Math 131B-1: Analysis

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Final Exam, March 16, 2016

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **180 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs. You may loose points in the lack of justification of your answers.
- Theorems from the lectures and homework may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 4 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **26 points**, which means that there are **6 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Exercise 4	
Total	

Exercise 1 (6 points).

Let $(f_n)_{n\geq 0}$ and $(g_n)_{n\geq 0}$ be two sequences of functions defined as $f_n, g_n : [0,+\infty) \to \mathbb{R}$, $f_n(x) = \dfrac{1}{\sqrt{1+n^4x}}$ and $g_n(x) = xf_n(x)$ for all n non-negative integer.

(1) Determine the point-wise limit of $(f_n)_{n\geq 0}$ and deduce that the sequence cannot converge uniformly on $[0,\infty)$! Justify your answer!

(2) Characterize the intervals $I \subseteq [0,+\infty)$ where $(f_n)_{n\geq 0}$ is converging uniformly to its point-wise limit. Justify your answer!

(3) Let $0 < a < b$. What can we say about $\lim_{n\to+\infty} \int_a^b f_n(x)\,\mathrm{d}x$? Compute this limit. Justify your answer!

(4) Is the sequence $(f_n')_{n\geq 0}$ converging point-wisely on $[0,+\infty)$? If yes, determine its point-wise limit. What can we say about its uniform convergence on an interval as in (2)? Justify your answer!

(5) Show that $(g_n)_{n\geq 0}$ is converging uniformly on any interval of the form $[0,b)$ and $[0,b]$ where $0 < b < +\infty$. What is its uniform limit? Is the sequence converging uniformly or point-wisely on $[0,+\infty)$? Justify your answer!

(6) Using eventually Weierstrass' $M-$test show that $\sum_{n=0}^{+\infty} f_n(x)$ is converging uniformly to a continuous function F on any interval $I \subseteq [0,+\infty)$ determined in (2). Construct an upper bound for F, i.e. give an explicite $C > 0$ such that $F(x) \leq C$ for all $x \in I$. You may admit now that $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \pi^2/6$.

Solutions.

(1) The point-wise limit of $(f_n)_{n\geq 0}$ is the function $f : [0,+\infty)$ defined as $f(0) = 1$ and $f(x) = 0$ for all $x > 0$. Indeed, $f_n(0) = 1$ for all $n \geq 0$, hence $\lim_{n\to+\infty} f_n(0) = 1 = f(0)$. If $x > 0$ one has $0 \leq f_n(x) \leq \dfrac{1}{n^2\sqrt{x}}$, hence by the squeeze lemma $\lim_{n\to+\infty} f_n(x) = 0 = f(x)$ for all $x > 0$. On the other hand f_n is continuous on $[0,+\infty)$ for all $n \geq 0$ and if it would be uniformly convergent, the limit function (which should be the same as the point-wise limit) would have to be continuous. Since f has a discontinuity at 0, the convergence cannot be uniform on $[0,+\infty)$.

(2) Clearly if one restricts ourselves to any interval of the form $I = [a,+\infty) \subset [0,+\infty)$ where $a > 0$, (or $(a,+\infty)$ or $[a,b)$, (a,b), $[a,b]$, $(a,b]$ with $b > a$) the sequence $(f_n)_{n\geq 0}$ is converging uniformly to f on I. To see this, one can use the same inequality as in (1), i.e.

$$0 \leq f_n(x) \leq \frac{1}{n^2\sqrt{x}} \leq \frac{1}{\sqrt{a}n^2}$$

for all $x \in [a,+\infty)$. This implies in particular that $\sup_{x\in[a,+\infty)} |f_n(x)| \leq \dfrac{1}{\sqrt{a}n^2} \to 0$, as $n \to +\infty$, which proves our claim (this is similar for the other type of intervals as well).

(3) $(f_n)_{n\geq 0}$ converges uniformly to 0 on any interval of the form $[c,+\infty)$ with $c > 0$, and in particular for $0 < c \leq a$. $[a,b]$ is a compact interval in $[c,+\infty)$. Hence the theorem on the integration and uniform convergence applies in this case and one has that $\lim_{n\to+\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b 0\,\mathrm{d}x = 0$.

(4) f_n is differentiable on the whole $[0,+\infty)$ for all $n \geq 0$ with derivatives $f_n'(x) = -\dfrac{n^4}{2\sqrt{(1+n^4x)^3}}$. One has clearly that $f_n'(0)$ does not converge, hence $(f_n')_{n\geq 0}$ does not converge point-wisely on $[0,+\infty)$.

On the other hand if one restricts the sequence to any interval of the form $[a, +\infty)$ with $a > 0$ as in (2) (or to the other type of intervals considered in (2)), one has the estimate

$$|f_n'(x)| \leq \frac{n^4}{2n^6 x^{3/2}} \leq \frac{1}{2n^2 a^{3/2}}, \ \forall x \in [a, +\infty).$$

Hence the supremum norm of the sequence is tending to 0, which implies that $(f_n')_{n\geq 0}$ converges uniformly (hence point-wisely as well) to 0 on any interval of the form that was constructed in (2). Be aware that this is not because you can interchange the differentiation and the limit for uniformly convergent sequences of function, which is not true in general.

(5) One can write a similar estimate as in (2) for $g_n(x)$, i.e.

$$|g_n(x)| \leq \frac{x}{n^2\sqrt{x}} = \frac{\sqrt{x}}{n^2}, \ \forall x \in [0, +\infty).$$

This implies in particular that $|g_n(x)| \leq \frac{\sqrt{b}}{n^2}$, $\forall x \in [0, b]$, where $0 < b < +\infty$ is any fixed constant, which similarly implies that $(g_n)_{n\geq 0}$ converges uniformly to 0 (as $n \to +\infty$) on every interval of the form $[0, b)$ or $[0, b]$ (since the supremum norm converges uniformly to 0), where $0 < b < +\infty$. Observe also that we do not have a problem in 0 now, since $g_n(0) = 0$ for all $n \geq 0$.

On the other hand, it is clear that for all $n \geq 0$ since $\lim_{x\to+\infty} g_n(x) = +\infty$ one has $\sup_{x\in[0,+\infty)} g_n(x) = +\infty$. This shows that one cannot hope for uniform convergence on the whole $[0, +\infty)$. But we still have point-wise convergence to 0 for all $x \in [0, +\infty)$.

(6) To use Weierstrass' M-test, one needs to use the estimate developed in (2), i.e. $\|f_n\|_{L^\infty} = \sup_{x\in I} |f_n| \leq \frac{1}{\sqrt{a}n^2}$, where $I = [a, +\infty)$ for any $a > 0$. Since $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges, $\sum_{n=0}^{+\infty} \|f_n\|_{L^\infty}$ also converges, which implies by the M-test (since the f_n's are continuous functions) that $\sum_{n=0}^{+\infty} f_n$ also converges uniformly on I to a continuous function $F : I \to \mathbb{R}$. Setting $C = \frac{\pi^2}{6\sqrt{a}}$, this defines an upper bound and $F(x) \leq C$ for all $x \in I = [a, +\infty)$.

Exercise 2 (6+2 points)**.**

Let us consider the power series $S(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2}$.

(1) Compute the radius of convergence of S and deduce that S is continuous on $(-1, 1)$.

(2) Is S differentiable on $(-1, 1)$? If yes, find the power series expansion around 0 for S' and determine where is this series converging uniformly to S'!

(3) Using the power series expansion on $(-1, 1)$ of $\ln(1-x)$ (easily derived from the geometric series) show that $S'(x) = -\frac{\ln(1-x)}{x}$ for all $x \in (-1, 1)$, $x \neq 0$.

(4) Why is S' continuous at 0? Show that $\lim_{x\to 0} S'(x) = 1$ and $\lim_{x\to 0} -\frac{\ln(1-x)}{x} = 1$. Using Abel's theorem, show that S is continuous at 1 and at -1.

We consider the function $g : [0, 1] \to \mathbb{R}$ defined as $g(0) = g(1) = S(1) = \pi^2/6$ and

$$g(x) = \ln(x)\ln(1-x) + S(1-x) + S(x), \forall x \in (0, 1).$$

One can admit that g is continuous on $[0, 1]$.

(5) Computing the derivative of g on $(0, 1)$, deduce that it is constant on $[0, 1]$. What is the constant value of g?

(6) Admitting once again that $S(1) = \pi^2/6$ and computing $g(1/2)$ show that

$$\sum_{n=1}^{+\infty} \frac{1}{2^n n^2} = \frac{\pi^2}{12} - \frac{\ln^2(2)}{2}.$$

Bonus question: show that the function g is continuous on $[0, 1]$ and $\lim_{x\to 0^+} g(x) = \lim_{x\to 1^-} g(x) = S(1)$.

Solutions.

(1) $R = \frac{1}{\limsup_{n\to+\infty}(1/n^2)^{1/n}} = \frac{1}{\lim_{n\to+\infty} \frac{1}{n^{1/n}n^{1/n}}} = 1$. And since S is a power series centered at 0, by the continuity of the power series one has that S is well-defined and continuous on $(-1, 1)$ (and the series converges uniformly on any compact sub-interval of $(-1, 1)$ to S).

(2) By the theorem that we proved during the lectures, S is differentiable on $(-1, 1)$ and the power series expansion around 0 is given by $\sum_{n\geq 1} \frac{x^{n-1}}{n}$. This series in converging uniformly to $S'(x)$ on every compact sub-interval of $(-1, 1)$.

(3) We know that the geometric series $\sum_{n\geq 0} x^n$ converges uniformly on any compact sub-interval of $(-1, 1)$ to $\frac{1}{1-x}$, one deduces (by integrating) the power series expansion $-\ln(1-x) = \sum_{n=1} \frac{x^n}{n}$. Since the monomial x defines itself a power series, by the theorem on the multiplication of power series, one deduces that $xS'(x) = \sum_{n\geq 1} \frac{x^n}{n}$, which is clearly the expansion of $-\ln(1-x)$. Dividing the quality by $x \neq 0$, one has the equality that we needed to show.

(4) Since the power series of S' converges uniformly on every compact sub-intervals of $(-1, 1)$ to S', which is continuous on $(-1, 1)$, one easily has the continuity in 0, and by the point-wise limit one has $\lim_{x\to 0} S'(x) = S'(0) = 1$. By the equality in (3), this implies in particular that $\lim_{x\to 0} -\frac{\ln(1-x)}{x} = 1$.

Since both $\sum_{n\geq 1} \frac{1}{n^2}$ and $\sum_{n\geq 1} \frac{(-1)^n}{n^2}$ are (absolutely) convergent, Abel's theorem implies that S is continuous both in ± 1.

(5) Clearly g is differentiable on $(0,1)$ with $g'(x) = \frac{\ln(1-x)}{x} - \frac{\ln(x)}{1-x} - S'(1-x) + S'(x)$. Using (3), one has that $S'(x) = -\frac{\ln(1-x)}{x}$ and $S'(1-x) = -\frac{\ln(x)}{1-x}$ which imply that $g'(x) = 0$ on $(0,1)$. By the continuity of g at the endpoints $x = 0$ and $x = 1$, one has that $g(x) = S(1)$ for all $x \in [0,1]$.

Let us solve the *bonus question* now. The continuity of g on $(0,1)$ is clear; we now discuss the continuity at the boundary points. Using (4) and the continuity of S at 0 one has that $\lim_{x\to 0^+}(S(1-x)+S(x)) = S(1) = \lim_{x\to 1^-}(S(1-x)+S(x))$. One has to compute $\lim_{x\to 0^+} \ln(x)\ln(1-x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\ln(1-x)}}$, since this is an $\frac{\infty}{\infty}$ undetermined case, and both functions are differentiable for $x \in (0,1)$, one can une the L'Hôpital rule, and one has

$$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\ln(1-x)}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{\ln^2(1-x)}\frac{1}{1-x}} = \lim_{x\to 0^+} \frac{(1-x)\ln^2(1-x)}{x} = \lim_{x\to 0^+} \frac{\ln(1-x)}{x}(1-x)\ln(1-x) = 0,$$

using also the limit in (4). With the very same technique one can show that $\lim_{x\to 1^-} g(x) = 0$ as well, which together with the previous limits and the values of g at $x = 0$ and $x = 1$ show the continuity of g on $[0,1]$.

(6) Following the hint, $g(1/2) = \ln^2(1/2) + 2S(1/2)$ and by the point-wise limit of the power series of S at $1/2$ one has $S(1/2) = \sum_{n\geq 1} \frac{1}{n^2 2^n}$. By the fact that $g = S(1)$ on $[0,1]$, one has that

$$\sum_{n\geq 1} \frac{1}{n^2 2^n} = \frac{1}{2}\left(\frac{\pi^2}{6} - \ln^2(2)\right).$$

Exercise 3 (6 points)**.**

Let us consider a metric space (X,d). A subset $E \subseteq X$ is said to be *dense* if it has non-empty intersection with any open subset of X, i.e. $E \cap A \neq \emptyset$, for any $A \subseteq X$ open. (X,d) is said to be *separable* if it contains a countable *dense* set, i.e. there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that any $A \subseteq X$ open set contains at least one element of the sequence $(x_n)_{n\in\mathbb{N}}$.

(1) Let (X,d) be a metric space and suppose that $E \subset X$ is a proper dense subset of X. Show that for any element $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $d - \lim_{n\to+\infty} x_n = x$.

(2) Show that $E \subseteq X$ is a dense set in the metric space (X,d) if and only if $\overline{E} = X$. *Hint:* one direction is already proved in (1).

(3) Show that $(\mathbb{N}, d)$ and $(\mathbb{Q}, d)$ where d is the standard metric on $\mathbb{R}$ are separable spaces.

(4) Let (X, d_X) and (Y, d_Y) two metric spaces and $f : X \to Y$ is continuous. Assume that (X,d) is separable. Show that $f(X)$ is a separable metric space with the metric d_Y restricted to $f(X)$.

Hint: show that if you take a countable dense set in X, its image through f is countable dense. You may use at some point the property on pre-images of open sets through continuous functions.

(5) Let us consider (X, d_{disc}) where d_{disc} is the discrete metric on $X \neq \emptyset$, defined as $d_{\text{disc}}(x,y) = 0$, if $x = y$ and $d_{\text{disc}}(x,y) = 1$, if $x \neq y$. Show that (X, d_{disc}) is separable if and only if X is countable.

Hint: determine the open balls of radius 1/2 for instance in (X, d_{disc}) and use the definition of the separability.

(6) Show that any compact metric space is separable.

Hint: choose special open covers of the space with balls of vanishing radii. Then use the finite open sub-cover characterization of the compact space, and construct a countable dense set from the chosen balls.

Solutions.

(1) Now chose $\varepsilon = 1/n$, where $n \in \mathbb{N}$ is arbitrary, by the denseness of E in X (and by the axiom of choice) and by the fact that $B(x; 1/n)$ is open one has that $\exists x_n \in E$ such that $x_n \in B(x; 1/n)$. This precisely means that $\lim_{n\to+\infty} d(x_n, x) \to 0$, which shows that approximation.

(2) Let us suppose that E is dense and show that $\overline{E} = X$. Clearly $\overline{E} \subseteq X$. To show that $X \subseteq \overline{E}$. Pick $x \in X$, by the fact that E is dense, one has that for all $\varepsilon > 0$ there exists $x_\varepsilon \in E$ such that $x_\varepsilon \in B(x; \varepsilon)$ which shows that x is an adherent point of E. This is precisely a consequence of (1), as the hint stated.

For the other direction, let us suppose that $\overline{E} = X$ and show that E is dense. Since any point $x \in X$ is an adherent point of E, by definition this means that for all $\varepsilon > 0$ $B(x,\varepsilon) \cap E \neq \emptyset$. If A is open in X then any point $x \in A$ is interior, which means that $B(x,\varepsilon) \subseteq A$ for some $\varepsilon > 0$. Since $x \in \overline{E}$ we have seen that $E \cap B(x,\varepsilon) \neq \emptyset$, hence $E \cap A \neq \emptyset$ which shows that E is dense in X.

(3) Any countable space is separable, because its elements can be listed as a sequence. $\mathbb{N}$ and $\mathbb{Q}$ are countable, so they are separable.

(4) Let $E \subset X$ be countable and dense. Now let us show that $f(E)$ is countable and dense in $f(X)$. To do so, first it is clear that $f(E)$ is countable. Secondly to show that it is dense, take any open set $A \subseteq f(X)$, clearly the pre-image $f^{-1}(A)$ is open in X. By the fact that E is dense in X, there exists $x_0 \in E \cap f^{-1}(A)$. But this implies that $f(x_0) \in A$, hence $A \cap f(E) \neq \emptyset$. By the arbitrariness of A we conclude.

(5) Let us consider the metric space (X, d_{disc}). Clearly for all $x \in X$, $\{x\}$ is an open set, since it is equal to $B(x; 1/2)$ for instance. So if we consider a dense set E in X w.r.t. the discrete metric, then $E = X$ is the only possibility. For the separability one needs countable dense sets. Thus the above arguments show that a discrete metric space is separable if and only if it is countable.

(6) Let $n \in \mathbb{N}$. Clearly $\bigcup_{x \in X} B(x; 1/n)$ is an open cover of X. By the fact that X is compact there exists $N_n > 0$ such that $X \subseteq \bigcup_{n=1}^{N_n} B(x; 1/n)$. Label the centers of these balls by $(x_n)_{n=1}^{N_n}$. We proceed with this construction for all $n \in \mathbb{N}$, and set $E = \bigcup_{n \in \mathbb{N}} \{x_1, \ldots, x_{N_n}\}$. Clearly E is countable. Let us show that E is dense. Let $A \subseteq X$ be an open set. Clearly each $x \in A$ is an interior point, hence there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subseteq A$. So pick $x \in A$. Chose $n \in \mathbb{N}$ such that $1/n < \varepsilon/2$ and take the centers $\{x_1, \ldots, x_{N_n}\}$ of the finite cover of X with the balls with radius $1/n$. Since these balls cover the whole space, it is clear that $x \in B(x_i; 1/n)$ for some $i \in \{1, \ldots, N_n\}$. Since the radius was chosen such that $1/n < \varepsilon/2$, clearly $B(x_i, 1/n) \subset B(x, \varepsilon) \subseteq A$. By the fact that these centers are in E one has that $E \cap A \neq \emptyset$. By the arbitrariness of A we conclude that E is dense.

Exercise 4 (6 points)**.**

Let us consider the function $f(x) = x(1-x)$ for all $x \in [0,1)$ and extended with period 1 on the whole $\mathbb{R}$.

(1) Check whether the function f constructed above is in the class $C(\mathbb{R}/\mathbb{Z};\mathbb{R})$.

(2) Compute the Fourier coefficients $\hat{f}(n)$, $n \in \mathbb{Z}$, of f and write the Fourier series expansion of f.

Hint: you can work either with the *characters* of complex exponentials or with the formulation containing sin and cos functions.

(3) What can we say about the L^2-convergence of the series to f? What can we say about the uniform convergence of the series to f? Justify your answer.

(4) Using the Fourier series expansion of f, compute $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$.

(5) Using the Fourier series expansion of f, compute $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2}$.

(6) Using the Plancherel-Parseval theorem for the Fourier series of f, compute $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^4}$.

Solutions.

(1) Clearly f is continuous on $[0,1)$, by the periodic extension $f(1) = f(0) = 0$ and $\lim_{x\to 1^-} f(x) = 0 = f(1)$ hence it is continuous at $x = 1$. At any other point of $\mathbb{R}$ it is continuous by these two arguments. It is $\mathbb{Z}$-periodic by construction and it has real values, hence it is in the class $C(\mathbb{R}/\mathbb{Z};\mathbb{R})$.

(2) Using the definition of the Fourier coefficients of f with the help of the characters $e_n = e^{2n\pi i x}$, $n \in \mathbb{Z}$, one has for $n \in \mathbb{Z}$, $n \neq 0$

$$\begin{aligned}
\hat{f}(n) &= \int_0^1 f(x) e^{-2n\pi i x}\,\mathrm{d}x = \int_0^1 x e^{-2n\pi i x}\,\mathrm{d}x - \int_0^1 x^2 e^{-2n\pi i x}\,\mathrm{d}x \\
&= \left[x \frac{e^{-2n\pi i x}}{-2n\pi i} \right]_{x=0}^1 + \frac{1}{2n\pi i} \int_0^1 e^{-2n\pi i x}\,\mathrm{d}x \\
&+ \left[x^2 \frac{e^{-2n\pi i x}}{2n\pi i} \right]_{x=0}^1 - \frac{2}{2n\pi i} \int_0^1 x e^{-2n\pi i x}\,\mathrm{d}x = \frac{2}{(2n\pi i)^2} = -\frac{1}{2(n\pi)^2}
\end{aligned}$$

and for $n = 0$ we obtain that

$$\hat{f}(0) = \int_0^1 f(x)\,\mathrm{d}x = \frac{1}{6}.$$

These imply that

$$f(x) = \frac{1}{6} - \sum_{n=-\infty, n\neq 0}^{+\infty} \frac{1}{2(n\pi)^2} e^{2n\pi i x}.$$

If one wants to transform the terms of the series using Euler's formula — $e^{2n\pi i x} = \cos(2n\pi x) + i\sin(2n\pi x)$ — to have a form that looks as $f(x) = \frac{1}{6} - \sum_{n=1}^{+\infty} \left(a_n \cos(2n\pi x) + b_n \sin(2n\pi x)\right)$, one has by correspondence: $a_n = -\frac{1}{n^2\pi^2}$ and $b_n = 0$ for all $n \in \mathbb{N}$. This implies that f can be written as well as

$$f(x) = \frac{1}{6} - \sum_{n=1}^{+\infty} \frac{1}{n^2\pi^2} \cos(2n\pi x).$$

(3) Since $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{R})$, the Fourier series of f converges in the L^2 metric to f, by Fourier's theorem. Since $\sum_{n=-\infty}^{+\infty} |\hat{f}(n)| = \sum_{n=-\infty}^{+\infty} \frac{1}{2\pi^2 n^2} + 1/36$ is (absolutely) convergent, as a consequence of Fourier's theorem one has that the Fourier series of f converges uniformly to f.

(4) The uniform convergence of the Fourier series implies in particular point-wise convergence. Hence

$$0 = f(0) = \frac{1}{6} - \sum_{n=1}^{+\infty} \frac{1}{n^2\pi^2},$$

which implies the desired formula, i.e. $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(5) By the same reasoning as in (4), one has

$$\frac{1}{4} = f(1/2) = \frac{1}{6} - \sum_{n=1}^{+\infty} \frac{1}{n^2\pi^2}(-1)^n,$$

which implies $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} = \pi^2(1/4 - 1/6) = \frac{\pi^2}{12}$.

(6) Parseval-Plancherel theorem implies that $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \|f\|_{L^2}^2$. Let us compute the r.h.s.

$$\|f\|_{L^2}^2 = \int_0^1 x^2(1-x)^2 \, \mathrm{d}x = \int_0^1 (x^2 - 2x^3 + x^4) \, \mathrm{d}x = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}.$$

The l.h.s. is just $\frac{1}{36} + \sum_{n\geq 1} \frac{1}{2n^4\pi^4}$. These imply the formula

$$\sum_{n\geq 1} \frac{1}{n^4} = \pi^4(1/15 - 1/18) = \frac{\pi^4}{90}.$$