ON THE SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS AND BESSEL-TYPE SPECIAL FUNCTIONS ON THE LEVI-CIVITA FIELD

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Abstract. Because of the disconnectedness of a non-Archimedean ordered field in the topology induced by the order, it is possible to have non-constant functions with zero derivatives everywhere. In fact the solution space of the differential equation $y' = 0$ is infinite dimensional [11]. In this paper, we give sufficient conditions for a function on an open subset of the Levi-Civita field to have zero derivative everywhere and we use the nonconstant zero-derivative functions to obtain non-analytic solutions of systems of linear ordinary differential equations with analytic coefficients. Then we use the results to introduce Bessel-type special functions on the Levi-Civita field and we study some of their properties.

1. Introduction

Solutions of linear ordinary differential equations and some Bessel-type special functions on the Levi-Civita field $\mathcal{R}$ [5, 6] are presented. We recall that the elements of $\mathcal{R}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

Definition 1. *(λ, $\sim$, $\approx$, $=_r$) For $x \neq 0$ in $\mathcal{R}$, we let $\lambda(x) = \min(supp(x))$, which exists because of the left-finiteness of $supp(x)$; and we let $\lambda(0) = +\infty$.*

Given $x, y \neq 0$ in $\mathcal{R}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; and we say $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$.

Given $x, y \in \mathcal{R}$ and $r \in \mathbb{R}$, we say $x =_r y$ if $x[q] = y[q]$ for all $q \leq r$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The set $\mathcal{R}$ is endowed with formal power series multiplication and componentwise addition, which make it into a field [3] in which we can isomorphically embed $\mathbb{R}$ as

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a subfield via the map $\Pi : \mathbb{R} \to \mathcal{R}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

Definition 2. *(Order in $\mathcal{R}$) Let $x, y \in \mathcal{R}$ be given. Then we say $x \geq y$ if $x = y$ or $[x \neq y$ and $(x - y)[\lambda(x - y)] > 0]$.*

It is easy to check that the relation $\geq$ is a total order and $(\mathcal{R}, +, \cdot, \geq)$ is an ordered field (which will be denoted henceforth, simply, by $\mathcal{R}$). Moreover, the embedding Π in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order. The order induces an absolute value on $\mathcal{R}$ in the natural way: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. We also note here that λ, as defined above, is a valuation; moreover, the relation $\sim$ is an equivalence relation, and the set of equivalence classes (the value group) is (isomorphic to) $\mathbb{Q}$.

Besides the usual order relations, some other notations are convenient.

Definition 3. *($\ll, \gg$) Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.*

Definition 4. *(The Number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.*

It is easy to check that $d^q \ll 1$ if $q > 0$ and $d^q \gg 1$ if $q < 0$. Moreover, for all $x \in \mathcal{R}$, the elements of $\mathrm{supp}(x)$ can be arranged in ascending order, say $\mathrm{supp}(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_{j=1}^{\infty} x[q_j]d^{q_j}$, where the series converges in the topology induced by the absolute value [3].

Altogether, it follows that $\mathcal{R}$ is a non-Archimedean field extension of $\mathbb{R}$. For a detailed study of this field, we refer the reader to [10, 20] and the references therein. In particular, it is shown that $\mathcal{R}$ is complete with respect to the topology induced by the absolute value; that is, every Cauchy sequence of elements of $\mathcal{R}$ converges to an element of $\mathcal{R}$. In the wider context of valuation theory, it is interesting to note that the topology induced by the absolute value is the same as that introduced via the valuation λ, as was shown in [19].

It follows therefore that the field $\mathcal{R}$ is just a special case of the class of fields discussed in [9]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [8], and for an overview of the related valuation theory to the books by Krull [4], Schikhof [9] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [7].

Besides being the smallest ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and real closed, the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer [3]. Having infinitely small numbers, the errors in classical numerical methods can be made infinitely small and hence irrelevant in all practical applications. One such application is the computation of derivatives of real functions representable on a computer [16], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

In this paper we present some tools to construct a large class of solutions of $y' = 0$ on $\mathcal{R}$. Then as an application of that, we define and study the properties of Bessel-type special functions on (open subsets of) $\mathcal{R}$.

2. Preliminary results

2.1. **Matrix exponentials on $\mathcal{R}$.** For an easier study of systems of linear ordinary differential equations on $\mathcal{R}$, it is beneficial to introduce matrix exponentials on $\mathcal{R}$. We define matrices on $\mathcal{R}$ and matrix operations: addition, multiplication, determinant, just as we do in the real case.

Definition 5. *Let $\mathcal{M}_n(\mathcal{R})$ denote the set of all $n \times n$ matrices with entries in $\mathcal{R}$. For $A \in \mathcal{M}_n(\mathcal{R})$, we define $|\cdot| : \mathcal{M}_n(\mathcal{R}) \to \mathcal{R}$ by*

$$|A| = \max_{1 \le i,j \le n} \{|a_{ij}|\},$$

where $|a_{ij}| = \max\{a_{ij}, -a_{ij}\}$.

In the following we will deal only with square matrices whose entries are at most finite in absolute value; we will denote this class of matrices by $\mathcal{M}_n^f(\mathcal{R})$.

Definition 6. *Let $A \in \mathcal{M}_n^f(\mathcal{R})$, and for each $k \in \mathbb{N} \cup \{0\}$ let $c_k \in \mathcal{R}$ be given. We say that the series $\sum_{k=0}^{\infty} c_k A^k$ is convergent in $\mathcal{M}_n^f(\mathcal{R})$ if the series $\sum_{k=0}^{\infty} c_k |A^k|$ is convergent in $\mathcal{R}$ with respect to the weak topology discussed in* [3, 17, 12].

Definition 7. *Let $A \in \mathcal{M}_n^f(\mathcal{R})$ be given. We define the exponential of A by the series*

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k,$$

where A^0 is the $n \times n$ identity matrix I_n.

In the next theorem we will show that the series in Definition 7 converges in the sense of Definition 6.

Theorem 1. *For any $A \in \mathcal{M}_n^f(\mathcal{R})$, the series*

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

is always convergent; thus, e^A is well-defined.

Proof. Let $A = (a_{ij})$; and let $A^2 = (b_{ij})$. Then, by the way we perform matrix multiplication, we have that

$$|b_{ij}| \le n \left(\max_{1 \le i,j \le n} \{|a_{ij}|\} \right) \left(\max_{1 \le i,j \le n} \{|a_{ij}|\} \right) = n|A|^2,$$

for all $i, j \in \{1, \dots, n\}$. It follows that

$$\left|A^2\right| = \max_{1 \le i,j \le n} \{|b_{ij}|\} \le n|A|^2.$$

Using induction on k, it is then easy to show that

$$\left|A^k\right| \le n^{k-1} |A|^k, \forall k \in \mathbb{N}. \tag{2.1}$$

Since
$$\sum_{k=1}^{\infty} \frac{n^{k-1}}{k!}|A|^k = \frac{1}{n}\sum_{k=1}^{\infty}\frac{n^k}{k!}|A|^k = \frac{1}{n}\left(\sum_{k=1}^{\infty}\frac{1}{k!}(n|A|)^k\right)$$
converges in the weak topology of $\mathcal{R}$ to $\left(e^{n|A|}-1\right)/n$ (since $|A|$ is at most finite), it follows from Equation (2.1) and the properties of weak convergence of infinite series [17, 12] that
$$\sum_{k=0}^{\infty}\frac{1}{k!}\left|A^k\right| = 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left|A^k\right|$$
converges weakly in $\mathcal{R}$ and hence
$$\sum_{k=0}^{\infty}\frac{1}{k!}A^k$$
converges in the sense of Definition 6. □

Because power series on $\mathcal{R}$ can be differentiated term by term within their domain of convergence [19], we obtain the following theorem.

Theorem 2. *Let $A \in \mathcal{M}_n^f(\mathcal{R})$ and let $F : \mathcal{R} \to \mathcal{M}_n(\mathcal{R})$ be given by $F(t) = e^{tA}$. Then F is differentiable at each $t \in \mathcal{R}$, with derivative*
$$F'(t) = Ae^{tA}.$$

Proof. Since $F(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}A^k$, we obtain $F'(t)$ by differentiating the series term by term as a function of t:

$$F'(t) = \sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}A^k = \sum_{k=0}^{\infty}\frac{t^k}{k!}A^{k+1} = A\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k = Ae^{tA} = AF(t).$$

All series in the equation above are well-defined.

□

3. Main results

3.1. **Linear ordinary differential equations on $\mathcal{R}$.** One of the main goals of this paper is to obtain solutions of linear ordinary differential equations when the coefficients are analytic functions of the independent variable, including non-analytic solutions in addition to the analytic ones.

The basic idea of forming the non-analytic solutions of linear ordinary differential equations on $\mathcal{R}$ is based on the following theorem (proved in [11]) which shows that even the simplest differential equation $y' = 0$ over $\mathcal{R}$ has infinitely many linearly independent solutions on $[-1, 1] \subset \mathcal{R}$.

Theorem 3. *The solution space of the differential equation $y' = 0$ on $[-1, 1]$ is infinite dimensional.*

Proof. For each $n \in \mathbb{N}$, let $g_n : [-1, 1] \to \mathcal{R}$ be given by $g_n(x)[q] = x[q/(n+1)]$. Then we show that, for all $n \in \mathbb{N}$, g_n is differentiable on $[-1, 1]$ with $g_n'(x) = 0$ for all $x \in [-1, 1]$. So let $n \in \mathbb{N}$ be given. We first observe that $g_n(x+y) = g_n(x)+g_n(y)$ for all $x, y \in [-1,1]$. Now let $x \in [-1,1]$ and $\epsilon > 0$ in $\mathcal{R}$ be given. Let $\delta = \min\{\epsilon^2, d\}$, and let $y \in [-1,1]$ be such that $0 < |y-x| < \delta$. Then

$$\left|\frac{g_n(y)-g_n(x)}{y-x}\right| = \left|\frac{g_n(y-x)}{y-x}\right| \sim |y-x|^n \text{ since } g(y-x) \sim (y-x)^{n+1}.$$

Since $|y-x| < \min\{\epsilon^2, d\}$, we obtain that $|y-x|^n \ll \epsilon$. Hence

$$\left|\frac{g_n(y)-g_n(x)}{y-x}\right| < \epsilon \text{ for all } y \in [-1,1] \text{ satisfying } 0 < |y-x| < \delta.$$

It follows that g_n is differentiable at x, with $g_n'(x) = 0$. This is true for all $x \in [-1,1]$ and for all $n \in \mathbb{N}$. Hence g_n is a solution of the differential equation $y' = 0$ on $[-1,1]$ for all $n \in \mathbb{N}$. Next we show that the set $S = \{g_n : n \in \mathbb{N}\}$ is linearly independent on $[-1,1]$. So let $j \in \mathbb{N}$ and let $n_1 < n_2 < \cdots < n_j$ in $\mathbb{N}$ be given. We show that $g_{n_1}, g_{n_2}, \ldots, g_{n_j}$ are linearly independent on $[-1,1]$. So suppose that $c_1 g_{n_1} + c_2 g_{n_2} + \cdots + c_j g_{n_j} = 0$ for some $c_1, c_2, \ldots, c_j$ in $\mathcal{R}$; we show that $c_1 = c_2 = \cdots = c_j = 0$. Since $c_1 g_{n_1} + c_2 g_{n_2} + \cdots + c_j g_{n_j} = 0$, we obtain that $c_1 g_{n_1}(d) + c_2 g_{n_2}(d) + \cdots + c_j g_{n_j}(d) = 0$. Hence $c_1 d^{n_1} + c_2 d^{n_2} + \cdots + c_j d^{n_j} = 0$; from which we infer that $c_1 = c_2 = \cdots = c_j = 0$. □

Remark 1. *For each $n \in \mathbb{N}$, it is easy to check that the mapping g_n in the proof of Theorem 3 is an order preserving field automorphism of $\mathcal{R}$; this is a special property of non-Archimedean structures since it is well-known that the only field automorphism of $\mathbb{R}$ is the identity map* [13].

In Proposition 1, Proposition 2 and Proposition 3, we will give sufficient conditions for a nonconstant function to be the solution of the differential equation $y' = 0$ on an open subset of $\mathcal{R}$.

Proposition 1. *Let $M \subseteq \mathcal{R}$ be open and let $f : M \to \mathcal{R}$ be such that, for some fixed $p > 1$ in $\mathbb{Q}$ and for some positive $\eta \ll 1$ in $\mathcal{R}$, we have that*

$$\forall x, y \in M, \lambda(x-y) \geq \lambda(\eta) \Rightarrow |f(x) - f(y)| \sim |x-y|^p.$$

Then f is differentiable on M with derivative $f'(x) = 0$ for all $x \in M$; that is, f is a solution for the differential equation $y' = 0$ on M.

Proof. Let $x \in M$ and $\epsilon > 0$ in $\mathcal{R}$ be given. Since M is open, there exists $\delta_0 > 0$ in $\mathcal{R}$ such that $(x - \delta_0, x + \delta_0) \subset M$. Let

$$\delta = \min\{\delta_0, \epsilon^{\frac{2}{p-1}}, \eta\}.$$

Then for all $y \in \mathcal{R}$ satisfying $0 < |y-x| < \delta$, we have that $y \in M$ and $\lambda(y-x) \geq \lambda(\delta) \geq \lambda(\eta)$; and hence

$$\begin{aligned} \left|\frac{f(x)-f(y)}{x-y}\right| &\sim |x-y|^{p-1} < \delta^{p-1} \\ &= \min\left\{\delta_0^{p-1}, \epsilon^2, \eta^{p-1}\right\} \\ &\leq \min\left\{\epsilon^2, \eta^{p-1}\right\} \\ &\ll \epsilon. \end{aligned}$$

The last step is justified by the fact that if $\epsilon \ll 1$ then $\epsilon^2 \ll \epsilon$ and if ϵ is finite or infinitely large then $\eta^{p-1} \ll \epsilon$ since $\eta \ll 1$ and $p - 1 > 0$. So in both cases, we obtain that $\min\{\epsilon^2, \eta^{p-1}\} \ll \epsilon$. Thus, f is differentiable at x, for all $x \in M$, with $f'(x) = 0$. □

Proposition 2. *Let $M \subseteq \mathcal{R}$ be open and let $f : M \to \mathcal{R}$ be such that, for some fixed $p > 1$ in $\mathbb{Q}$ and for some positive $\eta \ll 1$ and positive α in $\mathcal{R}$, we have that*

$$\forall x, y \in M, \lambda(x-y) \geq \lambda(\eta) \Rightarrow |f(x) - f(y)| \leq \alpha |x-y|^p.$$

Then f is a solution for the differential equation $y' = 0$ on M.

Proof. Let $x \in M$ and $\epsilon > 0$ in $\mathcal{R}$ be given. Then there exists $\delta_0 > 0$ in $\mathcal{R}$ such that $(x - \delta_0, x + \delta_0) \subset M$. Let

$$\delta = \min \left\{ \delta_0, \eta, \left(\frac{\epsilon^2}{\alpha} \right)^{1/(p-1)}, \left(\frac{\eta}{\alpha} \right)^{1/(p-1)} \right\}.$$

Then $\delta > 0$ and for $y \in \mathcal{R}$ satisfying $0 < |y - x| < \delta$, we have that $y \in M$ and $\lambda(y - x) \geq \lambda(\delta) \geq \lambda(\eta)$; and hence

$$\begin{aligned} \left| \frac{f(x) - f(y)}{x - y} \right| &\leq \alpha |x-y|^{p-1} < \alpha \delta^{p-1} \\ &= \min \left\{ \alpha \delta_0^{p-1}, \alpha \eta^{p-1}, \epsilon^2, \eta \right\} \\ &\leq \min \left\{ \epsilon^2, \eta \right\} \\ &\ll \epsilon. \end{aligned}$$

This shows that f is differentiable at x, for all $x \in M$, with $f'(x) = 0$. □

Remark 2. *We note that Proposition 1 follows from Proposition 2 if we take α to be any infinitely large positive number.*

Definition 8. *Let $M \subseteq \mathcal{R}$ and $h : M \to \mathcal{R}$. We say that h is level preserving on M and write $h \in P(M)$ if $\forall x, y \in M$ satisfying $\lambda(x) = \lambda(y)$ and $x =_r y$ it follows that $\lambda(h(x)) = \lambda(h(y))$ and $h(x) =_q h(y)$, where $q \geq \lambda(h(x)) + r - \lambda(x)$.*

Example 1. *Let $f : \mathcal{R} \to \mathcal{R}$ be given by $f(x)[q] = x[q-1]$. Then it is easy to check that $f \in P(\mathcal{R})$.*

Proposition 3. *Let $M \subset \mathcal{R}$ be open and such that $\lambda(x) \geq 0$ for all $x \in M$; let $h : M \to \mathcal{R}$ be a level preserving function on M; and let $\alpha \in \mathbb{Q}, \alpha > 1$, be given. Then the function $f : M \to \mathcal{R}$, given by*

$$f(x)[q] = \begin{cases} h(x) \left[\frac{q\lambda(h(x))}{\lambda(x)\alpha} \right] & \text{if } \lambda(x) > 0 \\ \\ h(x) \left[q + \lambda(h(x)) \right] & \text{if } \lambda(x) = 0 \end{cases}, \tag{3.1}$$

is differentiable on M, with derivative $f'(x) = 0$ for all $x \in M$.

Proof. Let $x \in \mathcal{R}$ and $\epsilon > 0$ in $\mathcal{R}$ be given. Since M is open, there exists $\eta > 0$ in $\mathcal{R}$ such that $\eta \ll 1$ and $(x - \eta, x + \eta) \subset M$. Let

$$\delta = \min\{\epsilon^{\frac{2}{\alpha - 1}}, \eta\}.$$

Then $0 < \delta \ll 1$ and $(x - \delta, x + \delta) \subset M$. Now let $y \in M$ be such that $0 < |y - x| < \delta$, we will show that

$$\left| \frac{f(x) - f(y)}{x - y} \right| < \epsilon.$$

First we note that, since $|y - x| \ll 1$, either $\lambda(x) = 0 = \lambda(y)$ or $[\lambda(x) > 0$ and $\lambda(y) > 0]$.

First assume that $\lambda(x) > 0$ (and hence $\lambda(y) > 0$). We distinguish three cases.

Case 1: $\lambda(x) \neq \lambda(y)$. In this case, we have that $\lambda(f(x)) \neq \lambda(f(y))$. It follows, that

$$\left|\frac{f(x)-f(y)}{x-y}\right| \sim (x-y)^{\alpha-1}.$$

Hence

$$\begin{aligned}
\lambda\left(\frac{f(x)-f(y)}{x-y}\right) &= (\alpha-1)\lambda(y-x) \geq (\alpha-1)\lambda(\delta) \\
&= (\alpha-1)\max\left\{\frac{2}{\alpha-1}\lambda(\epsilon), \lambda(\eta)\right\} = \max\left\{2\lambda(\epsilon), (\alpha-1)\lambda(\eta)\right\} \\
&> \lambda(\epsilon).
\end{aligned}$$

Hence

$$\left|\frac{f(x)-f(y)}{x-y}\right| \ll \epsilon.$$

Case 2: $x \sim y$ and $x[\lambda(x)] \neq y[\lambda(y)]$. In this case the argument is similar to that of Case 1.

Case 3: $x =_r y$ for some $r \in \mathbb{Q},\ r \geq \lambda(x)$. Then obviously $\lambda(|x-y|) = r_+$ where r_+ is a rational number and $r_+ > r$. It follows that $\lambda(|f(x)-f(y)|) = \alpha r_+$. Thus,

$$\lambda\left(\frac{f(x)-f(y)}{x-y}\right) = \lambda(f(x)-f(y)) - \lambda(x-y) = (\alpha-1)r_+ > (\alpha-1)\lambda(\delta),$$

from which we obtain, as in Case 1. above, that

$$\left|\frac{f(x)-f(y)}{x-y}\right| \ll \epsilon.$$

Finally, if $\lambda(x) = 0 = \lambda(y)$ then the proof that

$$\left|\frac{f(x)-f(y)}{x-y}\right| < \epsilon$$

follows the same process as above (when $\lambda(x) > 0$) except that we will have to use the appropriate expression for f from Equation (3.1). $\square$

In the following definition we will introduce a notation for all functions that are differentiable with derivative equal to 0 everywhere on an open subset of $\mathcal{R}$; and we will use that notation throughout the rest of the paper.

Definition 9. *Let $M \subseteq \mathcal{R}$ be open and let $f : M \to \mathcal{R}$. We define the class of functions $D_0^1(M)$ as follows:*

$$D_0^1(M) = \left\{f : M \to \mathcal{R} | f \text{ is differentiable on } M,\ f'(x) = 0\ \forall x \in M\right\}.$$

3.2. **Systems of linear ordinary differential equations on $\mathcal{R}$.** In this section we will investigate the solutions of systems of linear ordinary differential equations on $\mathcal{R}$, using the functions of class D_0^1.

The main goal of this section is to obtain solutions of systems of linear ODE's given in the form:

$$Y'(t) = A(t)Y(t) + B(t),$$

where $Y(t)$ is a vector of dimension $n > 0$, $n \in \mathbb{N}$, which contains the unknown functions, $Y'(t)$ contains the derivatives of the functions from $Y(t)$, $A(t) \in \mathcal{M}_n^f(\mathcal{R})$, for all $t \in \mathcal{R}$, which contains the coefficient functions of the system and $B(t)$ is a vector of dimension n, which contains functions that ensure the inhomogeneity of the system.

In order to the realize this we will study a few cases, going from the most special to the most general ones.

Theorem 4. *Consider the linear homogeneous system of ordinary differential equations with constant coefficients which are at most finite in absolute value*

$$Y'(t) = AY(t).$$

Then the solution is given by

$$Y(t) = e^{At}C + e^{At}U_{na}(t),$$

where $C \in \mathcal{M}_{n,1}(\mathcal{R})$ *is a vector containing constants, and* $U_{na}(t) \in \mathcal{M}_{n,1}(D_0^1)$ *is a vector which contains functions of class* D_0^1.

Proof. We rewrite the system in the form:

$$Y'(t) - AY(t) = O_{n,1}$$

which is equivalent to

$$e^{-At}Y'(t) - e^{-At}AY(t) = O_{n,1}$$

or

$$\left(e^{-At}Y(t)\right)' = O_{n,1}.$$

It follows that $e^{-At}Y(t) = C + U_{na}(t)$, where $C \in \mathcal{M}_{n,1}(\mathcal{R})$ and the elements of $U_{na}(t)$ are functions of class D_0^1, and hence $Y(t) = e^{At}C + e^{At}U_{na}(t)$. □

Remark 3. *The theorem above shows that the solutions of linear homogeneous systems with constant coefficients over* $\mathcal{R}$ *are very similar to those of the real case, except that the solutions in the non-Archimedean case may also involve non-analytic functions with zero-derivative.*

Since we know how to integrate $\mathcal{R}$-analytic functions [14] in the Lebesgue-like theory developed in [18, 15], we will study next those inhomogeneous systems, where the functions which ensure the inhomogeneity are $\mathcal{R}$-analytic.

Theorem 5. *Consider the inhomogeneous system of linear ordinary differential equations with constant coefficients:*

$$Y'(t) = AY(t) + B(t)$$

on the interval $[a, b] \subset \mathcal{R}$, *where* $|A|$, $|a|$ *and* $|b|$ *are at most finite in absolute value and where* $B(t)$ *is a vector, which contains functions that are* $\mathcal{R}$*-analytic on* $[a, b]$. *Then the solution is given by*

$$Y(t) = e^{At}C + e^{At}U_{na}(t) + e^{At}\int_{[a,t]} e^{-As}B(s),$$

where $C \in \mathcal{M}_{n,1}(\mathcal{R})$ *and the elements of* $U_{na}(t)$ *are functions of class* D_0^1.

Proof. We rewrite the system in the form:

$$Y'(t) - AY(t) = B(t)$$

which is equivalent to

$$\left(e^{-At}Y(t)\right)' = e^{-At}B(t).$$

It follows that
$$e^{-At}Y(t) = C + U_{na}(t) + \int_{[a,t]} e^{-As}B(s),$$
where $C \in \mathcal{M}_{n,1}(\mathcal{R})$, and $U_{na}(t) \in \mathcal{M}_{n,1}(D_0^1)$, and hence
$$Y(t) = e^{At}C + e^{At}U_{na}(t) + e^{At}\int_{[a,t]} e^{-As}B(s).$$
Note that we have used the fact that $e^{At}B(t)$ is a vector whose components are products of functions that are $\mathcal{R}$-analytic on $[a,b]$ and hence the components themselves are $\mathcal{R}$-analytic on $[a,b]$. □

The proofs of the next two theorems are very similar to those of Theorem 4 and Theorem 5 above; and therefore Theorem 6 and Theorem 7 will be stated without proofs.

Theorem 6. *Consider the homogeneous system of linear ordinary differential equations with non-constant coefficients:*
$$Y'(t) = A(t)Y(t)$$
on $[a,b]$, where $|a|$ and $|b|$ are at most finite, and where $A(t)$ is an $n \times n$ matrix whose elements are $\mathcal{R}$-analytic on $[a,b]$ and such that $|A(t)|$ is at most finite for all $t \in [a,b]$. Then the solution is given by
$$Y(t) = e^{\int_{[a,t]} A(s)}C + e^{\int_{[a,t]} A(s)}U_{na}(t),$$
where $C \in \mathcal{M}_{n,1}(\mathcal{R})$ and $U_{na}(t) \in \mathcal{M}_{n,1}(D_0^1)$.

Theorem 7. *Consider the inhomogeneous system of linear ordinary differential equations with non-constant coefficients:*
$$Y'(t) = A(t)Y(t) + B(t)$$
on $[a,b]$, where $|a|$ and $|b|$ are at most finite; where $A(t)$ is an $n \times n$ matrix whose elements are $\mathcal{R}$-analytic on $[a,b]$ and such that $|A(t)|$ is at most finite for all $t \in [a,b]$; and where $B(t)$ is a vector whose components are functions that are $\mathcal{R}$-analytic on $[a,b]$. Then the solution is given by
$$Y(t) = e^{\int_{[a,t]} A(s)}C + e^{\int_{[a,t]} A(s)}U_{na}(t) + e^{\int_{[a,t]} A(s)}\int_{[a,t]} e^{-\int_{[a,s]} A(r)}B(s),$$
where $C \in \mathcal{M}_{n,1}(\mathcal{R})$ and $U_{na}(t) \in \mathcal{M}_{n,1}(D_0^1)$.

3.3. **Bessel-type special functions on $\mathcal{R}$.** In this section we will study Bessel-type special functions on $\mathcal{R}$, with the help of the solutions for systems of linear ordinary differential equations that we developed in Subsection 3.2. We will introduce such functions with the following problem.

Problem 1. *Consider the differential equation*
$$t^2y'' + ty' + (t^2 - \nu^2)y = 0. \tag{3.2}$$
Let's study the solutions of the equation on $[a,1] \subset \mathcal{R}$, where $0 < a < 1$, a is finite, and $\nu \in \mathbb{Q} \setminus \mathbb{Z}$.

We call Equation (3.2) the Bessel equation of order ν, and we will study its solutions below.

It's easy to check, like in real case, that the functions

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{t}{2}\right)^{2n} \tag{3.3}$$

and

$$J_{-\nu}(t) = \left(\frac{t}{2}\right)^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{t}{2}\right)^{2n} \tag{3.4}$$

are two linearly independent analytic solutions of Equation (3.2), for all $t \in [a, 1]$, where Γ is the Euler's Gamma function. Moreover, it follows from Corollary 3.10 in [17] that the power series in Equation (3.3) and Equation (3.4) converge weakly in $[a, 1]$.

The main objective of Problem 1 is to construct solutions of Equation (3.2) that involve functions of class D_0^1, and to study their properties.

Let

$$\begin{cases} w_1 = y \\ w_2 = y' \end{cases}$$

and consider the following system of two linear ordinary differential equations:

$$\begin{cases} w_1' = w_2 \\ w_2' = \left(1 - \frac{\nu^2}{t^2}\right) w_1 + \frac{1}{t} w_2 \end{cases}$$

which we can write in matrix form as

$$W'(t) = A(t)W(t), \tag{3.5}$$

where

$$W(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}; \quad W'(t) = \begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix}; \quad A(t) = \begin{pmatrix} 0 & 1 \\ 1 - \frac{\nu^2}{t^2} & \frac{1}{t} \end{pmatrix}.$$

Because the elements of $A(t)$ are $\mathcal{R}$-analytic functions on $[a, 1]$ and they are at most finite in absolute value for all $t \in [a, 1]$, we can use Theorem 6 to write the solutions of the system in Equation (3.5) which are also the solutions of Equation (3.2).

Let

$$D(t) = \int_{[a,t]} A(s) = \begin{pmatrix} 0 & t-a \\ t - a + \nu^2(\frac{1}{t} - \frac{1}{a}) & \ln\frac{t}{a} \end{pmatrix},$$

where the function ln is $\mathcal{R}$-analytic on $[a, 1]$.

Thus, the solution of Equation (3.5) and hence of Equation (3.2) on $[a, 1]$ has the form

$$W(t) = e^{D(t)}C + e^{D(t)}U_{na}(t),$$

where $C \in \mathcal{M}_{2,1}(\mathcal{R})$ and $U_{na}(t) \in \mathcal{M}_{2,1}(D_0^1)$.

Taking into consideration that the analytic part of the solution has the form $c_1 J_\nu(t) + c_2 J_{-\nu}(t)$, we conclude that in the first row of the matrix $e^{D(t)}$ we can take the entries to be

$$D_{11} = J_\nu(t) \text{ and } D_{12} = J_{-\nu}(t).$$

With $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, it follows that the first component of $W(t)$, which is $y = w_1(t)$, has the form:

$$y = c_1 \left[J_\nu(t) + J_\nu(t) g_\nu(t)\right] + c_2 \left[J_{-\nu}(t) + J_{-\nu}(t) g_{-\nu}(t)\right],$$

where $g_\nu, g_{-\nu} \in D_0^1([a,1])$. Now we can define Bessel functions of the first kind on $\mathcal{R}$.

Definition 10. *For $0 < a < 1$, $a \in \mathcal{R}$ finite, we define the functions $\mathcal{J}_\nu, \mathcal{J}_{-\nu} : [a,1] \subset \mathcal{R} \to \mathcal{R}$ by*

$$\mathcal{J}_\nu(t) = J_\nu(t) + J_\nu(t) g_\nu(t), \ \text{and}$$
$$\mathcal{J}_{-\nu}(t) = J_{-\nu}(t) + J_{-\nu}(t) g_{-\nu}(t),$$

where $\nu \in \mathbb{Q} \setminus \mathbb{Z}$ and $g_\nu, g_{-\nu} \in D_0^1([a,1])$.

We call the functions $\mathcal{J}_\nu, \mathcal{J}_{-\nu}$ Bessel functions of the first kind and of order ν, and $-\nu$, respectively.

Next, we study some properties of the Bessel functions $\mathcal{J}_\nu$ and $\mathcal{J}_{-\nu}$.

Theorem 8. *Using the notations from Problem 1 and Definition 10 above, the following two statements are true.*

$$(a) \qquad \frac{2\nu}{t}\mathcal{J}_\nu(t) = \Big(J_{\nu-1}(t) + J_{\nu+1}(t)\Big)\Big(1 + g_\nu(t)\Big)$$
$$(b) \qquad 2\mathcal{J}_\nu'(t) = \Big(J_{\nu-1}(t) - J_{\nu+1}(t)\Big)\Big(1 + g_\nu(t)\Big)$$

for all $t \in [a,1]$.

Proof. Using Equation (3.3), we obtain that

$$\left(\frac{J_\nu(t)}{t^\nu}\right)' = \frac{1}{2^\nu} \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n-1}}{(n-1)!\Gamma(n+\nu+1) 2^{2n-1}} = -\frac{J_{\nu+1}(t)}{t^\nu}$$

Then, using the fact that

$$\mathcal{J}_\nu(t) = J_\nu(t) + J_\nu(t) g_\nu(t)$$

and $g_\nu'(t) = 0, \forall t \in [a,1]$, we obtain that

$$\left(\frac{\mathcal{J}_\nu(t)}{t^\nu}\right)' = -\frac{J_{\nu+1}(t)}{t^\nu} - \frac{J_{\nu+1}(t)}{t^\nu} g_\nu(t),$$

from which we obtain that

$$\mathcal{J}_\nu'(t)\frac{1}{t^\nu} + \mathcal{J}_\nu(t)\frac{-\nu}{t^{\nu+1}} = -\frac{J_{\nu+1}(t)}{t^\nu} - \frac{J_{\nu+1}(t)}{t^\nu} g_\nu(t),$$

or

$$\mathcal{J}_\nu'(t) = \frac{\nu}{t}\mathcal{J}_\nu(t) - J_{\nu+1}(t) - J_{\nu+1}(t) g_\nu(t) \tag{3.6}$$

Similarly, we can show that

$$\frac{1}{t}\Big(t^\nu \mathcal{J}_\nu(t)\Big)' = t^{\nu-1} J_{\nu-1}(t) + t^{\nu-1} J_{\nu-1}(t) g_\nu(t)$$

from which we obtain that

$$\frac{1}{t}\Big(\nu t^{\nu-1}\mathcal{J}_\nu(t) + t^\nu \mathcal{J}_\nu'(t)\Big) = t^{\nu-1} J_{\nu-1}(t) + t^{\nu-1} J_{\nu-1}(t) g_\nu(t),$$

or

$$\mathcal{J}_\nu'(t) = J_{\nu-1}(t) - \frac{\nu}{t}\mathcal{J}_\nu(t) + J_{\nu-1}(t)g_\nu(t) \tag{3.7}$$

If we subtract Equation (3.7) from Equation (3.6), we get statement (a) of the theorem; and if we add the two equations, we get (b). □

Remark 4. *Just as we did in Theorem 8, we obtain other recursive relations for $\mathcal{J}_\nu$ and $\mathcal{J}_{-\nu}$ that would extend the classical recursive relations for J_ν and $J_{-\nu}$ from Real Calculus to the non-Archimedean calculus on $\mathcal{R}$.*

In the following subsection, we will introduce the so-called generalized Bessel functions of the first kind on $\mathcal{R}$.

3.4. **Generalized Bessel functions of the first kind on $\mathcal{R}$.** We give some basic definitions based on those given by Á. Baricz ([2]) in the classical case (real and complex). As before, let $0 < a < 1$, $a \in \mathcal{R}$ finite; let $p \in \mathbb{Q} \setminus \mathbb{Z}$; and let $b, c \in \mathbb{R}$ in the rest of this paper.

Definition 11. *The differential equation*

$$t^2 w''(t) + btw(t) + [ct^2 - p^2 + (1-b)p]w(t) = 0 \tag{3.8}$$

will be referred to as the generalized Bessel equation of order p [2]*; any solution of it will be called a generalized Bessel function of order p.*

The generalized Bessel functions permit the study of Bessel functions, spherical Bessel functions and modified Bessel functions together. That is why it is very important to extend this kind of functions to the field $\mathcal{R}$.

Remark 5. *As done in the classical case* [2]*, it can be easily verified that the function*

$$w_p(t) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma(p+n+\frac{b+1}{2})} \left(\frac{t}{2}\right)^{2n+p}$$

is a solution of Equation (3.8). Moreover, when $c = b = 1$ we get the Bessel function of the first kind of order p discussed in the previous subsection.

Similarly to Definition 10, we introduce generalized Bessel functions of the first kind on $\mathcal{R}$ as follows:

Definition 12. *Let $0 < a < 1$, $a \in \mathbb{R}$ finite; $p \in \mathbb{Q} \setminus \mathbb{Z}$; and $b, c \in \mathbb{R}$. Then the function*

$$\mathcal{W}_p(t) = w_p(t) + w_p(t)g_p(t),$$

where $g_p \in D_0^1([a,1])$, and

$$w_p(t) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma(p+n+\frac{b+1}{2})} \left(\frac{t}{2}\right)^{2n+p},$$

is a solution of Equation (3.8). We call $\mathcal{W}_p(t)$ a generalized Bessel function of order p on $[a,1]$.

The proof of the following theorem is similar to that of Theorem 8 as well as to the proof of the corresponding result in the classical case ([2], Lemma 1.1); therefore we state the theorem without proof here.

Theorem 9. *Using the notation in Definition 12, the following statements are true.*

$$(a)\quad \frac{2p+b-1}{t}\mathcal{W}_p(t) = \Big(w_{p-1}(t) + cw_{p+1}(t)\Big)\Big(1 + g_p(t)\Big)$$

$$(b)\quad (2p+b-1)\mathcal{W}'_p(t) = \Big(pw_{p-1}(t) - (p+b-1)cw_{p+1}(t)\Big)\Big(1 + g_p(t)\Big),$$

for all $t \in [a, 1]$.

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