Math 164-1: Optimization

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First Midterm, April 20, 2016

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- You may use either a pen or a pencil to write your solutions. However, if you use a pencil I will withheld your paper for **two** weeks after grading it.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs. You may loose points in the lack of justification of your answers.
- Theorems from the lectures and homework may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 3 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **27 points**, which means that there are **7 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Total	

Exercise 1 (13 points)**.**

Let us consider the rhombus $\Omega \subset \mathbb{R}^2$ bounded by the straight lines $y = x$, $y = -x$, $y = \sqrt{2} + x$ and $y = \sqrt{2} - x$. We consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given as $f(x, y) = x^2 + y^2 - \sqrt{2}y + 9/2$. We aim to find all the local minimizers and maximizers of f on Ω. Note that by Weierstrass' theorem (Ω is compact and f is continuous on Ω), both global minimizers and maximizers exist.

(1) Select all the candidates from the interior of Ω both for local minimizers and maximizers.

(2) Using second order sufficient conditions decide whether the selected points in (1) are local minimizers, maximizers or neither of them.

(3) Compute the feasible directions at $(0, 0)$. Using first and second order necessary conditions, decide whether $(0, 0)$ is a good candidate for local maximizer of f on Ω or not. Compute the value of f at $(0, 0)$ and compare it with the values of f at the other three vertices of Ω.

(4) Find the global maximizers of f on the side of Ω where $y = x$. Compute the value of f at these points. *Hint:* do not use Lagrange multipliers!

(5) Characterize and draw the following level sets of f: $LS_5 := \{(x, y) \in \mathbb{R}^2 : \ f(x, y) = 5\}$ and $LS_6 := \{(x, y) \in \mathbb{R}^2 : \ f(x, y) = 6\}$.

(6) Find all the global maximizers and minimizers of f on Ω. *Hint:* you may use some of the geometrical properties of f – from the previous point – and point (3).

Exercise 2 (8 points)**.**

Let $A \in \mathbb{R}^{n\times n}$, $n \geq 1$, be a symmetric matrix. We would like to find the minimum and the maximum of the so-called *Rayleigh quotient* associated to the matrix A. To do so, we define the Rayleigh quotient as the the function $R : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, defined as $R(x) = \dfrac{x^\top Ax}{\|x\|^2}$. Here $\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}$ denotes the usual euclidean norm on $\mathbb{R}^n$.

(1) Show that $R(cx) = R(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R} \setminus \{0\}$.

(2) Using (1), show that looking for the local maximizers and minimizers of R on $\mathbb{R}^n \setminus \{0\}$ is equivalent to look for the local extremizers of R on the set $\Omega := \{x \in \mathbb{R}^n : \ \|x\|^2 = 1\}$.

(3) Using eventually the theory of Lagrange multipliers, show that the possible candidates for the local extremizers of R on Ω are the eigenvectors (normalized to be points in Ω) of A.

(4) Characterize the tangent space $T(x^1)$ and the normal space $N(x^1)$ to Ω at x^1, where $x^1 \in \Omega$ is an eigenvector of A associated to the smallest eigenvalue λ_1 of A.

(5) Assuming that one can order the eigenvalues (with possible multiplicities) of A as $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{n-1} < \lambda_n$, find the global minimizers and maximizers of R on Ω. Compute the minimum and the maximum of R.

Hint: you may plug in the candidates from (3) into R to find the global extremizers. Another possibility is to work with second order sufficient conditions from the Lagrangian theory.

Exercise 3 (6 points).

Let us consider the domain $\Omega \subset \mathbb{R}^2$ defined as $\Omega := \{(x, y) \in \mathbb{R}^2 : x \in [1, e];\ 0 \leq y \leq \ln(x)\}$, where e denotes the base of the natural logarithm ln. We aim to find the global minimizers and maximizers of the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = e^{xy}$ on Ω. Here also we observe that we are optimizing a continuous function on a compact set, hence the existence of a global minimizer and a global maximizer is given by the Theorem of Weierstrass.

(1) Show that there are no candidates for local extremizers of f in the interior of Ω and deduce that the global extremizers lie on $\partial\Omega$.

(2) Decompose $\partial\Omega$ into three pieces and find the global maximizers and minimizers of f on each of the pieces. Explain why cannot we use the theory of Lagrange multipliers for equality constraints on each piece of the boundary separately! *Hint*: Observe that finding the extremizers on each piece of the boundary reduces the problem to a 1D problem.

(3) Deduce from (2) the global minimizers and maximizers of f on Ω. Compute the maximal and minimal values as well.