Math 164-1: Optimization

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Midterm, November 16, 2015

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision. You may loose points in the lack of justification of your answers.
- Theorems from the lectures may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has **3 problems** and is worth **20 points**. Adding up the indicated points you can observe that there are **26 points**, which means that there are **6 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Total	

Exercise 1 (15 points)**.**

(1) Describe Newton's method in 1D to approximate the roots and local optima (minimizers/maximizers) of C^1, respectively C^2 functions.

(2) We aim now to use Newton's method to approximate the maximizer of $f : [-\pi/2, \pi/2] \to \mathbb{R}$, $f(x) = \cos(x)$. What are the optimizers of this function and what is the optimal value? Are the optimizers local or global? What about the uniqueness of the optimizers? Why is this problem equivalent to look for roots of the function $g : [-\pi/2, \pi/2] \to \mathbb{R}$, $g(x) = \sin(x)$?

(3) Initiate the algorithm (for g) with $x_0 \in [-\pi/2, \pi/2]$ such that $x_0 > 0$. Show that $x_1 < 0$. Show that in general $x_k \cdot x_{k+1} \leq 0$, for all $k \in \mathbb{N}$.

Hint: study the sign and growth properties of the function $h_1 : [-\pi/2, \pi/2] \to \mathbb{R}$, $h_1(x) = x - \tan(x)$.

(4) Show that if $x_k \in [-\pi/4, \pi/4]$ one has that $|x_{k+1}| < |x_k|$.

Hint: study the sign and growth properties of the function $h_2 : [-\pi/4, \pi/4] \to \mathbb{R}$, $h_2(x) = 2x - \tan(x)$.

(5) Show that the algorithm converges for all $x_0 \in [-\pi/4, \pi/4]$.

(6) Show that the order of convergence (if it exists) is at least 2. Is the interval in (5) optimal, i.e. could we choose x_0 outside of this interval (but of course not outside of $[-\pi/2, \pi/2]$) and still have the convergence? Justify your answer!

Hint for the convergence proof and order of convergence: a possible way is to use a second order (exact, i.e. with reminder term) Taylor expansion for $\sin(0)$ around x_k, then use the construction of the sequence x_k and try to give an upper bound for the term $\frac{|\sin(\xi_k)|}{2|\cos(x_k)|}$, where ξ_k is between 0 and x_k. Other correct proofs are also accepted!

Exercise 2 (7 points)**.**

We aim to solve numerically the following system of linear equations for $x = [x_1 \;\; x_2]^T \in \mathbb{R}^2$:

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{E}$$

We use the notation $Ax = b$ for (E) in the followings.

(1) Why does a solution for (E) exist? We claim that finding a solution of (E) (if there is any) is equivalent to the problem of finding a minimizer of the function

$$\mathbb{R}^2 \ni x \mapsto \frac{1}{2}\|Ax - b\|^2. \tag{F}$$

Why is this the case?

(2) We will use the conjugate gradient method to solve numerically the system (E). Show that the function in (F) is a quadratic one, generated by a positive definite, symmetric matrix from $\mathbb{R}^{2\times 2}$.

(3) Using the initial guess $x^0 = [0 \;\; 0]^T$, develop the steps of the conjugate gradient algorithm and show that it converges in at most 2 steps. Check if you have found indeed a solution of the system (E). If it is the case, is it unique? Why?

Exercise 3 (4 points)**.**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 convex function. We suppose that the function has a unique minimizer and construct the following algorithm to approximate it:

$$x^{k+1} = x^k - \alpha_k B \nabla f(x^k),$$

where

$$B := \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}$$

with $b \in \mathbb{R}$ and

$$\alpha_k := \text{argmin}_{\alpha \in \mathbb{R}} f(x^k - \alpha B \nabla f(x^k)). \tag{1}$$

At some x^k during the algorithm, let us suppose to obtain $\nabla f(x^k) = [1 \;\; 2]^T$. What is the maximal range for $b \in \mathbb{R}$ in this case that implies that $\alpha_k \geq 0$ (where α_k is given in (1))? For all b in the found range, is the matrix B positive definite?

Hint: use the fact that the graph of a convex function lies always above its tangent plane at any point (in particular at x^k) and use the construction of x^{k+1} and α_k.