Math 164 (Lec 2): Optimization

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Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- You may use either a pen or a pencil to write your solutions. However, if you use a pencil I will withheld your paper for **two** weeks after grading it.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs. You may lose points in the lack of justification of your answers.
- Theorems from the lectures and homework may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 3 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **26 points**, which means that there are **6 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- The problems are not necessarily ordered with respect to easiness!
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Total	

Exercise 1 (9 points)**.**

Let $\delta \in \mathbb{R}$ be a parameter, and consider the functions $f_\delta, g_\delta : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f_\delta(x, y) = x^2 + 2\delta xy + 2y^2 - x + y, \text{ and } g_\delta(x, y) = f_\delta(x, y) + x - y.$$

(1) Show that if $\delta \in (-\sqrt{2}, \sqrt{2})$, then both f_δ and g_δ have a unique minimizer on $\mathbb{R}^2$. Determine these minimizers in terms of δ. *Hint:* write down first and second order optimality conditions, and study the Hessians of f_δ and g_δ.

(2) Let $\delta = \sqrt{2}$. Show that in this case $g_{\sqrt{2}}$ has infinitely many global minimizers. Determine the geometric location (i.e. a set $S_{\sqrt{2}} := \{(x, y) \in \mathbb{R}^2 : (x, y) \text{ is a global minimizer of } g_{\sqrt{2}}\}$) of these global minimizers in terms of the eigenvalues/eigenvectors of $D^2 g_{\sqrt{2}}$.

(3) Let $\alpha > 0$. We want to approximate some of the global minimizers of $g_{\sqrt{2}}$ using a steepest descent (gradient descent) algorithm with fixed step size α. For any $(x^0, y^0) \in \mathbb{R}^2$ write down the first iteration using this algorithm, i.e. write (x^1, y^1) in terms of (x^0, y^0) and α.

(4) Show that there is a unique $\alpha > 0$ independent of the initial guess, such that using the gradient descent algorithm from (3) with this α and any $(x^0, y^0) \notin S_{\sqrt{2}}$ initial guess, (x^1, y^1) determined in (3) will lie in $S_{\sqrt{2}}$. Compute the value of this α. *Hint:* use the geometric definition of $S_{\sqrt{2}}$ and check what condition should be satisfied for (x^1, y^1) in order to lie on $S_{\sqrt{2}}$.

Solution

First let us write

$$f_\delta(X) = \frac{1}{2}X^\top Q_\delta X - b \cdot X, \text{ and } g_\delta(X) = \frac{1}{2}X^\top Q_\delta X,$$

where we denote $X = (x; y)^\top$ and

$$Q_\delta = \begin{pmatrix} 2 & 2\delta \\ 2\delta & 4 \end{pmatrix} \quad \text{and} \quad b = (1 \quad -1)^\top.$$

Observe that Q_δ is a symmetric matrix, hence all its eigenvalues are real.

(1) The candidates for the optimizers of f_δ and g_δ should satisfy the FONC, i.e. $Q_\delta X = b$, and $Q_\delta X = 0$ respectively. Let us check now the second order conditions, for observe that $D^2 f_\delta = D^2 g_\delta = Q_\delta$. Let us compute the eigenvalues λ_1, λ_2 of Q_δ, these are the solutions of $(2 - \lambda)(4 - \lambda) - 4\delta^2 = 0$, or equivalently $\lambda^2 - 6\lambda + 8 - 4\delta^2 = 0$, from where

$$\lambda_{1,2} = \frac{6 \pm \sqrt{36 - 4(8 - 4\delta^2)}}{2},$$

from where is is easy to see that if $8 - 4\delta^2 > 0$, then both eigenvalues are positive. This is exactly the case, when $\delta \in (-\sqrt{2}, \sqrt{2})$, so for this range Q_δ is invertible and positive definite, hence the candidates are unique and global minimizers. In the case of f_δ, this will be

$$\left(\frac{2 + \delta}{2(2 - \delta^2)}, \frac{1 + \delta}{2(\delta^2 - 2)}\right)^\top$$

and in the case of g_δ this is $(0, 0)^\top$.

(2) Just using the previous formula for the eigenvalues, in the case of $\delta = \sqrt{2}$, the eigenvalues of $Q_{\sqrt{2}}$ are $\lambda_1 = 0$ and $\lambda_2 = 6$, so the matrix is still positive semi-definite, however, we are not able to use the SOSC. Still, by FONC the candidates are such that $Q_{\sqrt{2}} X = 0$, which translates to

$$x + \sqrt{2}y = 0. \tag{1}$$

On the other hand the positive semi-definiteness means that $g_{\sqrt{2}}(X) = \frac{1}{2}X^\top Q_{\sqrt{2}}X \geq 0$, and $g_{\sqrt{2}}(X) = 0$ for all $X = (x, y)^\top$ such that (1) holds. This implies that all these vectors are actually global minimizers of $g_{\sqrt{2}}$ and the geometric location of these points is nothing else but the eigenspace associated to the $\lambda_1 = 0$ eigenvalue of $Q_{\sqrt{2}}$, with other words, these are all eigenvectors corresponding to $\lambda_1 = 0$.

(3) The definition of (x^1, y^1) reads as

$$(x^1, y^1)^\top = (x^0, y^0)^\top - \alpha(2x^0 + 2\sqrt{2}y^0;\;\; 2\sqrt{2}x^0 + 4y^0)^\top = [x^0 - \alpha(2x^0 + 2\sqrt{2}y^0);\;\; y^0 - \alpha(2\sqrt{2}x^0 + 4y^0)]^\top$$

(4) In order to obtain that (x^1, y^1) is a global minimizer of $g_{\sqrt{2}}$, it should satisfy (1), that is $x^1 + \sqrt{2}y^1 = 0$. Let us write this condition in terms of α. This reads as

$$x^0 - \alpha(2x^0 + 2\sqrt{2}y^0 = -\sqrt{2}[y^0 - \alpha(2\sqrt{2}x^0 + 4y^0)],$$

which after rearranging and using that $x^0 + \sqrt{2}y^0 \neq 0$, one obtains that $\alpha = 1/6$. Which means that independently of the initial guess, using this α we will always obtain a global minimizer in 1 step.

Exercise 2 (8 points)**.**

(1) Let us consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = (x + 1) \arctan(y).$$

Find all the local minimizers and maximizers of f on the set $S = \{(x, y) \in \mathbb{R}^2 : x^2 + 2x + y^2 - 3 = 0 \text{ and } x^2 - 2x + y^2 - 3 = 0\}$. Find the global minimizers and maximizers of f on S as well. Are they unique? Justify your answer!

Hint: it is easier to begin describing the geometry of S and solve the problem without Lagrange multipliers.

(2) Let $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2+2x+y^2-3 \leq 0 \text{ and } x^2-2x+y^2-3 \leq 0\}$. Consider moreover $g : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$g(x, y) = 4x^2 + y^2.$$

Find the global minimizers and maximizers of g on Ω. Are they unique? Justify your answers!

Hint: you may use either KKT multipliers (with the first and second order conditions), or Lagrange multipliers (with first and second order conditions; here you may need to deal separately with the interior case). Nevertheless, first understand the geometry of Ω and the structure of g. A clever Lagrangian approach is shorter.

Solution

(1) Actually, S is the intersection of two circles, the first one is $(x + 1)^2 + y^2 = 4$ a circle of radius 2 around $(-1, 0)$, and the second one is $(x - 1)^2 + y^2 = 4$ a circle of radius 2 around $(1, 0)$. Geometrically, one can see that these circles have two intersection points, namely $X_1 = (0, \sqrt{3})$ and $X_2 = (0, -\sqrt{3})$. Hence S has only these two points, so one can just compute $f(X_1) = \arctan(\sqrt{3}) = \pi/3$ and $f(X_2) = \arctan(-\sqrt{3}) = -\pi/3$. Thus X_1 is the unique global maximizer and X_2 is the unique global minimizer of f on S. These are trivially local extemizers as well.

(2) Observe that Ω is the intersection of the two disks, for which the boundary circles were described in (1). To select the possible candidates for local extremizers from the interior of Ω, FONC gives us that

$$\nabla f(x, y) = 0 = (8x;\ 2y)^\top,$$

which implies that $X_0 = (0, 0)$ is the only candidate from the interior, and the Hessian matrix of f is always positive definite, hence X_0 is a strict local minimizer.

Now let us check the two pieces of the boundary. First, for the left circle (the right piece of the boundary), one has that $y^2 = 4 - (x + 1)^2$, with $x \in [0, 1]$, hence here f becomes of one variable,

$$f(x, y) = 4x^2 + 4 - (x + 1)^2 = 3x^2 - 2x + 3$$

and $x \in [0, 1]$. This function is clearly decreasing on $[0, 1/3]$ increasing on $[1/3, 1]$, hence its minimizer is $x = 1/3$, corresponding to the points $X_1 = (1/3, -2\sqrt{5}/3)$ and $X_2 = (1/3, 2\sqrt{5}/3)$ where the function values are the same $8/3$. The local maximum points on this piece of the boundary are $x = 0$, which corresponds to the points $X_3 = (0, -\sqrt{3})$ and $X_4 = (0, \sqrt{3})$ where the function values are 3, and $x = 1$ which corresponds to the point $X_5 = (1, 0)$ with function value 4.

One can represent similarly the left piece of the boundary, corresponding to the circle with center at $(1, 0)$. Here $y^2 = 4 - (x - 1)^2$ with $x \in [-1, 0]$ and similarly, f will depend only on one variable,

$$f(x, y) = 4x^2 + 4 - (x - 1)^2 = 3x^2 + 2x + 3$$

and $x \in [-1, 0]$. This function is clearly decreasing on $[-1, -1/3]$ and increasing on $[-1/3, 0]$, hence its minimizer is $x = -1/3$, corresponding to the points $X_6 = (-1/3, -2\sqrt{5}/3)$ and $X_7 = (-1/3, 2\sqrt{5}/3)$ where the function values are the same $8/3$. The local maximum points on this piece of the boundary are $x = 0$, which corresponds to the points $X_8 = (0, -\sqrt{3})$ and $X_9 = (0, \sqrt{3})$ where the function values are 3, and $x = -1$ which corresponds to the point $X_{10} = (-1, 0)$ with function value 4.

So collecting all the information from the interior and from the boundary, we could find the global extremizers only from the selected candidates. Comparing the function values at the different candidates, the unique global minimizer is $X_0 = (0, 0)$ with function value 0, and one has two global maximizers at $X_5 = (1, 0)$ and $X_{10} = (-1, 0)$ with function value 4.

To solve the problem using the techniques of KKT multipliers, denote the two inequality constraints by $h^+(x, y) \leq 0$ and $h^-(x, y) \leq 0$, where $h^+, h^- : \mathbb{R}^2 \to \mathbb{R}$ are defined as

$$h^{\pm} = x^2 \pm 2x + y^2 - 3.$$

We need to do two major cases, when we either minimize or maximize the function.

CASE 1 - Minimization

Actually this case can be "neglected", because we are interested in global optimizers. One can observe easily that $g(x, y) \geq 0$ and $g(x, y) = 0$ if and only if $(x, y) = (0, 0)$. Since $(0, 0)$ is an interior point of Ω, this will be the unique global minimizer.

But if you would like to do the KKT theory, you will see below that the only possible case will be $(0, 0)$ anyway.

The KKT FONC condition tells us that there exist $\mu^+, \mu^- \geq 0$ such that at a local minimizer

$$\begin{aligned} &\nabla f(x, y) + \mu^+ \nabla h^+(x, y) + \mu^- \nabla h^-(x, y) = 0, \\ &\mu^+ h^+(x, y) + \mu^- h^-(x, y) = 0. \end{aligned}$$

This can be written as the system

$$\begin{aligned} &8x + \mu^+(2x + 2) + \mu^-(2x - 2) = 0, \\ &2y + 2y\mu^+ + 2y\mu^- = 0, \\ &\mu^+(x^2 + 2x + y^2 - 3) + \mu_-(x^2 - 2x + y^2 - 3) = 0. \end{aligned}$$

Clearly, in the case when $\mu^{\pm} = 0$ one has that $X_0 = (0, 0)$, which corresponds to the interior case, where we already checked that it is a strict local minimizer.

Now, as *Case 2* set $\mu^+ = 0$ and $\mu^- > 0$ which implies that $x^2 - 2x + y^2 - 3 = 0$, and from equation 2, one has that $y = 0$. From these two, one obtains that $x = -1$ or $x = 3$. The first one would imply that $\mu^- = -2$, which is not possible, and the second one $\mu^- = -6$ which is not possible either.

As *Case 3* set $\mu^+ > 0$ and $\mu^- = 0$ which implies that $x^2 + 2x + y^2 - 3 = 0$, and from equation 2, one has that $y = 0$. These two imply that either $x = -3$ or $x = 1$. The first one would imply $\mu^+ = -6$, which is not possible, and the second one $\mu^+ = -2$ which is not possible either.

As *Case 4* set $\mu^+ > 0$ and $\mu^- > 0$ which implies that $x^2 + 2x + y^2 - 3 = 0$ and $x^2 + 2x + y^2 - 3 = 0$ which selects the candidates $(0, \pm\sqrt{3})$. In both cases, one could divide the second equation by $2y$, which would imply that $\mu^+ + \mu^- = -1$ which is impossible, since there are two positive number.

So from here one obtains that the only possibility is $X_0 = (0, 0)$, which is clearly a strict global minimizer, hence it is unique.

CASE 2 - Maximization

In this case one has to change f to $-f$ and perform similar analysis as before.

The KKT FONC condition tells us that there exist $\mu^+, \mu^- \geq 0$ such that at a local minimizer

$$\begin{aligned} &-\nabla f(x, y) + \mu^+ \nabla h^+(x, y) + \mu^- \nabla h^-(x, y) = 0, \\ &\mu^+ h^+(x, y) + \mu^- h^-(x, y) = 0. \end{aligned}$$

This can be written as the system

$$\begin{aligned} &-8x + \mu^+(2x + 2) + \mu^-(2x - 2) = 0, \\ &-2y + 2y\mu^+ + 2y\mu^- = 0, \\ &\mu^+(x^2 + 2x + y^2 - 3) + \mu_-(x^2 - 2x + y^2 - 3) = 0. \end{aligned}$$

Clearly, in the case when $\mu^{\pm} = 0$ one has that $X_0 = (0,0)$, which corresponds to the interior case, where we already checked that it is a strict local minimizer, so this cannot be maximizer.

Now, as *Case 2* set $\mu^+ = 0$ and $\mu^- > 0$ which implies that $x^2 - 2x + y^2 - 3 = 0$, and from equation 2, one has that either $y = 0$, or $\mu^- = 1$. From the first, one obtains that $x = -1$ or $x = 3$ (but this is not possible because it is not a feasible point). The first one would imply that $\mu^- = 2$. The second subcase was $\mu^- = 1$, which would result to $x = -1/3$, and hence $y = \pm 2\sqrt{5}/3$

So we can collect the candidates with the corresponding multipliers as follows

1. $x = -1, y = 0, \mu^+ = 0, \mu^- = 2,$
2. $x = -1/3, y = \pm 2\sqrt{5}/3, \mu^+ = 0, \mu^- = 1,$

As *Case 3* set $\mu^+ > 0$ and $\mu^- = 0$ which implies that $x^2 + 2x + y^2 - 3 = 0$, and from equation 2, one has that either $y = 0$ or $\mu^+ = 1$. The first implies that either $x = -3$ (which is not feasible) or $x = 1$. So $x = 1$ implies that $\mu^+ = 2$. And the second subcase was that $\mu^+ = 1$ which implies that $x = 1/3$ and hence $y = \pm 2\sqrt{5}/3$. So we can collect the candidates with the corresponding multipliers as follows

3. $x = 1, y = 0, \mu^+ = 2, \mu^- = 0,$
4. $x = 1/3, y = \pm 2\sqrt{5}/3, \mu^+ = 1, \mu^- = 0,$

As *Case 4* set $\mu^+ > 0$ and $\mu^- > 0$ which implies that $x^2 + 2x + y^2 - 3 = 0$ and $x^2 + 2x + y^2 - 3 = 0$ which selects the candidates $(0, \pm\sqrt{3})$. In both cases, one could divide the second equation by $2y$, which would imply that $\mu^+ + \mu^- = 1$. Using the first equation one obtains that $\mu^+ - \mu^- = 0$ so one has the multipliers $\mu^+ = 1/2 = \mu^-$. This gives the last class of candidates as

5. $x = 0, y = \pm\sqrt{3}, \mu^+ = 1/2, \mu^- = 1/2.$

Now one has to derive the second order conditions to see whether the candidates are local maximizers of not. The Hessian matrices are independent of the point, so we introduce

$$L(\mu^+, \mu^-) := \begin{pmatrix} -8 & 0 \\ 0 & -2 \end{pmatrix} + (\mu^+ + \mu^-) \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

To check the second order conditions, we do here only the subcase 1. The others are similar. One needs to compute the tangent space for SONC and the modified one for SOSC, which will be the same, since the h^- constraint is active only with positive multiplier. Geometrically, it is clear that this space is the y-axis (i.e. $x = 0$). Or, since $\nabla h^-(-1,0) = (-4,0)^\top$, from where you can also deduce this. Now

$$L(0,2) := \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix},$$

which tested against vectors of the form $(0,y)$ will give you $2y^2 > 0$, whenever $y \neq 0$, so $(-1,0)$ is a strict local maximizer of f.

If you finished all these second order conditions, you just compare the value of the function at the local maximizers, to obtain the global maximizers.

Exercise 3 (9 points).

John is travelling by car from a city to another (the distance between the two cities is given as $d > 0$ miles) and he wants to minimize his costs during this travel. First, time is precious, hence he wants to travel for as short time as possible. Secondly, he wants to consume the lowest possible amount of fuel. Help John to compute his optimal average speed in the following situations.

(1) John is a conscious driver, so he respects speed limitations. Suppose that on the trajectory between the two cities there is a constant speed limit of $v_L > 0$ miles/hour. John also estimated the fuel consumption for his car and he realized that for a *unit distance* the amount of fuel that he needs is proportional to the average speed, i.e. there exists an $\alpha > 0$ constant (that he can compute) such that the amount of fuel per unit distance is αv, where v is the average speed of the car on a unit distance. Compute the optimal average speed that minimizes the sum of the needed time and the consumed fuel in terms of d, v_L and α. Discuss cases with respect to the relationship between v_L and α.

Hint: one knows that the time of the travel is the ratio of the distance and the average speed. You may use the technique of KKT multipliers since we are looking for an optimal average velocity in the interval $[0, v_L]$.

(2) John has read a new study which says that the optimal consumption of his car is actually achieved at a higher speed, so after some research he has found that the consumed fuel on a *unit distance* is not exactly a linear function of the average speed but it is given by $\alpha(v)$, where $\alpha : [0, +\infty) \to [0, +\infty)$ is a smooth function, given by the expression

$$\alpha(v) = \gamma v(4/3 + \cos(\pi v/60)),$$

where $\gamma > 0$ is a given positive constant.

Observe that $\alpha(0) = 0$, α is strictly increasing at 0 and α has another local minimizer at $\overline{v} \approx 60$.

Let us assume that $v_L = 55$ miles/hour (which is the speed limit), hence if he tries to travel with $\overline{v}$ average speed, there is a chance that he gets a fine. John wants to know whether it would be optimal to travel above the speed limit even if he gets a fine. We assume that he always gets a fine, once he travels above the speed limit.

One knows also that the fine can be only given once on this trajectory, the value of it is depending on the average speed as $\beta(v - v_L)$, where $\beta : \mathbb{R} \to [0, +\infty)$ is the smooth function defined as

$$\beta(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ t^2, & \text{if } t > 0. \end{cases}$$

(i) Write down the optimization problem (involving the needed time, the cost of the fuel and the fine) with a mathematical language, and explain why is this a problem without constraints!

(ii) Explain what does it mean (in terms of the fuel and the amount of the fine) if John travels with an optimal average speed in (i) that is larger than v_L.

(iii) Write down the first order necessary optimality condition that has to be satisfied for the optimizer in (i).

(iv) Observe that the equation in (iii) is highly nonlinear, so propose an algorithm that can be used to find an approximation of its root. Write down a recursive relation between two consecutive terms of the approximation sequence.

(v) We are expecting an optimal average speed around 60 miles/hour. Hence initiate the algorithm in (iv) with $v^0 = 60$ and compute v^1.

Solution

(1) In this case the objective function is a simple function of the average speed, that is $f : (0, +\infty) \to (0, +\infty)$ defined as

$$f(v) = \frac{d}{v} + d\alpha v.$$

The feasible set is $\Omega := (0, v_L]$. Since $\lim_{v\downarrow 0} f(v) = +\infty$, we simply exclude $v = 0$ from our consideration. We would like to minimize this function.

Let us use the theory of KKT multipliers to solve this problem. For that set $g_1(v) = -v$ and $g_2(v) = v - v_L$, as real valued real functions and write the constraints as $g_1(v) \le 0$ and $g_2(v) \le 0$.

Now any candidate for local minimizer has to satisfy the KKT FONC, meaning that there exist $\mu_1, \mu_2 \ge 0$ such that at the optimizer v on has

$$-\frac{d}{v^2} + d\alpha - \mu_1 + \mu_2 = 0,$$
$$\mu_1(-v) + \mu_2(v - v_L) = 0.$$

Now let us distinguish the usual cases. *Case 1* if $\mu_1 > 0$ and $\mu_2 > 0$ is not possible, since it that case at the same time $v = 0 = v_L$, but $v_L > 0$.

Case 2: $\mu_1 = \mu_2 = 0$ implies that (having in mind that $v > 0$) $v = \frac{\sqrt{\alpha}}{\alpha}$.

If one has that $\frac{\sqrt{\alpha}}{\alpha} \le v_L$, then this is a good candidate. The condition is equivalent to

$$\alpha \ge 1/(v_L)^2. \tag{2}$$

In this case, this will be clearly a local minimizer, because $f''(\sqrt{\alpha}/\alpha) = \frac{2d}{(\sqrt{\alpha}/\alpha)^3} > 0$ and the second derivatives of the constraints vanish.

Case 3: $\mu_1 = 0$ and $\mu_2 > 0$. This implies that $v = v_L$, from where $\mu_2 = d(1/(v_L)^2 - \alpha)$ which is only possible, if this quantity is nonnegative, hence if

$$\alpha \le 1/(v_L)^2 \tag{3}$$

is satisfied. If so, this would be also a local minimizer, since $f''(v_L) > 0$.

Case 4: $\mu_1 > 0$ and $\mu_2 = 0$ cannot happen, since with would mean that $v = 0$ which cannot be optimal by a previous reasoning.

It is clear that the two conditions (2) and (3) are almost excluding in nature. From the modeling point of view, the first condition means that the consumption coefficient of the car is large, so it would be more reasonable to not travel with a high speed. In the second case the situation is opposite, so since one does not consume that much even at higher speeds, it is reasonable to travel with the highest possible speed which is the speed limit itself in this case. In the common situation is when $\alpha = 1/(v_L)^2$, both cases will give v_L as optimal average speed.

(2)(i) The new objective function will involve 3 terms, and it can be written as $h : (0, +\infty) \to (0, +\infty)$ defined as

$$h(v) = \frac{d}{v} + d\alpha(v) + \beta(v - v_L),$$

for which one can consider a problem without constraints because the upper bound constraint is now incorporated in β (he gets a fine, if he travels faster than the speed limit) and we just do not care about the lower bound constraint, if one gets a candidate that is negative, we just ignore it. And by the same reasoning as before, if v becomes small from above 0, then the function explodes.

(ii) If the optimal average speed is higher than the speed limit, that means that the optimal time at this speed, plus the cost of the fuel, plus the value of the fine is smaller, that the time and the cost of the fuel at v_L. Morally speaking, the cost of the fuel is less, so this will balance the value of the fine.

(iii) Since we consider only interior points, FONC reads as

$$-\frac{d}{v^2} + d\alpha'(v) + \beta'(v - v_L) = 0,$$

which can be expressed further as

$$-\frac{d}{v^2} + d\gamma(4/3 + \cos(\pi v/60) - v(\pi/60)\sin(\pi v/60)) + \beta'(v - v_L) = 0, \tag{4}$$

where

$$\beta'(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 2t, & \text{if } t > 0. \end{cases}$$

(iv) A possible algorithm could be Newton's algorithm to find a root of (4). (A secant algorithm for instance is also possible to find the root).

If we denote the LHS of that equation by a function $H(v)$ and taking an initial guess v^0, the sequence of approximations is defined as

$$v^{k+1} = v^k - \frac{H(v^k)}{H'(v^k)},$$

provided $H'(v^k) \neq 0$.

Let us compute $H'(v)$ now. One has

$$H'(v) = \frac{2d}{v^3} + d\gamma[-(2\pi/60)\sin(\pi v/60) - v(\pi/60)^2\cos(\pi v/60)] + \beta''(v - v_L).$$

Let us evaluate the functions in $v^0 = 60$. This means that

$$H(60) = -\frac{d}{60^2} + d\gamma[4/3 - 1] + 10 = -\frac{d}{60^2} + \frac{d\gamma}{3} + 10$$

and

$$H'(60) = \frac{2d}{60^3} + d\gamma(\pi^2/60) + 2,$$

thus the first step in Newton's algorithm reads as

$$v^1 = 60 - \frac{-\frac{d}{60^2} + \frac{d\gamma}{3} + 10}{\frac{2d}{60^3} + d\gamma(\pi^2/60) + 2}$$