Math 164 (Lec 1): Optimization

Instructor: Alpár R. Mészáros

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Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- You may use either a pen or a pencil to write your solutions. However, if you use a pencil I will withheld your paper for **two** weeks after grading it.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs. You may lose points in the lack of justification of your answers.
- Theorems from the lectures and homework may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 3 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **26 points**, which means that there are **6 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- The problems are not necessarily ordered with respect to easiness!
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Total	

Exercise 1 (7 points)**.**

(1) Let us consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x_1, x_2) = \sin(x_1)\cos(x_2).$$

Find all the local minimizers and maximizers of f on the set $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 - \cos(x_1) = 0 \text{ and } x_2 - \sin(x_1) = 0\}$. Find the global minimizers and maximizers of f on S as well. Are they unique? Justify your answer!

Hint: it is easier to begin describing the geometry of S and solve the problem without Lagrange multipliers.

(2) Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [\pi/4, 5\pi/4] \text{ and } \cos(x_1) \leq x_2 \leq \sin(x_1)\}$. Consider moreover $g : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$g(x_1, x_2) = 1 - x_2^2.$$

Find the global minimizers and maximizers of g on Ω. Are they unique? Justify your answers!

Hint: it is more complicated to use the KKT techniques, than to use Lagrange multipliers (+ dealing separately with the interior case). Nevertheless, first understand the geometry of Ω and the structure of g.

Exercise 2 (Aquathlon problem – 9 points)**.**

John is doing aquathlon, which is a sport combining running and swimming. The start is situated on the left riverside and the finish is situated on the right riverside of the same river. Hence, he needs to run along the river on the riverside, cross the river by swimming, then possibly run further on the other side. The sides of the river are parallel straight lines, and the width of the river is uniform everywhere, and it is $d > 0$ meters. It is also allowed to immediately cross the river by swimming, or run only on the left side and arrive to the finish by swimming. The length of each riverside (where the race takes place) measures $L = 500$ meters.

We suppose that,

$$d < \sqrt{3}L, \tag{width}$$

meaning that the river is not that wide, compared to the length of the race.

Knowing that John can run with a constant uniform speed of $v_r = 2\ m/s$, and he can swim with a constant uniform speed of $v_s = 1\ m/s$, he wants to minimize the time that he needs for this combined race. We also assume that he will swim along a straight line from one riverside to the other. Help John to find an optimal strategy, i.e. tell him how far to run on the left side, then in which angle to start swimming and then to run on the other side, in order to achieve the shortest possible time.

(1) Introducing some variables if it is necessary, write an objective function and a feasible set, where the objective function should be minimized.

Hint: one knows that the constant speed is given by ratio of the distance travelled and the duration of the time.

(2) Show that if there is an optimal strategy from the interior of the feasible set, then that is depending only on the angle in which John needs to start swimming. Deduce from this the non-uniqueness of the optimal strategy.

(3) Show that there exists an optimal strategy and describe all of them (for which distance should John run on the left side, then in which angle needs he to start swimming) and show that these strategies are independent of the width d of the river. Compute the corresponding optimal time as well.

(4) Explain, how the condition (width) enters into the optimization problem. What happens if this condition is not fulfilled?

Exercise 3 (10 points)**.**

Let us consider the functions $f_\delta, g_\delta : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f_\delta(x_1, x_2) = 2x_1^2 + 2x_1x_2 + \delta x_2^2 - 2x_1 - x_2, \text{ and } \; g_\delta(x_1, x_2) = 2x_1^2 + 2x_1x_2 + \delta x_2^2$$

where $\delta \in \mathbb{R}$ is a given parameter. We aim to study some optimization problems involving f_δ and g_δ for different values of the parameter δ.

(1) Show that for any $\delta > 1/2$, f_δ has a unique minimizer on $\mathbb{R}^2$. Compute this minimizer in terms of δ. *Hint:* write down first and second order conditions for f_δ and study when $D^2 f_\delta$ becomes positive definite.

(2) Write f_δ as $\frac{1}{2}x^\top Q_\delta x - b \cdot x$, for some $Q_\delta \in \mathbb{R}^{2\times 2}$ and $b \in \mathbb{R}^2$ to be determined. Let $\delta > 1/2$ and $x^0 = (0 \;\; 0)^\top$, perform one iteration (compute x^1) of the gradient descent algorithm for f_δ with optimal step-size initialized with x^0. Compare x^1 to the optimal solution computed in (1). Is x^1 closer than x^0?

(3) Show that $g_{1/2}$ has infinitely many global minimizers, that lie on a line. Determine this line, and show that it corresponds to the eigenspace generated by the eigenvector corresponding to the smallest eigenvalue of $D^2 g$.

(4) Show that g_0 does not have either local minimizers or local maximizers.