Math 164-1: Optimization – Homework 5

Due: October 30, 2015

Exercise 1. *We want to study the gradient descent algorithm for convex functions. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function of class $C^2(\mathbb{R}^d)$ such that $\mathbb{R}^d \ni x^* = \mathrm{argmin}_{x \in \mathbb{R}^d} f(x)$. Answer to the following questions. Justify your answers.*

(1) Show that f is convex if and only if $D^2 f(x) \geq 0$ (positive semidefinite matrix) for all $x \in \mathbb{R}^d$.

Hint: Use a precise Taylor expansion of order 2 for f and the fact that f is above each tangent plane touching the graph of f.

(2) Let f be *strongly convex*, i.e. there exists $m > 0$ such that $D^2 f(x) \geq mI_d$ (in the sense of symmetric matrices, meaning that $D^2 f(x) - mI_d$ is positive semidefinite) for all $x \in \mathbb{R}^d$ (here $I_d \in \mathbb{R}^{d \times d}$ denotes the identity matrix). Show that if f is strongly convex (with constant $m > 0$), than

$$f(y) \geq f(x) + \nabla f(x) \cdot (y - x) + \frac{m}{2}\|y - x\|^2, \ \ \forall x, y \in \mathbb{R}^d.$$

(3) Show that

$$f(x^*) \geq f(x) - \frac{1}{2m}\|\nabla f(x)\|^2$$

and

$$\|x - x^*\| \leq \frac{2}{m}\|\nabla f(x)\|,$$

for all $x \in \mathbb{R}^d$.

(4) Let us suppose now that $D^2 f(x)$ uniformly bounded by a constant $M > 0$, i.e. $D^2 f(x) - MI_d$ is negative semidefinite for all $x \in \mathbb{R}^d$. Show that

$$f(y) \leq f(x) + \nabla f(x) \cdot (y - x) + \frac{M}{2}\|y - x\|^2, \ \ \forall x, y \in \mathbb{R}^d.$$

(5) Supposing the existence of constants $m, M > 0$ in (2) and (4) respectively. Show that the gradient descent algorithm to find x^* converges for a fixed step size $\alpha = 1/M$ with the rate

$$f(x^k) - f(x^*) \leq C^k(f(x^0) - f(x^*)),$$

for $0 \leq C \leq 1 - m/M$. This means that the algorithm is globally convergent. Give an example of a function that satisfies the assumption of this point.

Exercise 2.

Exercises 8.7 and 8.12 from the book of Chong and Zak.

Exercise 3.

Exercises 8.8 and 8.10 from the book of Chong and Zak.

Exercise 4.

Exercises 8.18 and 8.19 from the book of Chong and Zak.