Math 164-1: Optimization

Instructor: Alpár R. Mészáros

Final Exam, June 9, 2016

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **180 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs. You may lose points in the lack of justification of your answers.
- Theorems from the lectures and homework assignments may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 5 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **26 points**, which means that there are **6 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Exercise 4	
Exercise 5	
Total	

Exercise 1 (4 points).

Let us consider the following linear program

$$\min\{2x_1 + 2x_2 + 2x_3\} \quad \text{s.t.} \begin{cases} x_1 + x_2 + x_3 &= 6, \\ 2x_1 + x_3 &= 5, \\ -x_2 + 3x_3 &= 7, \end{cases} \quad x_1, x_2, x_3 \geq 0. \tag{LP_1}$$

(1) Write a feasible solution for (LP_1)!

(2) Does (LP_1) have an optimal solution? If yes, is this solution unique? Determine all the optimal solutions and the value of the objective function as well!

(3) What would happen, if one would exchange the constraints $x_1, x_2, x_3 \geq 0$ in (LP_1) to $x_1, x_2, x_3 \leq 0$? Would this new problem have a solution?

Imagine now that one removes the last equality constraint and one considers

$$\min\{2x_1 + 2x_2 + 2x_3\} \quad \text{s.t.} \begin{cases} x_1 + x_2 + x_3 &= 6, \\ 2x_1 + x_3 &= 5, \end{cases} \quad x_1, x_2, x_3 \geq 0. \tag{LP_2}$$

(4) What does the set of constraints in (LP_2) represent geometrically? Is it convex? Determine all the feasible solutions for (LP_2)! Justify your answer!

(5) Determine all the optimal feasible solutions for (LP_2)! Do we still have uniqueness? Determine the value of the objective function for the feasible solutions! Justify your answers!

Hint: it is easier to use some geometrical arguments and the structure of (LP_2), than to use the simplex algorithm for instance.

Exercise 2 (6 points).

Find the triangles in the plane with a fixed given area $A > 0$ that have minimal perimeter. To this aim determine the lengths of their sides $a, b, c > 0$ in terms of A using the theory of Lagrange multipliers.

(1) Write down the first order optimality conditions and select the candidates for the optimizers!

(2) Eventually using second order optimality conditions, determine which from the selected candidates in (1) are indeed optimizers.

Hint: you may use Heron's formula for the area, i.e $A = \sqrt{p(p-a)(p-b)(p-c)}$, where $p := \dfrac{a+b+c}{2}$ is the semi-perimeter. Solutions, using other formulas will be also accepted.

Exercise 3 (6 points)**.**

Let us consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{2}x^\top Ax - b^\top x + \gamma\|x\|^2,$$

where $n \geq 1$ is an integer, $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $b \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$ and $\|x\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$. In the followings, we denote the eigenvalues (that are all real) of A with possible multiplicities ordered by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. We aim to study the minimization problem

$$\min_{x\in\mathbb{R}^n} f(x). \tag{P}$$

(1) Show that the condition $\lambda_1 + 2\gamma > 0$ is a sufficient condition that ensures that (P) has a unique solution. Determine this unique solution! *Hint:* you may use first and second order optimality conditions! Moreover you may use Rayleigh's inequality, saying that if $Q \in \mathbb{R}^{n\times n}$ is symmetric, then the minimum of Rayleigh's quotient $\dfrac{x^\top Qx}{\|x\|^2}$ is the smallest, while the maximum of this quotient is the largest eigenvalue of Q.

(2) Let us suppose that for some of the eigenvalues λ_i one has $\lambda_i + 2\gamma < 0$. Show that in this case (P) does not have a solution! *Hint:* select a sequence of vectors $(x^k)_{k\geq 0}$ (using for instance the eigenvectors associated to the eigenvalue λ_i) such that $f(x_k) \to -\infty$, as $k \to +\infty$. You may also use the Cauchy-Schwartz inequality (i.e. $a \cdot b \leq \|a\|\|b\|, \forall a, b \in \mathbb{R}^n$).

From now on, one supposes that the assumption in (1), i.e. $\lambda_1 + 2\gamma > 0$ is satisfied, hence (P) has a unique minimizer x^*. We aim to find this minimizer numerically.

(3) Discuss why could we use *Newton's algorithm* to find x^*. How many steps are necessary using Newton's algorithm to find x^* starting from any initial guess $x^0 \in \mathbb{R}^n$? Justify your answer!

(4) Explain why can we use the *conjugate direction* algorithm developed during the lectures to find x^*! With respect to which matrix need we choose the conjugate directions? Justify your answer!

(5) Suppose that $A = I_n$ is the identity matrix. Construct a set of n vectors that are conjugate w.r.t. the matrix determined in (4)!

(6) Suppose once again that $A = I_n$. Write down the updates in the conjugate direction algorithm to find x^* starting from and initial point x^0 using the conjugate directions from (5)!

(7) Explain what is happening geometrically while proceeding the algorithm in (6)! Supposing that one knows x^*, construct an initial guess x^0 for which the algorithm in (6) terminates in precisely 2 steps.

Exercise 4 (5 points)**.**

A small company in a country far-far away named Appel wants to attribute three tasks (let us say: changing the locks on some of the office doors; ordering the files in some of the offices and assembling some electronic devices) to its workers (John the locksmith and Jane the accountant) in a way that it maximizes the *productivity*. We know that both John and Jane can work on fractions of each of the tasks and based on previous experience, the company knows exactly how productive John and Jane is for the different tasks. This can be seen from the following *productivity matrix*

$$P = \begin{pmatrix} 5 & 1 & 3 \\ 0 & 4 & 2 \end{pmatrix}.$$

The two rows of this matrix represent the productivity of John and Jane, respectively in the different tasks, e.g. John has a productivity 5 when changing the locks, while 1 for ordering the files, while Jane has productivity 0 when changing locks and productivity 2 for assembling the devices, and so on.

We know moreover the total number of each tasks: there are 3 locks to be changed, 2 offices with files to be ordered and 10 electronic devices to be assembled.

The company is aiming to find the optimal values for the variables $\gamma_{ij} \geq 0$, $i \in \{1,2\}$ and $j \in \{1,2,3\}$ which represent how much John and Jane works on the different tasks to obtain the highest possible global productivity. For instance if we get $\gamma_{13} = 7.4$ that means that John should assemble 7.4 electronic devices, if we get $\gamma_{22} = 1.7$ that means that Jane should order the files in 1.7 offices, etc.

So the company is looking for γ_{ij}'s that solve the problem

$$\max \sum_{i=1}^{2} \sum_{j=1}^{3} P_{ij} \gamma_{ij},$$

under the constraints that all the tasks are completed. Our job is to find these optimal quantities and let the company know how to distribute the different tasks among John and Jane.

(1) Write the above problem as a linear program, i.e. write the objective function that has to be maximized together with the constraints that should be satisfied. Explain how did you obtain them!

(2) Is the LP from (1) in standard form (in the sense of our lectures)? If not, transform it into a standard form. Justify your answer!

(3) Transform the LP from (1) into canonical form (in the sense of our lectures). Give two different set of basic variables and write a basic feasible solution for each cases.

(4) Use the simplex algorithm to solve the LP from (1), at each step write the reduced cost coefficients and a basic feasible solution. Write the optimal solution and the value of the objective function at the optimizer.

(5) Is the optimizer that you have found in (4) unique? Justify your answer! Interpret the solution that you have found in (4)! What do you observe from the "economical point of view"? For this, answer the following questions: what would be different if we would change the coefficients in the matrix P? Do you observe the same phenomena? Would be something different if one would have more than one locksmith or accountant at the company with similar skills?

Exercise 5 (5 points)**.**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. We define the function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ as

$$f^*(y) = \max_{x \in \mathbb{R}^n} \{y \cdot x - f(x)\},$$

whenever the maximizer exists and $+\infty$ otherwise. We aim to study some properties of f^*.

(1) Show that if $f(0) = 0$ then $f^*(y) \geq 0$ for all $y \in \mathbb{R}^n$.

(2) Show that $x \cdot y \leq f(x) + f^*(y)$ for all $x, y \in \mathbb{R}^n$.

(3) Let $p > 1$ and $f(x) = \frac{1}{p}\|x\|^p$. Show that f^* is well-defined (i.e. its value is always finite) and $f^*(y) = \frac{1}{q}\|y\|^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. *Hint:* use for instance the Cauchy-Schwartz inequality for the first part (i.e. $a \cdot b \leq \|a\|\|b\|, \forall a, b \in \mathbb{R}^n$) and the first order necessary optimality condition for the second part. For the second part you should first understand what is happening when $n = 1$.

(4) Using the previous points, show Young's inequality, i.e. $x \cdot y \leq \frac{1}{p}\|x\|^p + \frac{1}{q}\|y\|^q$ for all $x, y \in \mathbb{R}^n$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(5) Let us define $f^{**} = (f^*)^*$. Suppose that f^* and f^{**} are well-denied (i.e. finite) everywhere in $\mathbb{R}^n$. Show that $f^{**}(x) \leq f(x)$ for all $x \in \mathbb{R}^n$.