Math 164-1: Optimization – Homework 1

Due: October 2, 2015

Exercise 1. *Read carefully Chapters 2, 3, 4, 5 from the book of Chong and Zak and recall the known notions from linear algebra, geometry and calculus.*

Exercise 2. *Exercises 2.1 and 2.9 from the book of Chong and Zak.*

(i) (Exercise 2.1) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and rank $\mathbf{A} = m$. Show that $m \leq n$.

(ii) (Exercise 2.9) Show that for any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$.

Hint: Write $\mathbf{x} = (\mathbf{x} - \mathbf{y}) + \mathbf{y}$, and use the triangle inequality. Do the same for $\mathbf{y}$.

Exercise 3. *Exercises 3.6 and 3.15 from the book of Chong and Zak.*

(i) (Exercise 3.6) Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$. Show that the eigenvalues of the matrix $\mathbf{I}_n - \mathbf{A}$ are $1 - \lambda_1, \ldots, 1 - \lambda_n$.

(ii) (Exercise 3.15) Is the quadratic form

$$\mathbf{x}^T \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix} \mathbf{x}$$

positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite?

Exercise 4. *Exercises 4.1, 4.2, 4.3 and 4.4 from the book of Chong and Zak.*

(i) (Exercise 4.1) Show that a set $S \subset \mathbb{R}^n$ is a linear variety if and only if for all $\mathbf{x}, \mathbf{y} \in S$ and $\alpha \in \mathbb{R}$, we have $\alpha\mathbf{x} + (1 - \alpha)\mathbf{y} \in S$.

(ii) (Exercise 4.2) Show that the set $\{\mathbf{x} \in \mathbb{R}^n : \|x\| \leq r\}$ is convex, where $r > 0$ is a given real number and $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$ is the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$.

(iii) (Exercise 4.3) Show that for any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$, the set (linear variety) $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ is convex.

(iv) (Exercise 4.4) Show that the set $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}\}$ is convex (where $\mathbf{x} \geq \mathbf{0}$ means that every component of $\mathbf{x}$ is nonnegative).

Exercise 5. *Exercises 5.1, 5.2, 5.3 and 5.4 from the book of Chong and Zak.*

(i) (Exercise 5.1) Show that a sufficient condition for $\lim_{k\to\infty} \mathbf{A}^k = \mathbf{O}$ is $\|\mathbf{A}\| < 1$.

(ii) (Exercise 5.2) Show that for any matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$,

$$\|\mathbf{A}\| \geq \max_{1\leq i\leq n} |\lambda_i(\mathbf{A})|.$$

Hint: Use Exercice 5.1.

(iii) (Exercise 5.3) Consider the function

$$f(\mathbf{x}) = (\mathbf{a}^T\mathbf{x})(\mathbf{b}^T\mathbf{x}),$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{x}$ are n-dimensional vectors.

a. Find $\nabla f(\mathbf{x})$.

b. Find the Hessian $\mathbf{F}(\mathbf{x})$.

(iv) (Exercise 5.4) Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^2$ by $f(\mathbf{x}) = x_1^2/6 + x_2^2/4$ and $g(t) = [3t + 5, 2t - 6]^T$. Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(t) = f(g(t))$. Evaluate $\frac{dF}{dt}(t)$ using the chain rule.