Math 164: Optimization – Homework 1

Due: September 30, 2016

Exercise 1.

Read carefully Chapters 2, 3, 4, 5 from the book of Chong and Zak and recall the known notions from linear algebra, geometry and calculus. If you do not have the textbook yet, please make sure that you recall the notions of gradient, Jacobian, Hessian, Taylor expansion of functions depending on several variables; convex set, line segments, planes, hyperplanes in $\mathbb{R}^d$; eigenvalues, eigenvectors of matrices; positive definite matrices, quadratic forms, etc. The wikipedia could be a very good source as well.

Exercise 2 (Exercises 2.1 and 2.9 from the book of Chong and Zak)**.**

(1) (Exercise 2.1) Let $A \in \mathbb{R}^{m\times n}$ and rank $A = m$. Show that $m \leq n$.

(1) (Exercise 2.9) Show that for any two vectors $x, y \in \mathbb{R}^n$, $|\|x\| - \|y\|| \leq \|x - y\|$.
Hint: Write $x = (x - y) + y$, and use the triangle inequality. Do the same for y.

Exercise 3 (Based on Exercises 3.6 and 3.15 from the book of Chong and Zak)**.**

(1) (Exercise 3.6) Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$. Show that the eigenvalues of the matrix $I_n - A$ are $1 - \lambda_1, \ldots, 1 - \lambda_n$, where I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$.

(2) (based on Exercise 3.15) Is the quadratic form

$$x \ni \mathbb{R}^2 \mapsto x^\top \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix} x$$

positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite? Justify your answer! Compute the gradient and the Hessian of the application above.

Exercise 4 (Exercises 5.1, 5.2, 5.3 and 5.4 from the book of Chong and Zak)**.**

(1) (Exercise 5.1) Show that a sufficient condition for $\lim_{k\to\infty} A^k = O_n$ is $\|A\| < 1$.

(2) (Exercise 5.2) Show that for any matrix $A \in \mathbb{R}^{n\times n}$,

$$\|A\| \geq \max_{1\leq i\leq n} |\lambda_i(A)|.$$

Hint: Use Exercice 5.1.

(3) (Exercise 5.3) Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = (a^\top x)(b^\top x),$$

where a, b and x are n-dimensional vectors.

a. Find $\nabla f(x)$.

b. Find the Hessian $D^2 f(x)$.

(4) (Exercise 5.4) Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^2$ by $f(x) = x_1^2/6 + x_2^2/4$ and $g(t) = [3t + 5, 2t - 6]^\top$. Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(t) = f(g(t))$. Evaluate $\frac{dF}{dt}(t)$ using the chain rule.

Exercise 5 (Feasible directions, directional derivatives, local and global optimizers)**.**

(1) Let $\Omega = [-1, 1] \subset \mathbb{R}$. And let $f : \Omega \to \mathbb{R}$ be defined as $f(x) = x^2 - 2$.

Give 3 feasible directions at each of the points $-1; 1/2 \in \Omega$. Compute $\frac{\partial f}{\partial e}(x)$, where e's are the feasible directions that you considered before at the points $x \in \{-1, 1/2\}$. Give a local and a global minimizer and maximizer of f on Ω. Do the local optimizers coincide with the global ones? Are the global optimizers unique? Justify your answer!

(2) Let $\Omega = ([0, 1] \times [0, 1]) \cup ([1, 2] \times \{0\}) \subset \mathbb{R}^2$ (a unit square and a line segment in $\mathbb{R}^2$). Let $f : \Omega \to \mathbb{R}$ be defined as $f(x, y) = x^2 + y^2$.

Give 3 feasible directions at each of the points $(0, 0); (3/2, 0) \in \Omega$. Compute $\frac{\partial f}{\partial e}(x)$, where e's are the feasible directions that you considered before at the points $x \in \{(0, 0), (3/2, 0)\}$. Give a local and a global minimizer and maximizer of f on Ω. Do the local optimizers coincide with the global ones? Are the global optimizers unique? Justify your answer!