Math 251A: Introductory PDE
Homework #2

Due (recommended): November 1st, 2017

Exercise 1 (Harmonic functions in anisotropic settings)**.**

Let $H : \mathbb{R}^d \to [0, \infty)$ be such that it verifies the properties:

(a) $H \in C^\infty(\mathbb{R}^d \setminus \{0\})$;

(b) [positive homogeneity] $H(tx) = tH(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$;

(c) [Hessian bound for H^2] The Hessian matrix $D^2H^2(x)$ is positive definite for all $x \neq 0$.

H is said to be *absolutely homogeneous* if in addition

(b') $H(tx) = |t|H(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.

Suppose that H satisfies (a)-(c).

(1) Show that H is convex and H^2 is a strictly convex function.

(2) Show that $H(x) > 0$ for all $x \neq 0$.

(3) Show that $x \cdot \nabla H(x) = H(x)$ and $D^2(H^2/2)(x)x \cdot x = H^2(x)$ for all $x \in \mathbb{R}^d \setminus \{0\}$.

(4) Show that $\nabla H(tx) = \nabla H(x)$ and $D^2H^2(tx) = D^2H^2(x)$ for all $x \in \mathbb{R}^d \setminus \{0\}$, $t > 0$.

Given H, we define the differential operator $\Delta_H : C^2(\mathbb{R}^d) \to \mathbb{R}$[1] as

$$\Delta_H(u) = \operatorname{div}(H(\nabla u)\nabla H(\nabla u)).$$

Notice that if $H = |\cdot|$, the euclidean norm in $\mathbb{R}^d$, then Δ_H is the standard Laplace operator. We introduce moreover the *polar transform* $H_0 : \mathbb{R}^d \to [0, +\infty)$ of H by the formula

$$H_0(x) = \sup_{y \neq 0} \frac{x \cdot y}{H(y)}.$$

Let once again H satisfy (a)-(c).

(5) Show that he polar transform H_0 associated to H also satisfies (a)-(c).

(6) Show that $H(\nabla H_0(x)) = 1$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

(7) Show that $H(y) = \max\limits_{x \neq 0} \dfrac{x \cdot y}{H_0(x)}$, i.e. $(H_0)_0 = H$.

(8) Show that $\nabla H(\nabla H_0(x)) = \dfrac{x}{H_0(x)}$ for all $x \in \mathbb{R}^d \setminus \{0\}$.

(9) Let

$$u_\pm(x) = \begin{cases} \frac{1}{\omega_{d-1}} \frac{H_0(\pm x)^{2-d}}{d-2}, & \text{if } d > 2; \\ -\frac{1}{\omega_1} \ln H_0(\pm x), & \text{if } d = 2. \end{cases}$$

If δ_0 denotes the Dirac measure at the origin, then

(i) $-\Delta_H u_- = \delta_0$;

(ii) $-\Delta_H u_+ = \delta_0$ if and only if H is absolutely homogeneous.

[1] Notice that since this operator is nonlinear, Poisson representation and Green's functions cannot be expected in general.

Hint and remark: notice that ω_d denotes the volume of the unit ball given by H^0, i.e. $\{x \in \mathbb{R}^d : H^0(x) < 1\}$. Also, when you need to construct balls, it is always natural to work with balls w.r.t. H^0.

Now suppose that H satisfies (a)-(b')-(c). Let us introduce the function $K_H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$K_H(x, y) = x \cdot y - H_0(x)H(y)\nabla H_0(x) \cdot \nabla H(y). \tag{1}$$

(10) Show that if H and H_0 are such that $K_H(x, y) = 0$ for all $x, y \in \mathbb{R}^d$, then any Δ_H harmonic function satisfies the mean value property (here also, the balls are w.r.t. H^0). Is this property still true if one replaces (b') by (b)?

(11) Show that the following statements are equivalent:

  (i) H is Euclidean, i.e. there exists a symmetric positive definite matrix A such that $H(x) = \sqrt{Ax \cdot x}$ for all $x \in \mathbb{R}^d$.

  (ii) $K_H(x, y) = 0$, for all $x, y \in \mathbb{R}^d$.

(12) [2] Show that if for some $\Omega \subset \mathbb{R}^d$, $u : \Omega \to \mathbb{R}$ is a Δ_H-harmonic function that satisfies the mean value property for spheres, then $K_H(x, y) = 0$ for all $x, y \in \mathbb{R}^d$, thus by (11), H is induced by an inner product in $\mathbb{R}^d$. *Hint:* try to work with the fundamental solutions far from the origin and consider a proof by contradiction.

Exercise 2.

We will consider the same framework as in Exercise 1. Let us denote $B^0_r(x) := \{x \in \mathbb{R}^d : H^0(x) < r\}$ the open ball corresponding to H^0.

(1) We would like to solve $-\Delta_H u = f$ in $\Omega := B^0_1(0)$ with homogeneous 0 boundary condition on $\partial\Omega$. Write the corresponding energy functional for which this equation can be seen as the first order optimality condition. Show that the minimization problem has a solution in a suitable (Sobolev) space. Which is the least regularity for f that ensures that the corresponding equation has a solution in weak sense? Is the solution unique? Is it possible to obtain the existence of weak solutions via a Lax-Milgram type argument? *Hint:* you have to address the l.s.c. and convexity of the corresponding energy functional in the space that you chose.

(2) Find the explicit solution to the problem in (1), in the case when $f = 2d$. *Hint:* what would be the solution in the case when H is the standard Euclidean norm?

Exercise 3 (Approximation in Morrey/Campanato spaces and other properties)**.**

(1) Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain, and let $1 \le p < +\infty$, $0 < \lambda < d$. Construct a function in the Morrey space $L^{p,\lambda}(\Omega)$ that cannot be approximated by continuous functions in the Morrey norm. *Hint:* construct first some elementary non-trivial functions in the Morrey space.

(2) What can we say about the separability of Morrey spaces in (1)? Try to prove your claim.

(3) The Campanato space $\mathcal{L}^{p,d}(\Omega)$ is called the space of functions with *bounded mean oscillation*, thus one denotes it by $BMO(\Omega)$. Show that $L^\infty(\Omega) \subseteq BMO(\Omega)$. Give an example of a $BMO(\Omega)$ function which is not essentially bounded, hence showing that $L^\infty(\Omega)$ is a proper subset of $BMO(\Omega)$. *Hint:* think to a log function in 1D.

To be continued...

[2] A solution of this problem will imply an ice cream/beer for the author.