Math 32B-1: Calculus of Several Variables – Homework 3

Due: January 22, 2016

Exercise 1 (Based on Rogawski-Adams)**.**

(1) Let $f : [a, b] \to \mathbb{R}$ be a real valued real function that is continuous on $[a, b]$. We define $F, G : [a, b] \to \mathbb{R}$ as $F(x) = \int_a^x f(t)\,\mathrm{d}t$ and $G(x) = \int_a^x \int_a^t f(s)\,\mathrm{d}s\,\mathrm{d}t$ for all $x \in [a, b]$. Using the Fundamental Theorem of Calculus, show that $F'(x) = f(x)$, $G'(x) = F(x)$ and $G''(x) = f(x)$, for all $x \in [a, b]$.

(2) Let us fix $x_0 \in (a, b)$. Show (also geometrically, with a picture) that $G(x_0)$ is nothing else, but the double integral of f on a triangle in $\mathbb{R}^2$. By changing carefully the order of integration, also show that $G(x_0) = \int_a^{x_0} (x_0 - s)f(s)\,\mathrm{d}s$.

Exercise 2 (Based on Rogawski-Adams)**.**

(1) Find the volume of the solid object in $\mathbb{R}^3$ above the paraboloid surface $z = x^2 + y^2$ and below $z = 9$.

(2) Describe the domain of integration (also try to draw a picture) in $\mathbb{R}^3$ and compute the integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xy\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

Exercise 3.

(1) Let $D \subset \mathbb{R}^3$ be the right pyramid with base $[-1, 1] \times [-1, 1]$ in the xOy plane and hight 3. Draw the domain D. Then compute its volume using integration. Explain in details how you think.

(2) Now cut the pyramid at hight 1 with a plane parallel to the xOy plane. Using the previous idea, compute the volumes of the two pieces of the pyramid.

(3) Let $D \subset \mathbb{R}^3$ be the right cone with base given by the set $\{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 \leq 4\}$ in the xOy plane and hight 5. Draw this object and compute its volume using integration (you can assume that you know how to compute the area of a disk). Explain in details how you think.

(4) Using integration, compute the surface area of the above described cone. Explain in details how you think.

Hint: this last point is the trickiest one. Be careful, integrating the circumferences of the composing circles does not give you the right answer. (Why?) Instead think how the area is changing, as the hight is changing. For this you need the area of a little strip from the surface of the cone. Another possibility is to use the slant hight of the cone as one of your integration bound.

Exercise 4 (From Rogawski-Adams)**.**

Let $0 < a < b$ two given reals. Let D be the strip-like domain in $\mathbb{R}^2$ bounded by the curves $x = 0$, $y = 0$, $y = a - x$ and $y = b - x$. Let us define moreover $G : \mathbb{R}^2 \to \mathbb{R}^2$ as $G(u, v) = (u - uv, uv)$.

(1) Show that the image of the horizontal line $v = c$ is $y = \frac{c}{1-c}x$ if $c \neq 1$ and it is the $y-$axis if $c = 1$.

(2) Determine the images of vertical lines in the uOv plane.

(3) Compute the Jacobian determinant of G at an arbitrary point (u, v).

(4) Compute the area of D using two methods: first see your domain as a vertically or horizontally simple one (you may also decompose it into two pieces) and compute the volume as an iterated double integral. Then use the change of variable formula applied to G and compute once again the area.

(5) Compute $\int\int_D xy\,\mathrm{d}A$ using the previous two methods again.