Math 32B-1: Calculus of Several Variables

Instructor: Alpár R. Mészáros

Second Midterm, February 24, 2016

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Discussion section (choose one): 1A 1B 1C 1D

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision. You may loose points in the lack of justification of your answers.
- Theorems from the lectures may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has **3 problems** and is worth **20 points**.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Total	

Exercise 1 (6 points)**.**

(1) Compute $\displaystyle\int\int\int_D (x+y)\,\mathrm{d}V$, where $D := \{(x,y,z) \in \mathbb{R}^3 : \ (x-1)^2 + y^2 + z^2 \leq 4\}$.

Hint: you may use spherical coordinates.

(2) Find the total mass of a solid right circular cone with radius 3 and height 4, for which the density of each point of it is given by the distance from the base.

Hint: you may use cylindrical coordinates, and mind that the radius is always depending on the actual height.

Solutions

(1) Since D is a ball centered at $(1,0,0)$ with radius 2, we use the spherical coordinates $x = 1 + r\sin\varphi\cos\theta$, $y = r\sin\varphi\sin\theta$ and $z = r\cos\varphi$, where $r \in [0,2]$, $\theta \in [0,2\pi]$ and $\varphi \in [0,\pi]$. Since the Jacobian determinant is the same as for the usual spherical coordinates, one obtains that

$$\begin{aligned}
\int\int\int_D (x+y)\,\mathrm{d}V &= \int_0^2 \int_0^{2\pi} \int_0^{\pi} (1 + +r\sin\varphi\cos\theta + r\sin\varphi\sin\theta) r^2 \sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_0^2 \int_0^{2\pi} \int_0^{\pi} (r^2\sin\varphi + r^3\sin^2\varphi\cos\theta + r^3\sin^2\varphi\sin\theta)\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_0^{2\pi} \int_0^{\pi} ((8/3)\sin\varphi + 4\sin^2\varphi\cos\theta + 4\sin^2\varphi\sin\theta)\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&= \frac{32\pi}{3}
\end{aligned}$$

(2) The distance of any point (x,y,z) from the base is exactly the height z, hence the total mass of the cone denoted by C is given by $\displaystyle\int\int\int_C z\,\mathrm{d}V$. We use cylindrical coordinates to describe C. For this one has to compute first the radius as a function of the height z, this is $r(z) = \dfrac{3}{4}(4-z)$. Hence the integral becomes

$$\begin{aligned}
\int\int\int_C z\,\mathrm{d}V &= \int_0^4 \int_0^{2\pi} \int_0^{r(z)} zr\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \pi \int_0^4 zr(z)^2\,\mathrm{d}z = \frac{9\pi}{16} \int_0^4 z(4-z)^2\,\mathrm{d}z \\
&= \frac{9\pi}{16} \int_0^4 (16z - 8z^2 + z^3)\,\mathrm{d} = \frac{9\pi}{4^2} 4^4 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{1}{12}\pi 3^2 4^2 = 12\pi
\end{aligned}$$

Exercise 2 (7 points).

Alice decided that she goes hiking. The path that she has chosen can be described analytically in $\mathbb{R}^2$ as follows. She is starting at the point $a = (-1, 0)$ and goes up on the upper half part of the unit circle (centered at the origin) until $(0, 1)$ then she goes down on the same circle until $b = (1, 0)$. From b she then goes up on a straight line to the point $c = (2, 1)$. Then she continues straight to the point $d = (3, 1)$ and lastly she descends on the straight line to $e = (4, 0)$. We assume that on the whole road she is affected only by the gravitational field, which is given by the constant vector field $G = (0, -2)$.

(1) Give a parametrization for each piece of path (you have 4 pieces: the circular, the straight up, the horizontal and the straight down ones) such that Alice has constant speed on each piece of path, and she needs π time units for the circular piece, and 1 time unit on any other piece.

(2) Compute the total work she needs to perform in order to arrive from a to e on the path with the parametrization described in (1) and knowing that she is only affected by the gravitational field G.

Hint: you may use the additive property of line integrals with respect to concatenations of curves. You may use any other properties of the line integrals or the geometry of the problem in your solution.

(3) When she arrives to the point e, she meets Ben, who decided to hike on the very same path, but starting from e and towards a (using the opposite parametrization of the one in (1)). Knowing Alice's work and the fact that he is also affected only by the same gravitational field G, what is the total amount of work that Ben has to perform on this road? Justify your answer!

(4) After finishing the hike, Alice realizes that the gravitational field "helps her" always when she is going downwards. Would the amount of work she performed change if she doubled her speed on the path from d to e? If yes, how does the total amount of work change in this case? Justify your answer!

Solutions

(1) Let us denote the four pieces of Alice's path by C_{ab}, C_{bc}, C_{cd} and C_{de}. C_{ab} is the upper half of a circle oriented in the inverse trigonometric sense, so its parametrization is given by

$$C_{ab} = \{(\cos(\pi - t), \sin(\pi - t)) : \ t \in [0, \pi]\}.$$

This parametrization clearly uses π time units and Alice has speed 1 on this portion of path. For the straight line segments we use the natural parametrizations

$$C_{bc} = \{(1 + t, t) : \ t \in [0, 1]\}; \quad C_{cd} = \{(2 + t, 1) : \ t \in [0, 1]\}; \quad C_{de} = \{(3 + t, 1 - t) : \ t \in [0, 1]\}.$$

Every of these parametrizations use 1 time unit and the speed of Alice is $\sqrt{2}$, 1 and $\sqrt{2}$ on these paths respectively.

(2) The total (net) work that she needs is given by negative of the sum of the integrals

$$W = -\int_{C_{ab}} G \cdot \mathrm{d}r - \int_{C_{bc}} G \cdot \mathrm{d}r - \int_{C_{cd}} G \cdot \mathrm{d}r - \int_{C_{de}} G \cdot \mathrm{d}r.$$

We compute now all the four integrals separately:

$$\int_{C_{ab}} G \cdot \mathrm{d}r = \int_0^\pi (0, -2) \cdot (\sin(\pi - t), -\cos(\pi - t))\,\mathrm{d}t = 2\int_0^\pi \cos(\pi - t)\,\mathrm{d}t = -2(\sin(0) - \sin(\pi)) = 0,$$

which is very logical, since this part of the problem is symmetric to the $y-$axis, the G is constant.

For the piecewise linear path the tangential vectors are constants, these are actually the velocity of Alice and are given by $(1, 1)$, $(1, 0)$ and $(1, -1)$ respectively. Hence the integrals become

$$\int_{C_{bc}} G \cdot \mathrm{d}r = \int_0^1 (0, -2) \cdot (1, 1)\,\mathrm{d}t = -2;$$

$$\int_{C_{cd}} G \cdot \mathrm{d}r = \int_0^1 (0, -2) \cdot (1, 0)\, \mathrm{d}t = 0;$$

$$\int_{C_{de}} G \cdot \mathrm{d}r = \int_0^1 (0, -2) \cdot (1, -1)\, \mathrm{d}t = 2.$$

The work on the path C_{bc} is clearly the opposite of the one on C_{de} and the one on C_{cd} is 0, since G is always orthogonal to the tangent vector. Hence the total work $W = 0$.

(3) Since Ben plans to go on the very same path, just from e to a, he will need the opposite of the work needed by Alice, which is 0 for him as well.

(4) If Alice would double her speed on C_{de}, she would need half of the time, hence the work on this path would be just

$$-\int_{C_{de}} G \cdot \mathrm{d}r = -\int_0^{1/2} (0, -2) \cdot 2(1, -1)\, \mathrm{d}t = -2.$$

And the total (net) work would be the same.

Exercise 3 (7 points).

Let us consider the surface K of the cube centered at the origin with side length 3 in $\mathbb{R}^3$; i.e. the surface of $\{(x,y,z) \in \mathbb{R}^3 : -3/2 \le x \le 3/2;\ -3/2 \le y \le 3/2;\ -3/2 \le z \le 3/2\}$.

(1) Compute the *unit normal vectors* $N(P)$ of K pointing outward, where P is an arbitrary point varying on the 6 facets of the cube.

Hint: discuss 6 cases. You do not have to discuss the cases when P is on the edges or on the vertices of K.

Now let us consider the intersection of K with the $xy-$plane, i.e. $C = K \cap \{z = 0\}$ which defines a closed piecewise linear curve, given by the edges of the planar square centered at the origin with side length 3. We orient C clockwise.

(2) Compute $\int_C N \cdot \mathrm{d}r$! Justify your answer!

(3) Let us consider the following vector field defined as $F(x,y) = \left(\frac{1}{(2-x)^2(2-y)}, \frac{1}{(2-x)(2-y)^2} \right)$. Show that F a conservative vector field on a domain of $\mathbb{R}^2$ that should be determined! Find a potential function of F on its domain.

(4) Compute $\int_C F \cdot \mathrm{d}r$ and $\int_{\tilde{C}} F \cdot \mathrm{d}r$, where C was defined previously and $\tilde{C}$ is the line segment joining the point $(-3/2, -3/2)$ to $(3/2, 3/2)$ (oriented from $(-3/2,-3/2)$ towards $(3/2,3/2)$) i.e. a diagonal of K.

Solutions

(1) Since all the facets of the cube are parts of flat planes, the unit outward normal vectors are constants on each of them. With the usual orientation of the xyz-system of coordinates, these are given by

$$\begin{aligned}
N(P) &= (0,0,1), && \text{on the upper facet,}\\
N(P) &= (0,0,-1), && \text{on the bottom facet,}\\
N(P) &= (1,0,0), && \text{on the front facet,}\\
N(P) &= (-1,0,0), && \text{on the back facet,}\\
N(P) &= (0,1,0), && \text{on the right facet,}\\
N(P) &= (0,-1,0), && \text{on the left facet.}
\end{aligned}$$

(2) By the construction $N(P)$ is always orthogonal to the facets, hence it is orthogonal to the tangent vectors of the curve C as well, which gives $\int_C N \cdot \mathrm{d}r = 0$.

(3) F clearly is not defined, when $x = 2$ or $y = 2$, more precisely on the straight lines $l_1 := \{(2, y) : y \in \mathbb{R}\}$ and $l_2 := \{(x, 2) :\ x \in \mathbb{R}\}$. Hence F is well-defined and differentiable everywhere in $\mathbb{R}^2$ except in the points of l_1 and l_2. Computing the cross partial derivatives, we find

$$\frac{\partial F_1}{\partial y}(x,y) = \frac{1}{(2-x)^2(2-y)^2} = \frac{\partial F_2}{\partial x}(x,y),$$

which clearly implies that F is conservative on $\mathbb{R}^2 \backslash (l_1 \cup l_2)$. A potential function of F is $f : \mathbb{R}^2 \backslash (l_1 \cup l_2) \to \mathbb{R}$, defined as $f(x,y) = \frac{1}{(2-x)(2-y)}$ and $F(x,y) = \nabla f(x,y)$ for all $(x,y) \in \mathbb{R}^2 \setminus (l_1 \cup l_2)$.

(4) Since C is a closed curve and no points from l_1 or l_2 are contained in the interior of the region enclosed by C, and by the fact that F is conservative there, one obtains that $\int_C F \cdot \mathrm{d}r = 0$. The path $\tilde{C}$ does not intersect with l_1 or l_2, hence $\int_{\tilde{C}} F \cdot \mathrm{d}r = f(3/2, 3/2) - f(-3/2, -3/2) = 4 - 4/49 = 4 \cdot \frac{48}{49}$