$$

But

$$\int_{t}^{\infty} e_a(t_0, \sigma(s))\Delta s = -\frac{1}{a} \int_{t}^{\infty} \frac{-a}{(1 + \mu(s)a)e_a(s, t_0)}\Delta s =$$

$$= -\frac{1}{a} \left[\lim_{s \to \infty} \frac{1}{e_a(s, t_0)} - \frac{1}{e_a(t, t_0)}\right] = \frac{1}{ae_a(t, t_0)},$$

so

$$|z(t) - y(t)| \leq \frac{\varepsilon}{|a|}, \quad \forall t \in [t_0, \infty)$$

and the equation 2.1 is Ulam-Hyers stable.

*Remark* 2.10. If $\mathbb{T} = \mathbb{R}$ and $a > 0$, or $\mathbb{T} = \mathbb{Z}$ and $a > 1$ this situation occurs.

$\square$

*Remark* 2.11. If we consider the inhomogeneous equation $y^{\Delta} = ay + f$ and the perturbed equation $z^{\Delta} = az + f + h$, the difference of the solutions is not depending on $f$, so the Ulam-Hyers stability automatically is transferred to the inhomogeneous equation.

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*Proof of the theorem 2.6.* If $y$ is a solution of the equation

$$y^{\Delta(n)} - \sum_{k=1}^{n} a_k y^{\Delta(k)} = h(t)$$

and $\lambda_n$ is a root of the corresponding characteristic equation, then the function $z = y' - \lambda_n \cdot y$ is satisfying an $(n-1)^{th}$ order inhomogeneous equation with constant coefficients for which the roots of the characteristic equations are $\lambda_1, \ldots, \lambda_{n-1}$ and the inhomogeneity is the same. Hence by an inductive argument there exists a solution $y_1$ of the corresponding homogeneous equation and a constant $c_1$, such that $|z(t) - y_1(t)| < c_1 \cdot \varepsilon$, $\forall t \in [t_0, \infty)$. Applying Theorem 2.5 for the inhomogeneous equation $y' - \lambda_n y = y_1$, we deduce the existence of a function $y_2$ with the properties $y_2^\Delta = \lambda_n y_2 + y_1$ and $|y - y_2| < c_2 \cdot c_1 \varepsilon$, $\forall t \in [t_0, \infty)$. But $y_1$ being the solution of the $(n-1)^{th}$ order equation $y_2$ is the solution of the initial $n^{th}$ order equation, so we have the Ulam-Hyers stability of the initial equation. $\square$

## 3. Ulam-Hyers stability of some integral equations

In this section we study the Ulam-Hyers stability of the integral equation

$$u(t_1, t_2) = w(t_1, t_2) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} a(s_1, s_2) u(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2 + \int_{a_1}^{t_1} b(s_1, t_2) u(s_1, t_2) \Delta_1 s_1,$$

$$(3.1)$$

and of the more general equation

$$u(t) = w(t) + \int_a^t a_1(s_1) u(s_1) \Delta s_1 +$$

$$+ \int_a^t \int_a^{s_1} a_2(s_2) u(s_2) \Delta s_2 \Delta s_1 + \cdots + \int_a^t \int_a^{s_1} \ldots \int_a^{s_{n-1}} a_n(s_n) u(s_n) \Delta s_n \ldots \Delta s_1.$$

Equations of this types appear when we transform higher order dynamic equations to fixed point problems, hence the Ulam-Hyers stability of these equations also provides information on the Ulam-Hyers stability of the dynamic equations. The main difference consists in the fact that using integral equations, the Ulam-Hyers stability is obtained in a well chosen metric space, neither the conditions, nor the conclusions of these theorems are not the same as in the classical framework (the closeness of the approximative solution and the exact solution is not measured in the classical sense).

### 3.1. Preliminary results

We use some definitions and results from the recent article [2]. We operate with the extended metric from [2], which is defined for functions with several variables based on the work of C.C. Tisdell and A. Zaidi in [37]. This is needed

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in order to prove that our operators are Picard operators, (more precisely contractions) in some well chosen metric space. Suppose that $\mathbb{T}_1, \mathbb{T}_2$ are timescales, $\alpha, \beta > 0$ are real constants and define

$$\| \cdot \|_{\alpha,\beta} : C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n) \to \mathbb{R}_+ \tag{3.2}$$

$$\|u\|_{\alpha,\beta} = \sup_{\substack{s_1 \in [a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \\ s_2 \in [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}}} \frac{\|u(s_1, s_2)\|}{e_\alpha(s_1, a_1) \cdot e_\beta(s_2, a_2)} \tag{3.3}$$

for all $u \in C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$ and

$$d_{\alpha,\beta} : C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n) \times C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)) \to \mathbb{R}_+ \tag{3.4}$$

by

$$d_{\alpha,\beta}(u, v) = \|u - v\|_{\alpha,\beta}, \tag{3.5}$$

for all $u, v \in C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$.

**Lemma 3.1** ([2]). *If $\alpha, \beta > 0$, and $\sigma_1(b_1) < \infty, \sigma_2(b_2) < \infty$, we have the following properties:*

1. *$d_{\alpha,\beta}$ is a metric on $C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$;*
2. *$C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$ is a complete metric space with $d_{\alpha,\beta}$;*
3. *$\|\cdot\|_{\alpha,\beta}$ is a norm on $C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n)$ and it is equivalent to $\| \cdot \|_{0,0}$.*
4. *$C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R}^n, \| \cdot \|_{\alpha,\beta})$ is a Banach space.*

For the simplicity of notation in what follows we use the following notations: $X := C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}, \mathbb{R})$, $D_1 := [a_1, \sigma_1(b_1)]_{\mathbb{T}_1}$, $D_2 := [a_2, \sigma_2(b_2)]_{\mathbb{T}_2}$.

Using this Bielecki type (or "TZ") metric, we prove the following properties:

**Theorem 3.2.** *If $w, a, b \in X$, $\sigma_1(b_1) < \infty, \sigma_2(b_2) < \infty$, the operator $A : X \to X$ defined by*

$$A(u)(t_1, t_2) = w(t_1, t_2) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} a(s_1, s_2)u(s_1, s_2)\Delta_1 s_1 \Delta_2 s_2 + \int_{a_1}^{t_1} b(s_1, t_2)u(s_1, t_2)\Delta_1 s_1, \tag{3.6}$$

*is well defined and there exist $\alpha, \beta > 0$ such that $A$ is a contraction on $(X, d_{\alpha,\beta})$.*

*Proof.* We denote by

$$M_1 := \max \left\{ |a(t_1, t_2)| \big| (t_1, t_2) \in D_1 \times D_2 \right\} \text{ and}$$

$$M_2 := \max \left\{ |b(t_1, t_2)| \big| (t_1, t_2) \in D_1 \times D_2 \right\}.$$

Due to the given conditions this constants exist and also $M_1 < \infty$ and $M_2 < \infty$ as well. For $u, v \in X$ we have

$$|A(u)(t_1, t_2) - A(v)(t_1, t_2)| \leq \int_{a_1}^{t_1} \int_{a_2}^{t_2} |a(s_1, s_2)||u(s_1, s_2) - v(s_1, s_2)| \Delta_1 s_1 \Delta_2 s_2 +$$

$$+ \int_{a_1}^{t_1} |b(s_1, t_2)||u(s_1, t_2) - v(s_1, t_2)| \Delta_1 s_1$$

$$\leq M_1 \int_{a_1}^{t_1} \int_{a_2}^{t_2} \frac{|u(s_1, s_2) - v(s_1, s_2)|}{e_\alpha(s_1, a_1)e_\beta(s_2, a_2)} e_\alpha(s_1, a_1)e_\beta(s_2, a_2) \Delta_1 s_1 \Delta_2 s_2$$

$$+ M_2 \int_{a_1}^{t_1} \frac{|u(s_1, t_2) - v(s_1, t_2)|}{e_\alpha(s_1, a_1)e_\beta(t_2, a_2)} e_\alpha(s_1, a_1)e_\beta(t_2, a_2) \Delta_1 s_1$$

$$\leq M_1 \|u - v\|_{\alpha,\beta} \int_{a_1}^{t_1} \int_{a_2}^{t_2} e_\alpha(s_1, a_1)e_\beta(s_2, a_2) \Delta_1 s_1 \Delta_2 s_2$$

$$+ M_2 \|u - v\|_{\alpha,\beta} \int_{a_1}^{t_1} e_\alpha(s_1, a_1)e_\beta(t_2, a_2) \Delta_1 s_1$$

$$= M_1 \|u - v\|_{\alpha,\beta} \frac{(e_\alpha(t_1, a_1) - 1)(e_\beta(t_2, a_2) - 1)}{\alpha\beta} + M_2 \|u - v\|_{\alpha,\beta} \frac{e_\alpha(t_1, a_1) - 1}{\alpha} e_\beta(t_2, a_2)$$

$$\leq M_1 \|u - v\|_{\alpha,\beta} \frac{e_\alpha(t_1, a_1)e_\beta(t_2, a_2)}{\alpha\beta} + M_2 \|u - v\|_{\alpha,\beta} \frac{e_\alpha(t_1, a_1)}{\alpha} e_\beta(t_2, a_2)$$

If we divide our inequality by the positive term $e_\alpha(t_1, a_1)e_\beta(t_2, a_2)$, we obtain

$$\frac{|A(u)(t_1, t_2) - A(v)(t_1, t_2)|}{e_\alpha(t_1, a_1)e_\beta(t_2, a_2)} \leq \frac{M_1}{\alpha\beta} \|u - v\|_{\alpha,\beta} + \frac{M_2}{\alpha} \|u - v\|_{\alpha,\beta}$$

$$= \frac{M_1 + M_2\beta}{\alpha\beta} \|u - v\|_{\alpha,\beta}.$$

Taking the supremum over $(t_1, t_2) \in D_1 \times D_2$, this inequality implies

$$\|A(u) - A(v)\|_{\alpha,\beta} \leq \frac{M_1 + \beta M_2}{\alpha\beta} \|u - v\|_{\alpha,\beta}, \tag{3.7}$$

so $A$ is a contraction on $(X, d_{\alpha,\beta})$, if $\alpha\beta > M_1 + \beta M_2$. $\qquad \square$

*3.2. Main results*

Using the main lemma from [32]:

**Lemma 3.3.** *Let $(X, d)$ be a Banach space. If an operator $A : X \to X$ is a contraction with the positive constant $q < 1$, then $A$ is c-weakly Picard operator with the positive constant $c_A = \frac{1}{1-q}$. Moreover the fixed point equation (1.1) is Ulam-Hyers stable.*

we obtain the following result:

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**Theorem 3.4.** *Let $w, a, b \in X$, $\sigma_1(b_1) < \infty, \sigma_2(b_2) < \infty$. Then the integral equation*

$$u(t_1, t_2) = w(t_1, t_2) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} a(s_1, s_2) u(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2 + \int_{a_1}^{t_1} b(s_1, t_2) u(s_1, t_2) \Delta_1 s_1,$$

(3.8)

*is Ulam-Hyers stable on $D_1 \times D_2$.*

*Proof.* Using the notations from the proof of Theorem 3.2, we have $M_1 < \infty$ and $M_2 < \infty$, so the operator defined as

$$A(u)(t_1, t_2) = w(t_1, t_2) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} a(s_1, s_2) u(s_1, s_2) \Delta_1 s_1 \Delta_2 s_2 + \int_{a_1}^{t_1} b(s_1, t_2) u(s_1, t_2) \Delta_1 s_1,$$

(3.9)

is a contraction with the positive constant $q := \frac{M_1 + \beta M_2}{\alpha\beta}$, if we choose $\alpha, \beta$ such that $q < 1$. From the Lemma 3.3 we deduce that our operator $A$ is a c-weakly PO with the positive constant $c_A = \frac{\alpha\beta}{\alpha\beta - M_1 - \beta M_2}$ and from Theorem 1.18 we obtain the Ulam-Hyers stability of the equation (3.8). □

Using a similar argument we obtain the Ulam-Hyers stability of a more general integral equation.

**Theorem 3.5.** *Let $w, a, b \in X$, $\sigma_1(b_1) < \infty, \sigma_2(b_2) < \infty$. Further let be the functions $f, g \in C(D_1 \times D_2 \times \mathbb{R}, \mathbb{R})$ with a Lipschitz property in their last variables. Then the integral equation*

$$u(t_1, t_2) = w(t_1, t_2) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} a(s_1, s_2) f(s_1, s_2, u(s_1, s_2)) \Delta_1 s_1 \Delta_2 s_2$$

$$+ \int_{a_1}^{t_1} b(s_1, t_2) g(s_1, t_2, u(s_1, t_2)) \Delta_1 s_1,$$

*is Ulam-Hyers stable on $D_1 \times D_2$.*

As a consequence we give the following results for second order dynamic equations with constant coefficients.

**Theorem 3.6.** *For the real constants $c_1$ and $c_2$ we consider the second-order linear dynamic equation*

$$x^{\Delta\Delta}(t) + c_1 x^{\Delta}(t) + c_2 x(t) = 0,$$

(3.10)

*on a time scale interval $[a, b]_{\mathbb{T}}$. This equation is always Ulam-Hyers stable on $[a, b]_{\mathbb{T}}$.*

*Proof.* We integrate the equation (3.10) from $a$ to $t$ and we have:

$$x^{\Delta}(t) - x^{\Delta}(a) + c_1(x(t) - x(a)) + c_2 \int_a^t x(s) \Delta s = 0.$$

Now we integrate once again this equation from $a$ to $t$, and renaming the variables we get:

$$x(t) = x(a) - \left(x^\Delta(a) + c_1 x(a)\right) a + \left(x^\Delta(a) + c_1 x(a)\right) t -$$
$$- c_2 \int_a^t \int_a^s x(\xi)\Delta\xi\Delta s - c_1 \int_a^t x(s)\Delta s$$

Using the function $w(t) := x(a) - \left(x^\Delta(a) + c_1 x(a)\right) a + \left(x^\Delta(a) + c_1 x(a)\right) t$, we define the operator on $C[a, b]_{\mathbb{T}^\sigma}$

$$A(x)(t) = w(t) - c_2 \int_a^t \int_a^s x(\xi)\Delta\xi\Delta s - c_1 \int_a^t x(s)\Delta s. \qquad (3.11)$$

Equation 3.10 is equivalent to the equation $Ax = x$, so theorem 3.4 can be applied to obtain the Ulam-Hyers stability of the fixed point equation derived from $A$. Due to the equivalent transformations of (3.10), the boundedness of the interval and the boundedness of the exponential functions we also have the Ulam-Hyers stability of the dynamical equation 3.10. $\qquad\qquad \square$

**Theorem 3.7.** *Let* $p, q, f \in C_{rd}[a, b]_{\mathbb{T}}$ *and consider the following second-order inhomogeneous delta dynamic equation with variable coefficients:*

$$x^{\Delta\Delta}(t) + p(t)x^\Delta(t) + q(t)x(t) = f(t), \ t \in [a, b]_{\mathbb{T}}. \qquad (3.12)$$

*If* $p$ *is* $\Delta$ *differentiable and* $p = p^\sigma$ *on its domain, then the dynamic equation* (3.12) *is Ulam-Hyers stable.*

*Proof.* We use the same idea as in the proof of Theorem 3.6, what is, we want to construct an integral operator and we prove that it is c-weakly PO.

So we integrate 3.12 from $a$ to $t$:

$$x^\Delta(t) - x^\Delta(a) + \int_a^t p(s)x^\Delta(s)\Delta s + \int_a^t q(s)x(s)\Delta s = \int_a^t f(s)\Delta s.$$

Furthermore with partial integration using that $p = p^\sigma$ we have:

$$x^\Delta(t) - x^\Delta(a) + p(t)x(t) - p(a)x(a) - \int_a^t p^\Delta(s)x(s)\Delta s + \int_a^t q(s)x(s)\Delta s = \int_a^t f(s)\Delta s.$$

Integrating once more from $a$ to $t$ and arrange the terms we get:

$$x(t) = x(a) + \left(x^\Delta(a) + p(a)x(a)\right)(t - a) + \int_a^t \int_a^s f(\xi)\Delta\xi\Delta s$$
$$+ \int_a^t \int_a^s \left(p^\Delta(\xi) - q(\xi)\right) x(\xi)\Delta\xi\Delta s - \int_a^t p(s)x(s)\Delta s$$

So with $w(t) := x(a) + \left(x^\Delta(a) + p(a)x(a)\right)(t-a) + \int_a^t \int_a^s f(\xi)\Delta\xi\Delta s$ we define the operator $A$ on $C[a, b]_{\mathbb{T}}$ as it follows:

$$A(x)(t) := w(t) + \int_a^t \int_a^s \left(p^\Delta(\xi) - q(\xi)\right) x(\xi)\Delta\xi\Delta s - \int_a^t p(s)x(s)\Delta s.$$

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By applying Theorem 3.4 we obtain the Ulam-Hyers stability of the fixed point equation generated by $A$. Due to the boundedness of the interval and of the exponential functions we also obtain the Ulam-Hyers stability of the equation (3.12). $\qquad\square$

*Remark* 3.8. Theorem 3.6 and Theorem 3.7 can be obtained also without using Theorem 3.4 by using directly 3.3 and a metric space with functions having only one variable (see the proof of Theorem 3.9). These results are more general than Theorem 1.5. in [1].

Theorem 3.6 and Theorem 3.7 can also be generalized in order to imply the Ulam-Hyers stability of linear delta dynamic equations of order $n$.

**Theorem 3.9.** *Let* $w, a_1, a_2, \ldots, a_n \in C[a, \sigma(b)]_{\mathbb{T}}$, $\sigma(b) < \infty$. *Then the integral equation*

$$u(t) = w(t) + \int_a^t a_1(s_1)u(s_1)\Delta s_1 + \int_a^t \int_a^{s_1} a_2(s_2)u(s_2)\Delta s_2\Delta s_1$$

$$+ \cdots + \int_a^t \int_a^{s_1} \ldots \int_a^{s_{n-1}} a_n(s_n)u(s_n)\Delta s_n \ldots \Delta s_1,$$

*is Ulam-Hyers stable on* $[a, \sigma(b)]_{\mathbb{T}}$.

*Proof.* In the proof we use the same idea as in the proof of the Theorem 3.4, based on a one dimensional Bielecki type metric and norm in order to prove the contractive property of the defined operator. Let us define the integral operator $A : C[a, \sigma(b)]_{\mathbb{T}} \to C[a, \sigma(b)]_{\mathbb{T}}$, by

$$A(u)(t) = w(t) + \int_a^t a_1(s_1)u(s_1)\Delta s_1 + \int_a^t \int_a^{s_1} a_2(s_2)u(s_2)\Delta s_2\Delta s_1$$

$$+ \cdots + \int_a^t \int_a^{s_1} \ldots \int_a^{s_{n-1}} a_n(s_n)u(s_n)\Delta s_n \ldots \Delta s_1,$$

The functions $a_1, a_2, \ldots, a_n \in C[a, \sigma(b)]_{\mathbb{T}}$, so there exist positive real constants $M_1 < \infty, \ldots, M_n < \infty$ such that $|a_1(t)| < M_1, \ldots, |a_n(t)| < M_n$, for all $t \in [a, \sigma(b)]_{\mathbb{T}}$. If $u, v \in C[a, \sigma(b)]_{\mathbb{T}}$, we have

$$|A(u)(t) - A(v)(t)| \leq M_1 \int_a^t |u(s_1) - v(s_1)|\Delta s_1 +$$

$$+ M_2 \int_a^t \int_a^{s_1} |u(s_2) - v(s_2)|\Delta s_2\Delta s_1 + \cdots +$$

$$+ M_n \int_a^t \int_a^{s_1} \ldots \int_a^{s_{n-1}} |u(s_n) - v(s_n)|\Delta s_n \ldots \Delta s_1.$$

$$\leq M_1\|u - v\|_\alpha \frac{e_\alpha(t, a)}{\alpha} + M_2\|u - v\|_\alpha \frac{e_\alpha(t, a)}{\alpha^2} + \cdots + M_n\|u - v\|_\alpha \frac{e_\alpha(t, a)}{\alpha^n}$$

$$= ||u - v||_\alpha e_\alpha(t, a) \left( \frac{M_1}{\alpha} + \frac{M_2}{\alpha^2} + \cdots + \frac{M_n}{\alpha^n} \right).$$

Now we divide the inequality by the positive function $e_\alpha(t, a)$ and taking the supremum over $t \in [a, \sigma(b)]_\mathbb{T}$ we have

$$||A(u) - A(v)||_\alpha \le ||u - v||_\alpha \left( \frac{M_1}{\alpha} + \frac{M_2}{\alpha^2} + \cdots + \frac{M_n}{\alpha^n} \right) \qquad (3.13)$$

Let $M := \max\{M_1, M_2, \ldots, M_n\}$ and so we get

$$||A(u) - A(v)||_\alpha \le ||u - v||_\alpha M \frac{1}{\alpha} \frac{1 - \frac{1}{\alpha^n}}{1 - \frac{1}{\alpha}} \le ||u - v||_\alpha \frac{M}{\alpha - 1}. \qquad (3.14)$$

If $\frac{M}{\alpha-1} < 1$, the operator $A$ is a contraction, and due to Lemma 3.3 we have the c-weakly PO property of $A$. Moreover we also obtain the Ulam-Hyers stability of the integral equation and this implies the Ulam-Hyers stability of the $n^{th}$ order dynamic equation with constant coefficients (because of the boundedness of the interval and of the exponential function $e_a(t, a)$). $\qquad \qquad \square$

We can generalize the Theorem 3.9 in the following way:

**Theorem 3.10.** *Let $\mathbb{T}_1, \mathbb{T}_2, \ldots, \mathbb{T}_n$ be arbitrary time scales and $[a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \in \mathbb{T}_1, \ldots, [a_n, \sigma_n(b_n)]_{\mathbb{T}_n} \in \mathbb{T}_n$ time scale intervals such that $\sigma_1(b_1) < \infty, \ldots, \sigma_n(b_n) < \infty$. Denote $Y := C([a_1, \sigma_1(b_1)]_{\mathbb{T}_1} \times \cdots \times [a_n, \sigma_n(b_n)]_{\mathbb{T}_n}, \mathbb{R})$.*
*If $w, f_1, \ldots, f_n \in Y$, the operator $A : Y \to Y$ defined by*

$$A(u)(t_1, \ldots, t_n) = w(t_1, \ldots, t_n) + \int_{a_1}^{t_1} f_1(s_1, t_2, \ldots, t_n) u(s_1, t_2, \ldots, t_n) \Delta_1 s_1 +$$

$$+ \int_{a_1}^{t_1} \int_{a_2}^{t_2} f_2(s_1, s_2, t_3, \ldots, t_n) u(s_1, s_2, t_3, \ldots, t_n) \Delta_2 s_2 \Delta_1 s_1 + \cdots +$$

$$+ \int_{a_1}^{t_1} \int_{a_2}^{t_2} \ldots \int_{a_n}^{t_n} f_n(s_1, \ldots, s_n) u(s_1, \ldots, s_n) \Delta_n s_n \ldots \Delta_1 s_1$$

*is a c-weakly PO, moreover the fixed point equation $u = A(u)$ is Ulam-Hyers stable on $D_1 \times \cdots \times D_n$, where $D_i = [a_i, \sigma_i(b_i)]_{\mathbb{T}_i}, i = 1, \ldots, n$.*

The proof of this theorem uses a similar argument as the proof of Theorem 3.6 and is based on the norm

$$||u||_{\alpha_1, \ldots, \alpha_n} = \sup_{s_1 \in D_1, \ldots, s_n \in D_n} \frac{||u(s_1, \ldots, s_n)||}{e_{\alpha_1}(s_1, a_1) \ldots e_{\alpha_n}(s_n, a_n)} \qquad (3.15)$$

for all $u \in Y$. For this reason we omit the details.

*3.3. Ulam-Hyers stability of linear delta dynamic systems*

Here we study the Ulam-Hyers stability of linear delta dynamic systems

$$u^\Delta(t) = K(t)u(t) + F(t), \tag{3.16}$$

where $u(t) = (u_1(t), \ldots, u_n(t))^T$, with $u_1, \ldots, u_n : D \to \mathbb{R}$, $K(t) = (k_{ij}(t))_{i,j=1,\ldots,n}$ is a matrix of dimensions $n \times n$, $u^\Delta(t) = \left(u_1^\Delta(t), \ldots, u_n^\Delta(t)\right)^T$, $f_1, \ldots, f_n : D \to \mathbb{R}$ and $F(t) = (f_1(t), \ldots, f_n(t))^T$, on an arbitrary time scale interval $D := [a, \sigma(b)]_\mathbb{T}$. We use the one dimensional version of the Bielecki type metric (3.4) for the function space $C(D, \mathbb{R}^n)$. With this metric $d_\alpha$ the space $X = (C(D, \mathbb{R}^n), d_\alpha)$ is a Banach space.

**Theorem 3.11.** *If the functions $k_{ij}, f_i \in C_{rd}(D, \mathbb{R})$, $\forall i, j = 1, \ldots, n$, then the equation (3.16) is Ulam-Hyers stable on $D$.*

*Proof.* Without loss of generality we can assume, that $u(a) = 0$. Integrating the equation (3.16) from $a$ to $t$ we have

$$u(t) = \int_a^t K(s)u(s)\Delta s + \int_a^t F(s)\Delta s. \tag{3.17}$$

If we define the operator $A : C(D, \mathbb{R}^n) \to C(D, \mathbb{R}^n)$ by

$$A(u)(t) := \int_a^t F(s)\Delta s + \int_a^t K(s)u(s)\Delta s, \tag{3.18}$$

we need to prove, that there exists a positive constant $\alpha$ such that $A$ is a contraction on $X$. $k_{ij} \in C_{rd}(D, \mathbb{R}^n)$ implies that there exists a positive constant $M < \infty$, such that $||K(t)|| \le M, \forall t \in D$, where $|| \cdot ||$ is a matrix norm. If $u, v \in C(D, \mathbb{R}^n)$, we have

$$||A(u)(t) - A(v)(t)|| \le \int_a^t ||K(s)(u(s) - v(s))||\Delta s \le \int_a^t ||K(s)||\,||u(s) - v(s)||\Delta s$$

$$\le M \int_a^t \frac{||u(s) - v(s)||}{e_\alpha(s, a)} e_\alpha(s, a)\Delta s$$

$$\le \frac{M}{\alpha}||u - v||_\alpha e_\alpha(t, a)$$

Dividing the inequality by the positive function $e_\alpha(t, a)$, and taking the supremum over $t \in D$ we have

$$||A(u) - A(v)||_\alpha \le \frac{M}{\alpha}||u - v||_\alpha.$$

If $\frac{M}{\alpha} < 1$, the operator $A$ is a contraction, so by the Lemma 3.3 and Theorem 1.18 we deduce the Ulam-Hyers stability of the fixed point equation $u = A(u)$. Due to the equivalent transformations, and the boundedness of the interval we also have the Ulam-Hyers stability of the equation (3.16). $\square$

*Remark* 3.12. The previous results show that on a bounded intervals the Ulam-Hyers stability can be proved using a unified approach. On the other hand these results may not be relevant on some timescales (such as $\mathbb{Z}$). The theory of Picard operator can also be applied on unbounded intervals, where the generated function spaces are gauge spaces, but in generally we can establish only generalized Ulam-Hyers-Rassias stability and this is not the aim of this paper.

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