Math 164 (Lec 2): Optimization

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Midterm, October 26, 2016

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- You may use either a pen or a pencil to write your solutions. However, if you use a pencil I will withheld your paper for **two** weeks after grading it.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs. You may lose points in the lack of justification of your answers.
- Theorems from the lectures and homework may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 3 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **26 points**, which means that there are **6 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- The problems are not necessarily ordered with respect to easiness!
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Total	

Exercise 1 (9 points).

Let $\delta \in \mathbb{R}$ be a parameter, and consider the functions $f_\delta, g_\delta : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f_\delta(x,y) = x^2 + 2\delta xy + 2y^2 - x + y, \text{ and } g_\delta(x,y) = f_\delta(x,y) + x - y.$$

(1) Show that if $\delta \in (-\sqrt{2}, \sqrt{2})$, then both f_δ and g_δ have a unique minimizer on $\mathbb{R}^2$. Determine these minimizers in terms of δ. *Hint:* write down first and second order optimality conditions, and study the Hessians of f_δ and g_δ.

(2) Let $\delta = \sqrt{2}$. Show that in this case $g_{\sqrt{2}}$ has infinitely many global minimizers. Determine the geometric location (i.e. a set $S_{\sqrt{2}} := \{(x,y) \in \mathbb{R}^2 : (x,y) \text{ is a global minimizer of } g_{\sqrt{2}}\}$) of these global minimizers in terms of the eigenvalues/eigenvectors of $D^2 g_{\sqrt{2}}$.

(3) Let $\alpha > 0$. We want to approximate some of the global minimizers of $g_{\sqrt{2}}$ using a steepest descent (gradient descent) algorithm with fixed step size α. For any $(x^0, y^0) \in \mathbb{R}^2$ write down the first iteration using this algorithm, i.e. write (x^1, y^1) in terms of (x^0, y^0) and α.

(4) Show that there is a unique $\alpha > 0$ independent of the initial guess, such that using the gradient descent algorithm from (3) with this α and any $(x^0, y^0) \notin S_{\sqrt{2}}$ initial guess, (x^1, y^1) determined in (3) will lie in $S_{\sqrt{2}}$. Compute the value of this α. *Hint:* use the geometric definition of $S_{\sqrt{2}}$ and check what condition should be satisfied for (x^1, y^1) in order to lie on $S_{\sqrt{2}}$.

Exercise 2 (8 points)**.**

(1) Let us consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = (x + 1) \arctan(y).$$

Find all the local minimizers and maximizers of f on the set $S = \{(x, y) \in \mathbb{R}^2 : x^2 + 2x + y^2 - 3 = 0 \text{ and } x^2 - 2x + y^2 - 3 = 0\}$. Find the global minimizers and maximizers of f on S as well. Are they unique? Justify your answer!

Hint: it is easier to begin describing the geometry of S and solve the problem without Lagrange multipliers.

(2) Let $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2+2x+y^2-3 \leq 0 \text{ and } x^2-2x+y^2-3 \leq 0\}$. Consider moreover $g : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$g(x, y) = 4x^2 + y^2.$$

Find the global minimizers and maximizers of g on Ω. Are they unique? Justify your answers!

Hint: you may use either KKT multipliers (with the first and second order conditions), or Lagrange multipliers (with first and second order conditions; here you may need to deal separately with the interior case). Nevertheless, first understand the geometry of Ω and the structure of g. A clever Lagrangian approach is shorter.

Exercise 3 (9 points).

John is travelling by car from a city to another (the distance between the two cities is given as $d > 0$ miles) and he wants to minimize his costs during this travel. First, time is precious, hence he wants to travel for as short time as possible. Secondly, he wants to consume the lowest possible amount of fuel. Help John to compute his optimal average speed in the following situations.

(1) John is a conscious driver, so he respects speed limitations. Suppose that on the trajectory between the two cities there is a constant speed limit of $v_L > 0$ miles/hour. John also estimated the fuel consumption for his car and he realized that for a *unit distance* the amount of fuel that he needs is proportional to the average speed, i.e. there exists an $\alpha > 0$ constant (that he can compute) such that the amount of fuel per unit distance is αv, where v is the average speed of the car on a unit distance. Compute the optimal average speed that minimizes the sum of the needed time and the consumed fuel in terms of d, v_L and α. Discuss cases with respect to the relationship between v_L and α.

Hint: one knows that the time of the travel is the ratio of the distance and the average speed. You may use the technique of KKT multipliers since we are looking for an optimal average velocity in the interval $[0, v_L]$.

(2) John has read a new study which says that the optimal consumption of his car is actually achieved at a higher speed, so after some research he has found that the consumed fuel on a *unit distance* is not exactly a linear function of the average speed but it is given by $\alpha(v)$, where $\alpha : [0, +\infty) \to [0, +\infty)$ is a smooth function, given by the expression

$$\alpha(v) = \gamma v(4/3 + \cos(\pi v/60)),$$

where $\gamma > 0$ is a given positive constant.

Observe that $\alpha(0) = 0$, α is strictly increasing at 0 and α has another local minimizer at $\overline{v} \approx 60$.

Let us assume that $v_L = 55$ miles/hour (which is the speed limit), hence if he tries to travel with $\overline{v}$ average speed, there is a chance that he gets a fine. John wants to know whether it would be optimal to travel above the speed limit even if he gets a fine. We assume that he always gets a fine, once he travels above the speed limit.

One knows also that the fine can be only given once on this trajectory, the value of it is depending on the average speed as $\beta(v - v_L)$, where $\beta : \mathbb{R} \to [0, +\infty)$ is the smooth function defined as

$$\beta(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ t^2, & \text{if } t > 0. \end{cases}$$

(i) Write down the optimization problem (involving the needed time, the cost of the fuel and the fine) with a mathematical language, and explain why is this a problem without constraints!

(ii) Explain what does it mean (in terms of the fuel and the amount of the fine) if John travels with an optimal average speed in (i) that is larger than v_L.

(iii) Write down the first order necessary optimality condition that has to be satisfied for the optimizer in (i).

(iv) Observe that the equation in (iii) is highly nonlinear, so propose an algorithm that can be used to find an approximation of its root. Write down a recursive relation between two consecutive terms of the approximation sequence.

(v) We are expecting an optimal average speed around 60 miles/hour. Hence initiate the algorithm in (iv) with $v^0 = 60$ and compute v^1.