Math 164-1: Optimization

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Name:

Student ID:

Signature:

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision. You may lose points in the lack of justification of your answers.
- Theorems from the lectures may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 3 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **25 points**, which means that there are 5 **"bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Total	

Exercise 1 (12 points).

Let $\Omega \subset \mathbb{R}^3$, $\Omega := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ -1/2 \le x_i \le 1/2, \ i \in \{1, 2, 3\}\}$ be a unit cube in $\mathbb{R}^3$. Let us define the function $f : \mathbb{R}^3 \to \mathbb{R}$ as $f(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1$. We consider now the optimization problem

$$\max_{x\in\Omega} f(x) \tag{1}$$

Answer the following questions, showing full justification!

(1) Why does a maximizer exist in Problem 1?

(2) Why is the function f of class $C^2(\mathbb{R}^3)$? Compute the gradient $\nabla f(x)$ and the Hessian $D^2 f(x)$ for an arbitrary point $x \in \mathbb{R}^3$.

(3) Select all the candidates from the interior of Ω that could be local maximizers of f. Decide whether the selected candidates are local maximizers or not.

(4) Consider the point $y = (1/2, 0, 0) \in \Omega$. Is y an interior point of Ω? Characterize all the feasible directions $e = (e_1, e_2, e_3) \in \mathbb{R}^3$ at y (a picture could help). Is y a (strict) local maximizer of f?

(5) Consider the point $z = (1/2, 1/2, 1/2) \in \Omega$. Is z an interior point of Ω? Characterize all the feasible directions $e = (e_1, e_2, e_3) \in \mathbb{R}^3$ at z. Is z a (strict) local maximizer of f? What is the value of f at z?

(6) Is z from (5) a global maximizer of f? Determine another point $t \in \Omega$, $t \neq z$ which is a global maximizer of f.

Exercise 2 (8 points).

Let $\Omega \subset \mathbb{R}^2$, $\Omega := \left\{x = (x_1, x_2) \in \mathbb{R}^2 \;:\; x_1^2 + x_2^2 \leq 3\right\}$. We define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x_1, x_2) = x_1 x_2^2$. We wish to study the local minimizers and maximizers of f on Ω. Answer the following questions, showing full justification!

(1) Find all the candidates for local minimizers and maximizers in the interior of Ω.

(2) Study the points selected in (1) and decide whether they are (strict) local minimizers, (strict) local maximizers or neither of them.

(3) Characterize the feasible directions $e = (e_1, e_2) \in \mathbb{R}^2$ and $d = (d_1, d_2) \in \mathbb{R}^2$ at the points $x = (1, \sqrt{2})$ and $y = (-1, \sqrt{2})$ and determine whether x and y satisfy the first and second order necessary optimality conditions for maximizers and minimizers respectively!

(4) Is x a (unique) global maximizer? Is y a (unique) global minimizer?

Exercise 3 (5 points)**.**

Imagine that we have 3 events that have probability p_1, p_2 and p_3 respectively (these are real numbers in $[0, 1]$), moreover we know that $p_3 = 1 - p_1 - p_2$ and $p_3 \geq 0$.

Suppose that we want to find the discrete probability distribution (p_1^*, p_2^*, p_3^*) for these three events in such a way that (p_1^*, p_2^*) maximizes the Shannon-type entropy (a function widely used in information theory, statistical physics and in many other fields as a measure of uncertainty), defined as

$$E(p_1, p_2) := -p_1 \ln p_1 - p_2 \ln p_2,$$

where "ln" denotes the natural logarithm, i.e. of base $e \approx 2.71$. By the structure of the events one has always that

$$p_3^* = 1 - (p_1^* + p_2^*) \tag{2}$$

that is why we do not have to optimize also in p_3.

Since we are dealing with probabilities one has that $p_1, p_2, p_3 \in [0, 1]$ and by the structural condition (2) one has to impose also the constraint that $p_1+p_2 \leq 1$. Setting $\Omega := \{(p_1, p_2) \ : \ p_1, p_2 \in [0, 1], \ p_1 + p_2 \leq 1\}$, find all the interior local maximizers $(p_1^*, p_2^*) \in int(\Omega)$ of E, and determine the optimal value also for p_3^* using formula (2).

What can we say about the boundary points of Ω satisfying $p_1 + p_2 = 1$ and $p_1, p_2 \in (0, 1)$? Are there any, satisfying the first order necessary conditions for maximizers?