Math 251A: Introductory PDE
Homework #1

Due: Wednesday, October 18th, 2017

Exercise 1 (Distributions)**.**

Let $\Omega \subseteq \mathbb{R}^d$ be an open set. We consider $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ the space of all compactly supported functions defined on Ω with values in $\mathbb{R}$. One can put a topology on $\mathcal{D}(\Omega)$ by defining limits of functions as follows: we say that a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ converges to $\varphi \in \mathcal{D}(\Omega)$ if,

(i) there is a compact set $K \subset \Omega$, s.t. $\cup_{n\in\mathbb{N}}\mathrm{spt}(\varphi_n) \cup \mathrm{spt}(\varphi) \subseteq K$;

(ii) $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly as $n \to \infty$, for each multi-index α.

A linear functional on $\mathcal{D}(\Omega)$ which is continuous w.r.t. the previously described topology is called a *distribution*. We denote the space of distributions as $\mathcal{D}'(\Omega)$ (i.e. the continuous topological dual of $\mathcal{D}(\Omega)$).

Let $T \in \mathcal{D}'(\Omega)$ be a distribution. If $U \subseteq \Omega$ is an open set and $T\varphi = 0$ for all $\mathcal{D}(U)$, then we say that T *vanishes* on U. Let V the union of all open sets in Ω where T vanishes. Then $\Omega \setminus V$ is called the support of T.

(1) Show that $T \in \mathcal{D}'(\Omega)$ if and only if for any $K \subset \Omega$ compact there exists $C > 0$ and $N \in \{0\} \cup \mathbb{N}$ such that
$$|T\varphi| \leq C\|\varphi\|_N$$
for all $\varphi \in \mathcal{D}_K$, where $\mathcal{D}_K := \{\varphi \in \mathcal{D}(\Omega) : \ \mathrm{spt}(\varphi) \subseteq K\}$ and $\|\phi\|_N := \max\{|\partial^\alpha \varphi(x)| : x \in \Omega, |\alpha| \leq N\}$. If there exist $N \geq 0$ which can be used in the above description for all $K \subset \Omega$ compact (with possibly different C), the smallest of these is called the *order* of T. If no such N exists, then we call T of *infinite order*.

(2) Give examples of distribution with order 2 and ∞. Show that $T \in \mathcal{D}'(\Omega)$ is of order 0, if and only if it is a finite signed measure on Ω.

(3) Let $T \in \mathcal{D}'(\Omega)$ such that $T\varphi \geq 0$ for all $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$. Show that T is a finite positive measure on Ω.

(4) Show that if the support of $T \in \mathcal{D}'(\Omega)$ is a compact subset of Ω, then T has finite order.

(5) Characterize the distributions supported on finite sets.

(6) Show that every distribution on $\mathbb{R}^d$ is the limit of (in the topology on $\mathcal{D}'$) infinitely differentiable functions. *Hint:* construct a sequence of approximation by mollification. Be careful, when you define convolution of distributions.

Exercise 2 (Harmonic functions, Liouville-type theorems)**.**

(1) Show that when $d = 2$ (the dimension of the ambient space), any positive superharmonic function (defined on the whole space) is constant, while this is not necessary the case when $d \geq 3$. *Hint:* compare your superharmonic function to a family of harmonic functions obtained by "flattening" of the fundamental solution $-\log|x|$.

(2) Show that harmonic functions refined on the whole space, that vanish on an open set vanish everywhere (without knowing analiticity). For this, using the divergence theorem, show that if u is harmonic, then
$$\frac{\mathrm{d}}{\mathrm{d}r}\int_{|x|=1} u(rx)u(x/r)\mathrm{d}x = 0.$$

Conclude by scaling that

$$\int_{|x|=1} u(ax)u(bx)\mathrm{d}x = \int_{|x|=1} u^2(cx)\mathrm{d}x,$$

where $ab = c^2$. Using the previous results, conclude that if u vanishes in a neighborhood of the origin, then it vanishes identically.

Exercise 3 (Harnack inequality and regularity)**.**

(1) Show that if $u \geq 0$ is a harmonic function on $B_1(0)$, then there exists $C(d) > 0$ such that

$$\sup_{B_{1/2}} u \leq C(d)u(0).$$

Show that $C(d) \leq 2^d$.

(2) Using the Harnack inequality, show the oscillation decay estimate: if u is a not necessarily nonnegative harmonic function on $B_1(0)$, then

$$\mathrm{osc}_{B_{1/2}(0)} u \leq \frac{C(d)}{C(d)+1} \mathrm{osc}_{B_1(0)} u,$$

where for $u : A \to \mathbb{R}$, $\mathrm{osc}_A u := \sup_A u - \inf_A u$.

(3) Using the oscillation decay estimate, show that

$$\|u\|_{C^{0,\alpha}} \leq M(d) \sup_{B_1(0)} |u|,$$

where $(1/2)^\alpha = \frac{C(d)}{C(d)+1}$ and $M(d) > 0$ depends only on d.

(4) Show that if we have a one-sided bound on a harmonic function u defined on $\mathbb{R}^d$, e.g. $u(x) \geq f(|x|)$ for some radial function f, then we have an upper bound as well, i.e. there exists g radial such that $u(x) \leq g(|x|)$ where g has the same growth as f at ∞. Show that if u is harmonic and we can "touch it" below by a paraboloid of opening M, then we can touch it by above with a paraboloid of opening CM.