Math 164-1: Optimization – Homework 1

Due: April 1, 2016

Exercise 1.

Read carefully Chapters 2, 3, 4, 5 from the book of Chong and Zak and recall the known notions from linear algebra, geometry and calculus. If you do not have the textbook yet, please make sure that you recall the notions of gradient, Jacobian, Hessian, Taylor expansion of functions depending on several variables; convex set, line segments, planes, hyperplanes in $\mathbb{R}^d$; eigenvalues, eigenvectors of matrices; positive definite matrices, quadratic forms, etc. The wikipedia could be a very good source as well.

Exercise 2. *Exercises 2.1 and 2.9 from the book of Chong and Zak.*

(1) (Exercise 2.1) Let $A \in \mathbb{R}^{m\times n}$ and rank $A = m$. Show that $m \le n$.

(1) (Exercise 2.9) Show that for any two vectors $x, y \in \mathbb{R}^n$, $|\|x\| - \|y\|| \le \|x - y\|$.
Hint: Write $x = (x - y) + y$, and use the triangle inequality. Do the same for y.

Exercise 3. *Exercises 3.6 and 3.15 from the book of Chong and Zak.*

(1) (Exercise 3.6) Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$. Show that the eigenvalues of the matrix $I_n - A$ are $1 - \lambda_1, \dots, 1 - \lambda_n$, where I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$.

(2) (Exercise 3.15) Is the quadratic form

$$x \ni \mathbb{R}^2 \mapsto x^T \begin{bmatrix} 1 & -8 \\ 1 & 1 \end{bmatrix} x$$

positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite? Justify your answer!

Exercise 4. *Exercises 4.2, 4.3 and 4.4 from the book of Chong and Zak.*

(1) (Exercise 4.2) Show that the set $\{x \in \mathbb{R}^n : \|x\| \le r\}$ is convex, where $r > 0$ is a given real number and $\|x\| = \sqrt{x^T x}$ is the Euclidean norm of $x \in \mathbb{R}^n$.

(2) (Exercise 4.3) Show that for any matrix $A \in \mathbb{R}^{m\times n}$ and vector $b \in \mathbb{R}^m$, the set (linear variety) $\{x \in \mathbb{R}^n : Ax = b\}$ is convex.

(3) (Exercise 4.4) Show that the set $\{x \in \mathbb{R}^n : x \ge 0\}$ is convex (where $x \ge 0$ means that every component of x is nonnegative).

Exercise 5. *Exercises 5.1, 5.2, 5.3 and 5.4 from the book of Chong and Zak.*

(1) (Exercise 5.1) Show that a sufficient condition for $\lim_{k\to\infty} A^k = O_n$ is $\|A\| < 1$.

(2) (Exercise 5.2) Show that for any matrix $A \in \mathbb{R}^{n\times n}$,

$$\|A\| \ge \max_{1\le i\le n} |\lambda_i(A)|.$$

Hint: Use Exercice 5.1.

(3) (Exercise 5.3) Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = (a^T x)(b^T x),$$

where a, b and x are n-dimensional vectors.

a. Find $\nabla f(x)$.

b. Find the Hessian $\mathbf{F}(x) = D^2 f(x)$.

(4) (Exercise 5.4) Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^2$ by $f(x) = x_1^2/6 + x_2^2/4$ and $g(t) = [3t + 5, 2t - 6]^T$. Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(t) = f(g(t))$. Evaluate $\frac{dF}{dt}(t)$ using the chain rule.