ULAM-HYERS STABILITY OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN SOBOLEV SPACES

SZILÁRD ANDRÁS AND ALPÁR RICHÁRD MÉSZÁROS

ABSTRACT. In the present paper we study the Ulam-Hyers stability of some elliptic partial differential equations on bounded domains with Lipschitz boundary. We use direct techniques and also some abstract methods of Picard operators.

The novelty of our approach consists in the fact that we are working in Sobolev spaces and we do not need to know the explicit solutions of the problems or the Green functions of the elliptic operators. We show that in some cases the Ulam-Hyers stability of linear elliptic problems mainly follows from standard estimations for elliptic PDEs, Cauchy-Schwartz and Poincaré type inequalities or Lax-Milgram type theorems.

We obtain powerful results in the sense that working in Sobolev spaces, we can control also the derivatives of the solutions, instead of the known pointwise estimations. Moreover our results for the nonlinear problems generalize in some sense some recent results from the literature (see for example [8]).

1. Introduction

The problem of Ulam-Hyers stability has it roots back in the 1940's, when S. Ulam proposed a question regarding to the stability of group homomorphisms. This problem, among many others can be found in [16] and it is stated as follows:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$ does there exists a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta,\ \forall x, y \in G_1,$$

then there exists a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon,\ \forall x \in G_1?$$

This question of Ulam was answered partially by D. H. Hyers in [4] in the case of approximately additive functions and when the groups in the question are Banach spaces. This work started an avalanche in the theory of stability theory of functional equations, and since then many results have been obtained in this field, studying the Ulam-Hyers stability of ODEs, PDEs, integral equations, etc.

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We should mention also the work of T. M. Rassias, who generalized this notion of stability in [10], proving the Ulam-Hyers-Rassias stability of the Cauchy additive functional equation. We mention also two important papers, of P. Găvruţă [2] and V. Radu [9], which contributed to the development of the generalized Hyers-Ulam-Rassias stability.

A very good and deep insight to this theory can be found in the book [5] and in the two monographs of S.-M. Jung (see [6] and [7]).

On the other hand we can say that most of the methods (also in the above mentioned references) for studying the Ulam-Hyers or Ulam-Hyers-Rassias stability rely on some direct calculation, which cannot be adapted in many cases to other problems. However some results of I. A. Rus solved this question, more precisely he proposed a unified framework for studying this type of stability issues, with the help of the theory of Picard and weakly Picard operators (see [14]). This method is very elegant and powerful in the sense that it is not based on direct calculations, but on an "abstract" operatorial framework which is very general and easier to handle.

In this paper we mainly investigate the question of Ulam-Hyers stability and its generalizations of linear and non-linear elliptic PDEs, given by the following definition. The rigorous definition for the function spaces, the norms and more general elliptic problems will be given later on.

Definition 1.1. Let us consider the problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We say that the above equation is Ulam-Hyers stable, if there exists a positive constant C such that for every $\varepsilon > 0$ and for each solution v of the problem:

$$\begin{cases} -\Delta v = \bar{f}, & \text{in } \Omega \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $||f - \bar{f}|| \leq \varepsilon$, there exist a solution u of the original problem (1.1), such that

$$||u - v|| \leq C \cdot \varepsilon.$$

Remark 1.2. If in the previous definition we have $||u-v|| \leq \psi(\varepsilon)$, where $\psi : [0, \infty) \to [0, \infty)$ is a non-decreasing function with $\psi(0) = 0$, we say that the problem (1.1) is generalized Ulam-Hyers stable. For several other generalized Ulam-Hyers type stabilities see [14].

Our paper is organized as follows. In the Section 2 we present some preliminary results, definitions and the framework of Picard and weakly Picard operators for the study of Ulam-Hyers stability. Moreover we give some preliminary results from [8] concerning the generalized Ulam-Hyers stability of some non-linear elliptic PDEs.

In Section 3 we study linear elliptic PDEs in Sobolev spaces and we show that in many cases the Ulam-Hyers stability is not more than elliptic estimates and it can be obtained from standard inequalities, such as Hölder/Cauchy-Schwartz and Poincaré's inequalities. Moreover we will see that the Lax-Milgram lemma also implies the Ulam-Hyers stability.

Section 4 is devoted to non-linear elliptic problems. Here the main objective is the study of Ulam-Hyers stability of some Laplace equations with non-linearities having Lipschitz or more generalized regularity properties. The results from this section generalize some recent results from [8].

2. Preliminary results

First of all let us present some results from the theory of Picard and weakly Picard operators and their link to the question of Ulam-Hyers stability.

2.1. **Picard, weakly Picard and c-weakly Picard operators and their applications.** The Picard operator technique was used by many authors to study some functional nonlinear integral equations, Grönwall type lemmas, etc. See for example [1], [11]-[14]. We use the terminologies and notations from [11], [12], [13].

Let $(X, \to)$ be an L-space ([12]), $A : X \to X$ an operator. We denote by F_A the set of fixed points of A. We also denote $A^0 := 1_X$, $A^1 := A, \dots, A^{n+1} := A^n \circ A$, $n \in \mathbb{N}$ the iterate operators of the operator A. We also define $A^\infty(x) := \lim_{n\to\infty} A^n(x)$.

Definition 2.1 ([14])**.** By definition $A : X \to X$ is weakly Picard operator if the sequence of successive approximations, $A^n(x)$, converges for all $x \in X$ and the limit (which may be depend on x) is a fixed point of A.

Definition 2.2 ([11], [12], [13])**.** A is a Picard operator (briefly PO), if there exists $x_A^* \in X$ such that:
(i) $F_A = \{x_A^*\}$;
(ii) $A^n(x) \to x_A^*$ as $n \to \infty, \forall x \in X$.
Equivalently we can say, that if for a weakly Picard operator $A : X \to X$ $F_A = \{x_A^*\}$, then A is a PO.

The following class of weakly Picard operators is very important in the study of generalized Ulam-Hyers stability. Let (X, d) be a metric space.

Definition 2.3 ([14])**.** The $A : X \to X$ be a weakly Picard operator is said to be *ψ-weakly Picard operator* if

$$d(x, A^\infty(x)) \leq \psi(d(x, A(x))), \text{for all x} \in \text{X},$$

for a $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing function which is continuous in 0 and $\psi(0) = 0$. In the case when $\psi(t) = ct$ for a $c > 0$ a real number, we say that A is c-weakly Picard operator.

Analogously to the notion of Ulam-Hyers stability in the theory of functional equations we can define what we means by the Ulam-Hyers stability of a fixed point equation

Definition 2.4 ([14])**.** Let (X, d) be a metric space and $A : X \to X$ be an operator. By definition, the fixed point equation

$$x = A(x) \tag{2.1}$$

is said to be Ulam-Hyers stable if there exists a real number $c_A > 0$ such that: for each $\varepsilon > 0$ real number and each solution y^* of the inequality

$$d(y, A(y)) \leq \varepsilon,$$

there exists a solution x^* of the equation (2.1) such that

$$d(y^*, x^*) \leq c_A \cdot \varepsilon. \tag{2.2}$$

Remark 2.5. We remark that we can speak about *generalized Ulam-Hyers stability*, if in the above definition we change the inequality (2.2) by

$$d(y^*, x^*) \leq \psi(\varepsilon),$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function, which is continuous in 0 and $\psi(0) = 0$. It is obvious, that if $\psi(t) = ct$ for some $c > 0$, we get back the usual notion of Ulam-Hyers stability.

The generalized Ulam-Hyers stability is linked to the comparison functions, here is the definition of these.

Definition 2.6 ([13]). An increasing function $\varphi : [0, +\infty) \to [0, +\infty)$ is said to be a *comparison function*, if $\varphi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$. As a consequence we also have that $\varphi(t) < t$, for all $t > 0$, $\varphi(0) = 0$ and φ is continuous in 0.

A function $\varphi : [0, +\infty) \to [0, +\infty)$ is said to be a *strict comparison function*, if it is strictly increasing and

$$\sum_{n \geq 1} \varphi^n(t) < +\infty,$$

for all $t > 0$.

Two examples of strict comparison functions are $\varphi_1, \varphi_2 : [0, +\infty) \to [0, +\infty)$, $\varphi_1(t) = at$, for $a \in [0, 1)$ and $\varphi_2(t) = \frac{t}{1+t}$.

Now let us present the link between the theory of Picard operators and the Ulam-Hyers stability.

Theorem 2.7 ([14]). *Let (X, d) be a metric space. If $A : X \to X$ is a ψ-weakly Picard operator, then the fixed point equation* (2.1) *is generalized Ulam-Hyers stable.*

Remark 2.8. The power of this method relies in this main characterization, because if it is possible to define a fixed point equation from the functional equation we want to study, then if the defined operator is c-weakly Picard we have immediately the Ulam-Hyers stability of the desired equation. Hence by equivalent transformation, the stability of the primal problem.

Another property is the following.

Lemma 2.9 ([13, 15]). *Let (X, d) be a Banach space and $A : X \to X$ a ϕ-contraction, which is*

$$d(A(x), A(y)) \leq \phi(d(x, y)), \text{ for all } x, y \in X.$$

Then A is a PO. If we additionally suppose that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+, \psi(t) := t - \phi(t)$ is strictly increasing and onto, then the operator A is ψ-weakly PO, and we have that

$$d(x, x^*) \leq \psi^{-1}(d(x, A(x))), \forall x \in X,$$

where x^* is the unique fixed point of A.

Remark 2.10. If the operator $A : X \to X$ is a contraction with the positive constant $q < 1$, then A is c-weakly Picard operator with the positive constant $c_A = \frac{1}{1-q}$. Moreover the fixed point equation (2.1) is Ulam-Hyers stable.

2.2. **Preliminary results for non-linear problems.** Here we would like to present some preliminary results regarding to the stability of classical solutions of some non-linear elliptic PDEs, due to V. L. Lazăr in [8].

The main result from [8] reads as follows. Consider the following problem.

$$\begin{cases} \Delta u = f(x, u(x)), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where f in a continuous function on $\overline{\Omega} \times \mathbb{R}$.

Theorem 2.11 ([8]). *Let Ω be a bounded domain in $\mathbb{R}^d$ such that its border $\partial\Omega$ is sufficiently smooth. Denote by G the usual Green function corresponding to the Laplace operator. Suppose that:*

(i) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$;

(ii) *there exists $p \in C(\overline{\Omega}, [0, +\infty))$ with*

$$\sup_{x\in\overline{\Omega}} \int_{\Omega} G(x,s)p(s)ds \leq 1$$

and a comparison function $\varphi : [0, +\infty) \to [0, +\infty)$ such that for each $s \in \overline{\Omega}$ and each $u, v \in \mathbb{R}$ we have that

$$|f(s,u) - f(s,v)| \leq p(s)\varphi(|u-v|).$$

Then the problem (2.3) *has a unique solution $u^* \in C(\overline{\Omega}, \mathbb{R})$. Moreover if the function $\psi : [0, +\infty) \to [0, +\infty)$, $\psi(t) := t - \varphi(t)$ is increasing and onto, then the problem* (2.3) *is generalized Ulam-Hyers stable with the function ψ^{-1}, i.e. for each $\varepsilon > 0$ and for each ε-solution y^* of the problem* (2.3) *we have that*

$$|u^*(x) - y^*(x)| \leq \psi^{-1}(\varepsilon),$$

for all $x \in \overline{\Omega}$.

3. Ulam-Hyers stability of linear elliptic PDEs

In what follows, if we do not state otherwise $\Omega \subset \mathbb{R}^d$ is a bounded, open domain with Lipschitz boundary. First present the Ulam-Hyers stability of a model problem, namely Poisson's problem, with homogeneous boundary condition. The result is the following.

Theorem 3.1. *Let $f \in H^{-1}(\Omega)$. Then Poisson's problem with Dirichlet boundary conditions*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

is Ulam-Hyers stable with respect to weak solutions in $H_0^1(\Omega)$.

Now we spend some time presenting in details two possibilities for the proof of this theorem, to see the advantages and disadvantages of both methods.

First proof, with fixed point approach. The idea of the proof is to construct a fixed point equation as (2.1), then try to prove, that the operator is c-weakly Picard, finally we can conclude by the Theorem 2.7 the Ulam-Hyers stability of the desired problem.

At first we remark, that it is well known due to Riesz's representation theorem, that the problem (3.1) has a unique weak solution in $H_0^1(\Omega)$.

We define an abstract operator as follows. Let $A : H_0^1(\Omega) \to H_0^1(\Omega)$, which associates to an input $v \in H_0^1(\Omega)$ the unique weak solution of the modified problem

$$\begin{cases} -\Delta u + \lambda u = f + \lambda v, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

where $\lambda > 0$ is a positive real constant that will be chosen small enough. We give the precise assumptions on λ later.

By Riesz's representation theorem it is obvious again, that for any input $v \in H_0^1(\Omega)$ the problem (3.2) has a unique weak solution in $H_0^1(\Omega)$, due to the fact that the right hand side is an element in $H^{-1}(\Omega)$ (because for bounded domains, we can use a Sobolev embedding theorem, since there exists a continuous and compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and $L^2(\Omega) \subset H^{-1}(\Omega)$.) Hence the operator A is well defined.

Next we are going to prove, that A is a c-weakly Picard operator. Actually we can prove a stronger fact, namely that it is a contraction.

Let us write the weak formulation of the problem (3.2) and prove the claim. The problem (3.2) is equivalent to

(3.3)
$$\int_\Omega \nabla u(x) \cdot \nabla \varphi(x) dx + \lambda \int_\Omega u(x)\varphi(x) dx = \langle f, \varphi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \lambda \int_\Omega v(x)\varphi(x) dx,$$

$\forall \varphi \in H_0^1(\Omega)$.

Now we take two input functions $v_1, v_2 \in H_0^1(\Omega)$, and by definition let $u_i = A(v_i), i = 1, 2$ the unique weak solutions of (3.2). Our aim is to give an approximation to $||u_1 - u_2||_{H_0^1}$.

We write the corresponding weak formulations for v_1, u_1 and v_2, u_2 and take the difference. Without the loss of the generality we can use the same test function $\varphi \in H_0^1(\Omega)$ in both formulations. Hence we will have:

$$\int_\Omega \nabla(u_1 - u_2)(x) \cdot \nabla \varphi(x) dx + \lambda \int_\Omega (u_1 - u_2)(x)\varphi(x) dx = \lambda \int_\Omega (v_1 - v_2)(x)\varphi(x) dx,$$

$\forall \varphi \in H_0^1(\Omega)$.

Now let $\varphi = u_1 - u_2$, so we have

$$\int_\Omega |\nabla(u_1 - u_2)|^2(x) dx + \lambda \int_\Omega |u_1 - u_2|^2(x) dx = \lambda \int_\Omega (v_1 - v_2)(x)(u_1 - u_2)(x) dx.$$

This implies, that

$$||u_1 - u_2||^2_{H_0^1} \leq \lambda \int_\Omega (v_1 - v_2)(x)(u_1 - u_2)(x) dx.$$

Using the Cauchy-Schwarz inequality for the right hand side, we have

$$||u_1 - u_2||^2_{H_0^1} \leq \lambda ||v_1 - v_2||_{L^2} ||u_1 - u_2||_{L^2}.$$

Using twice Poincaré's inequality we have

$$||u_1 - u_2||^2_{H_0^1} \leq \lambda \cdot C_\Omega \cdot C_\Omega ||v_1 - v_2||_{H_0^1} ||u_1 - u_2||_{H_0^1},$$

where $C_\Omega > 0$ is the positive Poincaré constant, depending only on the geometry of Ω, and its value is actually $C_\Omega = \frac{1}{\lambda_1}$, where $\lambda_1 > 0$ is the least eigenvalue of $-\Delta$ on $H_0^1(\Omega)$.

Now dividing the inequality by $||u_1 - u_2||_{H_0^1}$, we have

$$||A(v_1) - A(v_2)||_{H_0^1} = ||u_1 - u_2||_{H_0^1} \leq \lambda \cdot C_\Omega^2 ||v_1 - v_2||_{H_0^1}.$$

So if we choose $\lambda < \frac{1}{C_\Omega^2} = \lambda_1^2$, then the operator A is a contraction.

So by the Theorem 2.7 and Theorem 2.9 we have the Ulam-Hyers stability of (3.2), hence the Ulam-Hyers stability of (3.1). $\square$

Second proof using elliptic estimations. Let $\varepsilon > 0$ be fixed. We consider the problem

$$\begin{cases} -\Delta v = g, & \text{in } \Omega \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $g \in H^{-1}(\Omega)$ and $||f - g||_{H^{-1}} \leq \varepsilon$. We know that the solution v exists and is unique in $H_0^1(\Omega)$. The same is true for the original problem (3.1), with the solution u. We want to show, that there exists a positive constant $C > 0$, independent of ε, such that

$$||u - v||_{H_0^1} \leq C\varepsilon.$$

We subtract the two equations and write the weak formulation of this equation, then take as test function $u - v \in H_0^1(\Omega)$. So we have

$$\int_\Omega |\nabla(u - v)|^2 \leq \langle f - g, u - v\rangle_{H^{-1}\times H_0^1}.$$

Now using an estimation for the right hand side, we will have

$$||u - v||_{H_0^1} \leq \varepsilon.$$

So the original problem is Ulam-Hyers stable. □

We can see that these type of argumentations in the previous proofs are very elementary and immediately. So this can be adapted easily to many other problems. The proofs of the following results are straightforward and based on the same ideas and methods, so we just state the results without the proofs.

Theorem 3.2. *Let $f \in H^{-1}(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$. Then Poisson's problem with Dirichlet boundary conditions*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

is Ulam-Hyers stable with respect to weak solutions in $H^1(\Omega)$.

Theorem 3.3. *Let $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega.)$ Then the problem*

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g, & \text{on } \partial\Omega \end{cases} \tag{3.6}$$

is Ulam-Hyers stable with respect to weak solutions in $H^1(\Omega)/\mathbb{R}$ if and only if

$$\int_\Omega f(x)dx + \int_{\partial\Omega} g(\sigma)d\sigma = 0. \tag{3.7}$$

Here we denote by $\int_{\partial\Omega} g(\sigma)d\sigma$ the $d-1$ dimensional surface integral of the function g in the Lebesgue sense.

Now let us state a result for the Ulam-Hyers stability of more general linear elliptic PDEs. For this first of all let us state the well-known Lax-Milgram lemma.

Lemma 3.4 (Lax-Milgram lemma). *Let X be a Hilbert space and $a(\cdot, \cdot)$ a bilinear form on X, which is*

(1) *continuous:* $|a(u, v)| \leq C||u||_X||v||_X, \forall u, v \in X$;
(2) *coercive:* $|a(u, u)| \geq c||u||_X^2, \forall u \in X$.

Then for any $f \in X^*$ there is a unique solution $u \in X$ to the equation

$$a(u, v) = f(v) \tag{3.8}$$

and it holds

$$||u||_X \leq \frac{1}{c}||f||_{X^*} \tag{3.9}$$

The following observation allows us to deduce the Ulam-Hyers stability of a general class of linear elliptic PDEs. The proof of it is obvious, so we omit it.

Theorem 3.5. *Each problem in a Hilbert space, which can be written as problem* (3.8) *is Ulam-Hyers stable due to the estimation* (3.9) *from the Lemma 3.4.*

4. Ulam-Hyers stability of non-linear elliptic PDEs

In this section we would like to study the Ulam-Hyers stability of some nonlinear elliptic problems. Let us consider the following model problem:

$$\begin{cases} -\Delta u(x) = f(x, u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

We remark that the equation is written for a.e. $x \in \Omega$. We study the Ulam-Hyers stability of this problem, assuming some conditions on the function f.

Theorem 4.1. *If for $f : \Omega \times \mathbb{R} \to \mathbb{R}$ there exists a non-decreasing function φ such that $[0, +\infty) \ni t \mapsto C_\Omega \varphi(C_\Omega t)$ is a comparison function, $\psi(t) := t - C_\Omega \varphi(C_\Omega t)$ is strictly increasing and onto and*

$$||f(\cdot, v) - f(\cdot, w)||_{L^2} \leq \varphi\left(||v - w||_{L^2}\right), \forall v, w \in H_0^1(\Omega), \tag{4.2}$$

then the problem (4.1) *is generalized Ulam-Hyers stable in $H_0^1(\Omega)$ with the function ψ, where C_Ω is the Poincaré constant of the domain Ω.*

Proof. We use some ideas as before in the case of linear problems. For a $v \in H_0^1(\Omega)$ let us define the problem

$$\begin{cases} -\Delta u(x) = f(x, v), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Let us take an operator $A : H_0^1(\Omega) \to H_0^1(\Omega)$ which associates for an input $v \in H_0^1(\Omega)$ the unique solution of the problem (4.3). So we transformed our elliptic problem to a fixed point problem. If we can prove that A is a ψ-weakly PO, for a comparison function ψ, then we obtain the generalized Ulam-Hyers stability of the fixed point problem and hence the Ulam-Hyers stability of the original elliptic problem.

Let us take two inputs $v_1, v_2 \in H_0^1(\Omega)$ and let us denote the corresponding solutions by $u_i := A(v_i), i = 1, 2$.

Writing the weak formulations and taking the difference, we have

$$\int_\Omega \nabla(u_1 - u_2) \cdot \nabla\varphi dx = \int_\Omega \Big(f(x, v_1) - f(x, v_2)\Big)\varphi dx, \tag{4.4}$$

$\forall \varphi \in H_0^1(\Omega)$.

We choose $\varphi := u_1 - u_2$ and we will have:

$$\begin{aligned} ||A(v_1) - A(v_2)||^2_{H_0^1} &= ||u_1 - u_2||^2_{H_0^1} \leq C_\Omega ||f(\cdot, v_1) - f(\cdot, v_2)||_{L^2} ||u_1 - u_2||_{H_0^1} \\ &\leq C_\Omega \varphi(||v_1 - v_2||_{L^2}) ||u_1 - u_2||_{H_0^1} \end{aligned}$$

$\leq C_\Omega \varphi(C_\Omega ||v_1 - v_2||_{H_0^1}) ||u_1 - u_2||_{H_0^1}$

Now dividing the above inequality by the positive quantity $||u_1 - u_2||_{H_0^1}$ (we have that $u_1 \neq u_2$ almost everywhere, if $v_1 \neq v_2$ almost everywhere)

From here we have that the operator A is ψ-weakly PO, where $\psi(t) = C_\Omega \varphi(C_\Omega t)$, hence by the Lemma 2.9 we have the desired property. □

5. Some concluding remarks

Remark 5.1. Let us take a strictly increasing and onto function $\psi : [0, +\infty) \to [0, +\infty)$, such that $0 < \psi'(t) < 1$, if $C_\Omega > 1$ and $1 - C_\Omega < \psi'(t) < 1$, if $0 < C_\Omega < 1$ for all $t > 0$. If we define the function $\varphi : [0, +\infty) \to [0, +\infty)$, $\varphi(t) = \frac{1}{C_\Omega^2} - \frac{1}{C_\Omega}\psi\left(\frac{1}{C_\Omega} t\right)$, this φ and ψ will satisfy the assumptions of the Theorem 4.1.

Remark 5.2. In the particular case when $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in the second variable, with a constant $0 < c < \frac{1}{C_\Omega^2}$, where C_Ω is the Poincaré constant, then the problem (4.1) is Ulam-Hyers stable in $H_0^1(\Omega)$.

Remark 5.3. We will have the same result, if we replace f by an operator $F : H_0^1(\Omega) \to L^2(\Omega)$ with the property that

$$||F(u) - F(v)||_{L^2} \leq c||u - v||_{H_0^1}, \forall u, v \in H_0^1(\Omega),$$

for a given $0 < c < \frac{1}{C_\Omega}$. The only difference is the condition for the constant c, because in this case we have to use just once Poincaré's inequality.

Remark 5.4. In comparison with the result (2.11) (from [8]), we can say that in some cases our result is more powerful. This is motivated by the following two observations:

- we obtain generalized Ulam-Hyers stability in the space $H_0^1(\Omega)$, which is better than a point-wise estimation because we control also the derivative of the solutions.
- Because of the fact that for the Poincaré constant C_Ω we have that $\frac{1}{C_\Omega^2} = \lambda_1$, where λ_1 is the least eigenvalue of the operator $-\Delta$ on $H_0^1(\Omega)$, we obtain the relation

$$C_\Omega^2 \leq ||G||_{L^2(\Omega \times \Omega)} \leq |\Omega| \sup_{x \in \overline{\Omega}} \int_\Omega G(x, \xi) d\xi,$$

  where G is the Green function of $-\Delta$ on Ω.

  Taking a concrete example for f, $f(x, s) = cs$ for a $c > 0$, using the Theorem 2.11, we immediately deduce that $p \equiv 1$ and $\varphi(t) = ct$, with $c < 1$. This implies that the assumption is satisfied if $\sup_{x \in \overline{\Omega}} \int_\Omega G(x, \xi) d\xi \leq 1$.

  Using the estimation between the Poincaré constant and the norm of the Green function, we see that using Theorem 2.11 we should have that $C_\Omega^2 \leq |\Omega|$, which is that $\frac{1}{C_\Omega^2} \geq \frac{1}{|\Omega|}$, or $\lambda_1 \geq \frac{1}{|\Omega|}$.

  By the celebrated Faber-Krahn inequality we know that in 2 dimensions we have

$$\lambda_1 \geq \frac{\pi j_{0,1}^2}{|\Omega|},$$

  where $j_{0,1}$ is the first positive zero of the Bessel function J_0 and $j_{0,1} \approx 2.4048$. From here it is clear that $\lambda_1 \geq \frac{1}{|\Omega|}$ is always satisfied. In [8] we have the restriction $c < 1$ for the constant, while by our approach we needed to have $c < \lambda_1$, which of course covers more cases, if the area of Ω is small enough (in order to have $\frac{\pi j_{0,1}^2}{|\Omega|} > 1$).

Remark 5.5. The recent article [3], mentioned by the referee, is related to our work, however its approach is different from ours.

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Department of Mathematics, Babeş-Bolyai University, Cluj-Napoca, Romania
E-mail address, Sz. András: `andrasz@math.ubbcluj.ro`

Laboratoire de Mathématiques d'Orsay, University of Paris-Sud, Orsay, France
E-mail address, A. R. Mészáros: `alpar.meszaros@math.u-psud.fr`