Math 164-1: Optimization

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Final exam, December 11, 2015

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **3 hours**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision. You may lose points in the lack of justification of your answers.
- Theorems from the lectures may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has **5 problems** and is worth **20 points**. Adding up the indicated points you can observe that there are **25 points**, which means that there are **5 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. Hence you do not have to worry if you cannot solve some problem. All scores higher than 20 will be considered as 20 in the gradebook.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Exercise 4	
Exercise 5	
Total	

Exercise 1 (Two dimensional linear programs – 4 points)**.**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x_1, x_2) = -6x_1 + 3x_2$. Using geometrical arguments we aim to solve the two linear programs associated to f,

$$\begin{aligned} &\min f(x_1, x_2) \\ &\text{s.t. } (x_1, x_2) \in \Omega \end{aligned} \tag{P_1}$$

and

$$\begin{aligned} &\max f(x_1, x_2) \\ &\text{s.t. } (x_1, x_2) \in \Omega, \end{aligned} \tag{P_2}$$

where $\Omega := \left\{(x_1, x_2) \in \mathbb{R}^2 : \ -2x_1 + x_2 \geq 5;\ 5x_1 + 3x_2 \geq 1;\ 2x_1 + 3x_2 \leq 7\right\}$. Draw the feasible set Ω, then determine the solutions of (P_1) and (P_2) giving also the values of f at these points. Justify your answer.

Exercise 2 (6 points)**.**

Let us consider the following linear program

$$
\begin{array}{ll}
\max & 2x_1 - 4x_2 + 3x_3 \\
\text{s.t.} & \left\{ \begin{array}{lll}
3x_1 + x_3 & \geq & -12 \\
x_1 + 2x_3 & \leq & 7 \\
x_1 + 2x_2 + 4x_3 & = & 10 \\
x_1, x_2, x_3 & \geq & 0
\end{array} \right.
\end{array} \tag{LP}
$$

(1) Write (LP) in the standard form.

(2) Transform the standard linear program into a canonical form.

(3) Use the simplex algorithm to solve (LP) starting with the previously obtained canonical tableau. At each step determine the reduced cost coefficients and the basic feasible solutions. Determine the optimal basic solution and the value of the objective function at this point.

(4) Determine the dual problem associated to the original (LP). Would it be easier to solve this problem? Explain why or why not?

Hint: note that you have both inequality and equality constraints, so writing the dual consider the equality constraint as two inequality constraints.

Exercise 3 (5 points)**.**

Let us consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x_1, x_2) = -x_1 - 2x_2 - 2x_1 x_2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

and the triangle defined as $\Omega := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0;\ x_2 \geq 0;\ x_1 + x_2 \leq 1\right\}$. We aim to find the local minimizers and maximizers of f on Ω.

(1) Explain why does a global minimizer and maximizer of f on Ω exist!

(2) Show that no local minimizers and local maximizers can exist in the interior of Ω.

(3) Determine the global minimizers and maximizers of the functions $g(x) = \frac{1}{2}x^2 - x$ and $h(x) = \frac{1}{2}x^2 - 2x$ when $x \in [0, 1]$.

(4) Using eventually the theory of Lagrange multipliers, find the global minimizers and maximizers of f on Ω.

Hint for (4): explain why is enough to introduce only one Lagrange multiplier, associated to only one part of the boundary, then work with only this multiplier.

Exercise 4 (5 points)**.**

Let $C := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1;\ x_2 \geq 0\}$ be the unit half ball in $\mathbb{R}^2$. We aim to write down a formula for the orthogonal projection of an arbitrary point $(a_1, a_2) \in \mathbb{R}^2$ onto C. With other words for $(a_1, a_2) \in \mathbb{R}^2$ given, we are looking for $(x_1^*, x_2^*) \in C$ such that the distance between (a_1, a_2) and (x_1^*, x_2^*) is minimal. Hence the above problem is equivalent to

$$\min_{(x_1,x_2)\in C} \left\{ \frac{1}{2}(x_1 - a_1)^2 + \frac{1}{2}(x_2 - a_2)^2 \right\} \tag{1}$$

Why does an optimizer exist for any $(a_1, a_2) \in \mathbb{R}^2$? You can admit here that any local minimizer of the above problem is a global minimizer. Using the theory of KKT multipliers solve the above problem, discussing the cases w.r.t. the geometrical location of the point $(a_1, a_2) \in \mathbb{R}^2$. Is the optimizer unique for a given (a_1, a_2)? Drawing a picture, explain geometrically as well what you obtained.

Hint: you should discuss 5 cases w.r.t. the geometrical location of (a_1, a_2).

Exercise 5 (5 points)**.**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = \frac{1}{2}ax_1^2 + \frac{1}{2}bx_2^2,$$

where $0 < a < b$ are two given reals. We consider the problem

$$\min_{(x_1,x_2)\in\mathbb{R}^2} f(x_1, x_2). \tag{2}$$

(1) Show that there exists a unique optimizer $(x_1^*, x_2^*) \in \mathbb{R}^2$ of f. Determine this point, using first and second order optimality conditions.

(2) Write down the Newton algorithm to find (x_1^*, x_2^*) and show that it is converging in one step for any initial condition $(x_1^0, x_2^0) \in \mathbb{R}^2$.

(3) Write down the gradient descent algorithm (steepest descent) with optimal step size to find the optimizer (x_1^*, x_2^*).

- Write (x_1^{k+1}, x_2^{k+1}) explicitly in terms of (x_1^k, x_2^k) and the optimal step size $\alpha_k > 0$.
- Write a condition for α_k to be satisfied in terms of (x_1^k, x_2^k) and (x_1^{k+1}, x_2^{k+1}).
- Let us assume that at some step $k > 0$ one has $\nabla f(x_1^k, x_2^k) \neq 0$. Show that the algorithm is converging if and only if either $x_1^k = 0$ or $x_2^k = 0$.
- Give a sufficient condition for $(x_1^0, x_2^0) \neq (0, 0)$ that implies the convergence of the algorithm. In how many steps can we reach the optimizer in this last case?