# Math 251A: Introductory PDE
# Homework #3

Due (recommended): November 22nd, 2017

**Exercise 1** (Some warm up problems and reading)**.**

(1) Read Chapter 5.5 from Giaquinta and Martinazzi.

(2) Exercise 5.16 (page 85) from Giaquinta and Martinazzi.

(3) Exercises 6.3 and 6.4. (page 98) from Giaquinta and Martinazzi.

**Exercise 2** (Caccioppoli's inequality revisited)**.**

Let $\Omega \subseteq \mathbb{R}^d$ be a smooth bounded domain.

**Part 1**

(1) Let $u, v : \Omega \to \mathbb{R}$ be differentiable functions such that $v \neq 0$ in $\Omega$. Show the identity

$$|\nabla u|^2 - \nabla(u^2/v) \cdot \nabla v = |\nabla u - (u/v)\nabla v|^2, \tag{1}$$

where $|\cdot|$ stands for the standard Euclidean norm in $\mathbb{R}^d$.

(2) Let $v > 0$ be a continuous (weakly) harmonic function on $\Omega$ and $\eta \in C_c^\infty(\Omega)$, $\eta \geq 0$. Using the previous identity, show

$$\int_\Omega |\eta \nabla v|^2 \mathrm{d}x \leq 4 \int_\Omega |v \nabla \eta|^2 \mathrm{d}x$$

and deduce Caccioppoli's inequality.

**Part 2**

(1) Let $u$ be a weakly $p$-harmonic function on $\Omega$ ($p > 1$), i.e. the weak solution of the equation

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0. \tag{2}$$

If we set $u = g$ on $\partial\Omega$, which is the least possible regularity for $g$ that ensures that (2) has a unique solution with this boundary condition? Using eventually calculus of variations, show that the previous problem has a unique solution in a suitable Sobolev space that you need to determine. *Hint:* you should address the l.s.c. and coercivity of the energy functional in particular.

(2) Write a suitable version of (1) involving $p > 1$, i.e. for functions as in (1)/Part 1 and $p > 1$ show that

$$|\nabla u|^p - \nabla(|u|^p/v^{p-1}) \cdot |\nabla v|^{p-2}\nabla v = \Phi_p(u, v),$$

where $\Phi_p(u, v) \geq 0$ has to be determined.

(3) Using this previous identity, show that for any positive solution of (2) and $\eta \geq 0$ smooth compactly supported in $\Omega$, one has

$$\int_\Omega |\eta \nabla v|^p \mathrm{d}x \leq p^p \int_\Omega |v \nabla \eta|^p \mathrm{d}x.$$

Deduce the corresponding Caccioppoli inequality.

(4) Imagine that one wants to solve (2) with and $f$ right hand side. Which is the least regularity for $f$ that allows us to construct a weak solution of the problem in the corresponding Sobolev space described in (1)? Does the obtained Caccioppoli inequality allow us to obtain second order Sobolev estimates provided $f \in L^q(\Omega)$ for a suitable $q > 1$? Can one iterate these estimates? Justify your answers.

**Exercise 3** (Neumann problems)**.**

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain. We denote the outward normal to $\partial\Omega$ by $n$.

(1) Let $q > 1$ and $f \in (W^{1,q'}(\Omega))'$, where $1/q + 1/q' = 1$. Show that the equation

$$-\nabla \cdot F = f$$

with $F \cdot n = 0$ on $\partial\Omega$ has at least one weak solution $F \in L^q(\Omega; \mathbb{R}^d)$. Write also the corresponding weak formulation and comment a bit on the boundary condition. What can we say about the uniqueness of the solution?

(2) Let $w \in L^q(\Omega; \mathbb{R}^d)$. Show that the problem

$$-\Delta u + \nabla \cdot w = 0 \tag{3}$$

with the boundary condition $(\nabla u - w) \cdot n = 0$ on $\partial\Omega$ has a weak solution $u \in W^{1,q}(\Omega)$. If one imposes $\int_\Omega u \mathrm{d}x = 1$, show that this solution is unique. *Hint:* use the previous point.

(3) Suppose that $w \in H^1(\Omega)$. Show that the solution $u$ of the problem (3) is $H^2(\Omega)$ up to the boundary.

(4) Let $b \in L^q(\Omega; \mathbb{R}^d)$. Show that the problem

$$-\Delta u + \nabla \cdot (bu) = 0 \tag{4}$$

with the boundary condition $(\nabla u - w) \cdot n = 0$ on $\partial\Omega$ has a weak solution $u \in W^{1,q}(\Omega)$.

(5) Suppose that $q > d$ and any $u$ solution to (4) for any $b \in L^q(\Omega; \mathbb{R}^d)$ with $\int_\Omega u \mathrm{d}x = c > 0$ is such that $u > 0$ on every open subset of $\Omega$. Show that $u > 0$ on $\overline{\Omega}$. Notice that if $u \in W^{1,q}(\Omega)$ and $q > d$, one can identify it with its Hölder continuous representative, thus it is meaningful to evaluate $u$ on $\partial\Omega$ in the usual sense. *Hint:* you may do a reflection argument, i.e. take $x_0 \in \partial\Omega$ and extend both $u$ and $b$ by reflection to a whole ball $B_r(x_0)$ in a way that these solve a similar equation to (4). Write a detailed proof!

**Exercise 4** (On Hölder regularity)**.**

(1) Show that $u \in C^{0,\alpha}(\Omega)$ if and only if there exists a constant $K > 0$ and for all $x \in \Omega$ there exists $C_x \in \mathbb{R}$ such that

$$\|u - C_x\|_{L^\infty(B_r(x))} \leq K r^\alpha.$$

Furthermore show that if $|C_x| + K \leq M$ for all $x \in \Omega$, then $\|u\|_{C^{0,\alpha}} \leq M$. Explain a bit geometrically this characterization of Hölder continuous functions.

(2) Show that $u \in C^{1,\alpha}(\Omega)$ if and only if for all $x \in \Omega$ there exists a linear function $l_x(y) = a_x + b_x \cdot (y - x)$ and a uniform constant $K > 0$ such that

$$\|u - l_x\|_{L^\infty(B_r(x))} \leq K r^{1+\alpha}.$$

Furthermore show that if $|a_x| + |b_x| + K \leq M$ for all $x \in \Omega$, then $\|u\|_{C^{1,\alpha}} \leq M$. What does this characterization mean geometrically?

(3) Formulate similar characterization with paraboloids for functions $u \in C^{2,\alpha}(\Omega)$.

(4) Explain why the condition $\|u - l_x\|_{L^\infty(B_r(x))} \leq Kr$ implies $C^{0,1}$ regularity (i.e. Lipschitz continuity) but not $C^1$. In particular, why does one need to take $\alpha \in (0, 1)$?

(5) Suppose that we can find a sequence of paraboloids $p_k = a_k + b_k \cdot x + \frac{1}{2}x^\top M_k x$ and an $0 < r < 1$ such that

$$\|u - p_k\|_{L^\infty(B_{r^i})} \leq K r^{i(2+\alpha)}$$

for a uniform constant $K$, $i \in \mathbb{N}$ and $\alpha \in (0,1)$. Show that $u \in C^{2,\alpha}(\Omega)$ with a norm that depends on $K$ and $r$.

(6) We investigate the case when $\alpha = 0$ in the previous characterization. Suppose that $p$ is a quadratic polynomial and $\eta$ a smooth cutoff function such that $\eta \equiv 1$ on $B_{1/2}(0)$. Then define

$$u(x) = \sum_k 2^{-2k}(\eta p)(2^k x).$$

Let $p_i$ be the first $i$ terms of the previous series. Show that

$$\|u - p_i\|_{L^\infty} \leq C 2^{-2i}$$

but $u$ is not $C^2$.