Math 164-1: Optimization – Homework 2

Due: April 8, 2016

Exercise 1. *Exercises 6.2, 6.3 and 6.4 from the book of Chong and Zak.*

(i) (Exercise 6.2) Find the minimizers and maximizers of the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x_1, x_2) = \frac{1}{3}x_1^3 - 4x_1 + \frac{1}{3}x_2^2 - 16x_2.$$

(ii) (Exercise 6.3) Let $\Omega \subseteq \mathbb{R}^d$ and $f : \Omega \to \mathbb{R}$. Show that if x^* is a global minimizer of f over Ω and if $x^* \in \Omega' \subset \Omega$, then x^* is a global minimizer of f over Ω'.

(iii) (Exercise 6.4) Let $\Omega, \Omega' \subseteq \mathbb{R}^d$ and $f : \Omega' \to \mathbb{R}$. Suppose that x^* is a local minimizer of f over Ω and $\Omega \subset \Omega'$. Show that if x^* is an interior point of Ω, then x^* is a local minimizer of f over Ω'. Show that the same conclusion cannot be made if x^* is not an interior point on Ω.

Hint: for the second part construct an example in $1D$ for instance.

Exercise 2. *Exercises 6.8 and 6.9 from the book of Chong and Zak.*

(i) (Exercise 6.8) Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x) = x^T \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} x + x^T \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 6.$$

(a) Find the gradient and the Hessian of f at the point $[1 \;\; 1]^T$.

(b) Find the directional derivative of f at $[1 \;\; 1]^T$ w.r.t. a unit vector in the direction of maximal rate of increase.

(c) Find an interior point that satisfies the first order necessary optimality condition (for a local minimizer) of f. Does this point satisfy the second order necessary optimality condition?

(ii) (Exercise 6.9) Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x_1, x_2) = x_1^2 x_2 + x_2^3 x_1.$$

(a) In what direction does the function f decrease most rapidly at the point $x^0 = [2 \;\; 1]^T$?

(b) What is the rate of increase of f at the point x^0 in the direction of maximum decrease of f?

(a) Find the rate of increase of f at the point x^0 in the direction $d = [3 \;\; 4]^T$.

Exercise 3. *Exercises 6.18 from the book of Chong and Zak.*

Suppose that we are given $n \geq 1$ real numbers $x_1, \ldots, x_n$. Find the number $\bar{x} \in \mathbb{R}$ (if it exists) such that the sum of the squared difference between $\bar{x}$ and the numbers above is minimized. In other words, solve

$$\min_{\bar{x} \in \mathbb{R}} \sum_{i=1}^{n} (\bar{x} - x_i)^2.$$

Exercise 4. *Exercises 6.11 and 6.12 from the book of Chong and Zak.*

(i) (Exercise 6.11) Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x_1, x_2) = -x_2^2$. Let us consider the minimization problem $\min_{x \in \Omega} f(x)$, where $\Omega := \{x \in \mathbb{R}^2 \,:\, |x_2| \leq x_1^2, x_1 \geq 0\}$.

(b) Does the point $[0 \;\; 0]^T$ satisfy the first order necessary condition for a local minimizer? In other words, is it true that $d \cdot \nabla f(0,0) \geq 0$ for all feasible directions d at $[0 \;\; 0]$?

(c) Is the point $[0 \;\; 0]^T$ local/strict local minimizer, a local/strict local maximizer or none of the above?

(ii) (Exercise 6.12) Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = 5x_2$ and let $\Omega = \{x \in \mathbb{R}^2 \;:\; x_1^2 + x_2 \geq 1\}$ and let us consider the problem

$$\min_{x \in \Omega} f(x).$$

(a) Does the point $x^* := [0 \;\; 1]^T$ satisfy the first order necessary condition?

(a) Does the point $x^* := [0 \;\; 1]^T$ satisfy the second order necessary condition?

(a) Is the point $x^* := [0 \;\; 1]^T$ a local minimizer?