Math 131B-1: Analysis

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Final exam, June 14th, 2017

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **180 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use a scratch paper, you should ask for it from one of the proctors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs. You may lose points in the lack of justification of your answers.
- Theorems from the lectures and homework assignments may be used in order to justify your solution. In this case state precisely the theorem you are using.
- The problems are not necessarily ordered w.r.t. difficulty.
- This exam has **4** problems and is worth **33 points**. Adding up the indicated points you can observe that there are **44 points**, which means that there are **11 "bonus" points**. This permits to obtain the highest score 33, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 33 will be considered as 33 in the gradebook.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Exercise 4	
Total	

Exercise 1 (8 points).

Let us consider the trigonometric functions $\sin, \cos : \mathbb{R} \to \mathbb{R}$. In this exercise you might use all the properties of these functions, which were derived during the lectures or showed in homework exercises. We define $\tan : \mathbb{R} \setminus \{(k+1/2)\pi : k \in \mathbb{Z}\} \to \mathbb{R}$ as

$$\tan(x) := \frac{\sin(x)}{\cos(x)}.$$

Notice that this function is well-defined, since cos is vanishing only at points of the form $(k+1/2)\pi$, $k \in \mathbb{Z}$.

(1) Show that tan is π-periodic and differentiable on its domain of definition. Give a formula for its derivative as well!

(2) Show the identity $\cos^2(x) = 1/(1+\tan^2(x))$ for all $x \in (-\pi/2, \pi/2)$.

(3) Show that tan is strictly increasing on $(-\pi/2, \pi/2)$. Compute

$$\lim_{x \to -\pi/2} \tan(x) \quad \text{and} \quad \lim_{x \to \pi/2} \tan(x).$$

(4) Argue why is arctan differentiable. Derive the formula for its derivative. *Hint:* you might use $\tan(\arctan(x)) = x$ for all $x \in \mathbb{R}$ and (2).

(5) Show that tan is a bijective map from $(-\pi/2, \pi/2)$ onto $\mathbb{R}$, hence $\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$ exists. We denote $\tan^{-1}$ as arctan.

(6) Using eventually the geometric series, write the power series expansion for $\arctan'$ (the derivative of arctan) with center 0 for all x such that $|x| < 1$.

(7) Using the appropriate theorem for power series, deduce a power series expansion for arctan on $(-1, 1)$. Argue why is arctan is real analytic on $(-1, 1)$. *Hint:* you might use a homework exercise to justify the last question.

(8) Using Abel's theorem and (7), show that arctan is continuous at $x = 1$, (although we know that the function is differentiable by (5), show the continuity of it at $x = 1$, as it is asked, using Abel's theorem). *Hint:* you have to discuss the convergence of an alternating series.

(9) Using the fact that $\tan(\pi/4) = 1$ and (8), write $\pi/4$ as the sum of a convergent series.

Solution

(1) From the definition of Trigonometric functions, we have

$$\cos(z+\pi) = \frac{e^{i(z+\pi)} + e^{-i(z+\pi)}}{2} = \frac{-e^{iz} - e^{-iz}}{2} = -\cos(z) \tag{1}$$

$$\sin(z+\pi) = \frac{e^{i(z+\pi)} - e^{-i(z+\pi)}}{2i} = \frac{-e^{iz} + e^{-iz}}{2i} = -\sin(z) \tag{2}$$

Using the properties above, we conclude that

$$\tan(z+\pi) = \frac{\cos(z+\pi)}{\sin(z+\pi)} = \frac{-\cos(z)}{-\sin(z)} = \tan(z) \tag{3}$$

Note that $\tan(z)$ is well defined on the domain. Moreover, it is differentiable since *sin* and *cos* are differentiable on the domain. Use Theorem 4.7.2 in the textbook and differentiation formula for fractions to conclude that

$$(\tan(z))' = \frac{(\sin(z))' \cos(z) - \sin(z)(\cos(z))'}{\cos^2(z)} = \frac{\sin^2(z) + \cos^2(z)}{\cos^2(z)} = \frac{1}{\cos^2(z)} \tag{4}$$

(2)

$$1 + \tan^2(x) = 1 + \frac{\sin^2(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} \tag{5}$$

(3) Let $f(x) := \tan(x)$. From part (1), we get

$$f'(x) = \frac{1}{\cos^2(x)} > 0 \tag{6}$$

Thus, tan is strictly increasing on $(-\pi/2, \pi/2)$.

Note that $\begin{cases} \sin(x) \to -1 \text{ and } \cos(x) \to +0 & \text{as } x \to -\pi/2 \\ \sin(x) \to 1 \text{ and } \cos(x) \to +0 & \text{as } x \to \pi/2 \end{cases}$

From this, we conclude that

$$\lim_{x \to -\pi/2} \tan(x) = -\infty, \qquad \lim_{x \to \pi/2} \tan(x) = +\infty \tag{7}$$

(4) arctan is differentiable since $(\tan(x))' = \frac{1}{\cos^2(x)}$ is non-vanishing on the domain. Use the Chain rule and part (2) to get

$$1 = (x)' = (\tan(\arctan(x))' = \frac{1}{\cos^2(\arctan(x))}(\arctan(x))' \tag{8}$$

$$(\arctan(x))' = \cos^2(\arctan(x)) = \frac{1}{1 + \tan^2(\arctan(x))} = \frac{1}{1 + x^2} \tag{9}$$

(5) In part (3), we showed that

$$\lim_{x \to -\pi/2} \tan(x) = -\infty, \qquad \lim_{x \to \pi/2} \tan(x) = +\infty \tag{10}$$

Note that tan is differentiable, in particular, it is continuous. Thus by mean value theorem, tan is a surjective (onto) map. On the other hand, tan is strictly increasing, and so it is injective (one to one). Therefore, tan is a bijective map from $(-\pi/2, \pi/2)$ onto $\mathbb{R}$

(6)

$$(\arctan(x))' = \frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \tag{11}$$

This formula is valid for $|-x^2| < 1$, i.e. $|x| < 1$, by geometric series.

(7) The radius of convergence of power series is preserved by differentiation and integration. Moreover, for each $r < 1$, $\sum_{n=0}^{\infty} x^{2n}$ converges uniformly on $(-r, r)$. It follows that

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} + C \tag{12}$$

Note that $\arctan(0) = 0$. This implies $C = 0$, $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$ for $x \in (-r, r)$. Since $r < 1$ is arbitrary, this is true for any $x \in (-1, 1)$. In particular, arctan is real analytic on $(-1, 1)$.

(8) At $x = 1$, we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \tag{13}$$

Let $a_n = \frac{1}{2n+1}$. Note that $\{a_n\}_{n=0}^{\infty}$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$. By alternating series test, $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ converges. Thus, Abel's theorem implies arctan is continuous at $x = 1$.

(9)

$$\pi/4 = \arctan(1) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \tag{14}$$

Exercise 2 (Fourier series – 9 points).

Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$ for all $x \in [0, 1)$, and then we extend it periodically to the whole real line. For a given metric space (X, d), we denote by $C(\mathbb{R}/\mathbb{Z}; X)$ the space of continuous functions defined on $\mathbb{R}$ with values in X that are $\mathbb{Z}$-periodic.

(1) Is f an element of $C(\mathbb{R}/\mathbb{Z}; \mathbb{R})$? Justify your answer!

(2) Find the Fourier coefficients of f and the Fourier series associated to f. *Hint:* you might use either the definition with the complex valued characters, or the equivalent definition with sin and cos.

(3) Is it possible to study the uniform convergence of the series found in (2), by Weierstrass' M-test? Justify your answer!

(4) Study the pointwise convergence of the series found in (2) at $x = 0$ and $x = 1$. What do you observe? Is the Fourier series of f converging uniformly to f on $\mathbb{R}$? Justify your answers!

(5) Show the pointwise convergence of the series at $x = 1/2$ and $x = 1/4$ to $f(1/2)$ and $f(1/4)$ respectively. *Hint:* for the latter one, you might use Exercise 1(9).

(6) Show that the Fourier series found in (2) is converging in the L^2 sense to f, meaning that the sequence of partial sums is converging w.r.t. the d_{L^2} metric to f. Recall that for two complex valued $\mathbb{Z}$-periodic square integrable functions g, h , we define the d_{L^2} metric as

$$d_{L^2}(g, h) := \left(\int_0^1 |g(x) - h(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

Hint: compute the integral by hand. You might use the fact that $\sum_{n\geq 1} \frac{1}{n^2} = \pi^2/6$.

Solution

(1) No. $\lim_{x\to 1-} f(x) = 1 \neq 0 = f(0) = f(1)$

(2) Note that $e^{2\pi in} = 1$ When $n \neq 0$,

$$\begin{aligned}
\hat{f}(n) &= \int_0^1 f(x)e^{-2\pi inx}dx \\
&= \int_0^1 xe^{-2\pi inx}dx \\
&= x \cdot \frac{1}{-2\pi in}e^{-2\pi inx}\Big|_0^1 - \int_0^1 1 \cdot \frac{1}{-2\pi in}e^{-2\pi inx}dx \\
&= (1 \cdot \frac{1}{-2\pi in}e^{-2\pi in} - 0) - \frac{1}{(2\pi in)^2}e^{-2\pi inx}\Big|_0^1 \\
&= -\frac{1}{2\pi in}
\end{aligned}$$

For $n = 0$, $\hat{f}(0) = \int_0^1 xdx = \frac{1}{2}$. Thus, the Fourier series associated to f is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx} = \frac{1}{2} + \sum_{n\neq 0} -\frac{1}{2\pi in}e^{2\pi inx} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{\pi n}\sin(2\pi nx) \tag{15}$$

(3) No.

$$|-\frac{1}{2\pi in}e^{2\pi inx}| = \frac{1}{2\pi n} \tag{16}$$

which is not summable. Therefore, we cannot apply the Weierstrass' M-test to study the uniform convergence of the series.

(4) At $x = 0$ and $x = 1$, $\sin(2\pi nx) = 0$. Thus, the series converges pointwisely to $\frac{1}{2}$. If the Fourier series of f were converging uniformly to f on $\mathbb{R}$, the function f should be continuous. On the other hand, in part (1), we observed that f is not continuous. Thus, the Fourier series of f does not converge uniformly to f on $\mathbb{R}$.

(5) At $x = \frac{1}{2}$, $\sin(2\pi nx) = 0$ for all $n \geq 1$. Thus, the series converges to $\frac{1}{2}$.

At $x = \frac{1}{4}$, $\sin(2\pi nx) = \begin{cases} (-1)^k & \text{if } n = 2k+1 \\ 0 & \text{if } n = 2k \end{cases}$

$$\frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin(2\pi n \frac{1}{4}) = \frac{1}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\pi} = \frac{1}{2} - \frac{1}{\pi} \cdot \frac{\pi}{4} = \frac{1}{4} \tag{17}$$

(6) Use $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ to conclude

$$(x - \frac{1}{2} + \sum_{n=1}^{k} \frac{1}{\pi n} \sin(2\pi nx))^2 = x^2 + \frac{1}{4} + (\sum_{n=1}^{k} \frac{1}{\pi n} \sin(2\pi nx))^2 - x + 2\sum_{n=1}^{k} \frac{x}{\pi n} \sin(2\pi nx) - \sum_{n=1}^{k} \frac{1}{\pi n} \sin(2\pi nx) \tag{18}$$

It is easy to check that

$$\int_0^1 \sin(2\pi nx)\sin(2\pi mx)dx = \begin{cases} \frac{1}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{19}$$

$$\int_0^1 x\sin(2\pi nx)dx = -\frac{1}{2\pi n} \qquad (n \neq 0) \tag{20}$$

$$\int_0^1 \sin(2\pi nx)dx = 0 \qquad (n \neq 0) \tag{21}$$

A combination of the equalities yields

$$\begin{aligned}
&\int_0^1 |x - \frac{1}{2} + \sum_{n=1}^{k} \frac{1}{\pi n} \sin(2\pi nx)|^2 dx \\
&= \int_0^1 x^2 - x + \frac{1}{4} dx + \sum_{n=1}^{k} \frac{1}{2}(\frac{1}{\pi n})^2 + 2\sum_{n=1}^{k} \frac{1}{\pi n}(-\frac{1}{2\pi n}) \\
&= \frac{1}{12} - \frac{1}{2\pi^2} \sum_{n=1}^{k} \frac{1}{n^2} \\
&\to \frac{1}{12} - \frac{1}{2\pi^2} \frac{\pi^2}{6} = 0 \qquad \text{as } k \to \infty
\end{aligned}$$

Exercise 3 (A generalization of the Arzelà-Ascoli theorem and its applications - 6+2.5+3.5=12 points)**.**

Let (X, d_X) be a metric space. Let us recall the following notions. We say that a set $E \subseteq X$ is *dense* in X, if for any $r > 0$ and any $x \in X$, $B(x; r) \cap E \neq \emptyset$. Equivalently, $E \subset X$ is dense if, for any $x_0 \in X$ there exists a sequence from E that converges to x_0. We say that the metric space (X, d) is *separable*, if it contains a countable dense set.

Let (X, d_X) be a *compact separable metric space* and (Y, d_Y) be a *compact metric space.* We consider a sequence $(f_n)_{n\in\mathbb{N}}$ of *equicontinuous* functions, i.e. $f_n : X \to Y$ for all $n \in \mathbb{N}$ and for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ s.t.

$$d_Y(f_n(x), f_n(y)) \leq \varepsilon, \quad \forall x, y \in X : \ d_X(x, y) \leq \delta_\varepsilon, \ \forall n \in \mathbb{N}.$$

Part 1

(1) Show that the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded. *Hint:* what do you know about the metric space (Y, d_Y)?

(2) Show that the sequence $(f_n)_{n\in\mathbb{N}}$ has a subsequence which is uniformly converging to some continuous function $f : X \to Y$. *Hint:* you should consider breaking the proof into several steps. Notice first that (X, d_X) is separable, hence it has a countable dense set $E \subseteq X$. Study the convergence of the sequences $\{(f_n(x))_{n\in\mathbb{N}} : x \in E\}$, pass to subsequences. Use a diagonal argument to construct a subsequence of the original sequence of functions which is converging at every element of E. Then use the fact that (X, d) is compact and that E is dense in X, to show that this subsequence is Cauchy. Lastly, is the space of continuous functions between X and Y complete? Why? Conclude!

Solution

(1) Since the space (Y, d_Y) is compact, hence it is (totally) bounded in particular. This means that there exists $C > 0$ and $y_0 \in Y$ s.t. $Y \subseteq B(y_0; C)$. In particular $f_n(x) \in B(y_0; C)$ for all $x \in X$ and for all $n \in \mathbb{N}$. This means that the sequence $(f_n)_n$ is uniformly bounded.

(2) We proceed by a diagonal argument as follows. Let us denote the countable dense set E of X by the sequence $(x_n)_{n\in\mathbb{N}}$, such that $x_i \in X$ for all $i \in \mathbb{N}$. Since $(f_n(x_1))_n$ is a sequence in a compact space, it has a convergent subsequence, that we denote by $(f_{n_1}(x_1))_{n_1}$. Similarly $(f_{n_1}(x_2))_{n_1}$ is a sequence in the same compact space, it has a convergent subsequence that we denote by $(f_{n_2}(x_2))_{n_2\in\mathbb{N}}$. Using the same procedure, after k steps, we have a subsequence $(f_{n_k})_{n_k}$ which is converging at the points $\{x_1, x_2, \ldots, x_k\}$. Let us define a subsequence of $(f_n)_n$ by a diagonal procedure,

$$g_i := f^i_{n_i}, \ \forall i \in \mathbb{N},$$

meaning that we take the i^{th} element of the sequence indexed by n_i (that we obtained at the i^{th} step). By this construction, $(g_i)_{i\in\mathbb{N}}$ is converging pointwisely on the dense set E. This implies in particular that for any $x_i \in E$, $(g_n(x_i))_n$ is a Cauchy sequence in (Y, d_Y), so for any $\varepsilon > 0$ there exists an index $N_{\varepsilon,x_i} \in \mathbb{N}$ such that

$$d_Y(g_n(x_i), g_m(x_i)) \leq \varepsilon, \ \forall n, m > N_{\varepsilon,x_i}. \tag{22}$$

Since the sequence is equicontinuous, one has that for any $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that

$$d_Y(g_n(x), g_n(y)) \leq \varepsilon, \ \forall x \in X, y \in B(x; \delta_\varepsilon), n \in \mathbb{N}. \tag{23}$$

Now, clearly $X \subseteq \bigcup_{x\in X} B(x; \delta_\varepsilon)$, which is an open cover of X. By the fact that X is compact, there is a number $m \in \mathbb{N}$ and $\{y_1, \ldots, y_m\} \subset X$ such that

$$X \subseteq \bigcup_{i=1}^{m} B(y_i; \delta_\varepsilon)$$

Since the set E is dense, for each $i \in \{1, \dots, m\}$, there exists $x_i \in E \cap B(y_i; \delta_\varepsilon)$, moreover, for any $y \in X$, there exists $i \in \{1, \dots, m\}$ such that $y \in B(y_i; \delta_\varepsilon)$. Now fix any $y \in X$ and take the corresponding y_i and x_i. We have

$$d_Y(g_n(y), g_m(y)) \leq d_Y(g_n(y), g_n(x_i)) + (g_n(x_i), g_m(x_i)) + d_Y(g_m(x_i), g_m(y)) \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,$$

where the first and the third ε is coming from (23) and the second inequality is due to (22), provided $n, m \geq \max\{N_{\varepsilon, x_i} : i \in \{1, \dots, m\}\}$. Thus, taking the supremum w.r.t. $y \in X$ the sequence $(g_n)_n$ is Cauchy w.r.t. the uniform convergence of continuous functions. Last, the space of continuous functions is complete, provided the image space is complete. (Y, d_Y) is compact, so in particular complete, hence the sequence $(g_n)_n$ converges uniformly to some continuous function.

Part 2

(1) Let us consider a sequence $(r_n)_{n\in\mathbb{N}}$ of continuously differentiable functions with values in $[-5,5]$, i.e. there is a $T>0$ given and $r_n:[0,T]\to[-5,5]$ are continuously differentiable functions for all $n\in\mathbb{N}$. We suppose that there exists $M>0$ such that

$$\int_0^T |r_n'(t)|^2\,\mathrm{d}t < M,\ \forall n\in\mathbb{N}.$$

Show that this sequence of functions has a subsequence that is converging uniformly to some continuous $r:[0,T]\to[-5,5]$. *Hint:* show that the sequence is equicontinuous by using the fundamental theorem of calculus and a Cauchy-Schwarz inequality, then conclude by Part 1.

Solution

Take $s,t\in[0,T],\ s<t$, then we have $r_n(t)-r_n(s)=\int_s^t r_n'(\tau)\,\mathrm{d}\tau$, from where using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|r_n(t)-r_n(s)| &= \left|\int_s^t r_n'(\tau)\,\mathrm{d}\tau\right| \le \int_s^t |r_n'(\tau)|\,\mathrm{d}\tau\\ &\le \left(\int_s^t \mathrm{d}\tau\right)^{1/2}\left(\int_s^t |r_n'(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} \le |t-s|^{1/2}\left(\int_0^T |r_n'(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} \le \sqrt{M}|t-s|^{1/2}.\end{aligned}$$

The last inequality implies that the sequence is equicontinuous. Indeed, for any $\varepsilon>0$, by choosing $\delta_\varepsilon:=\varepsilon^2/M$, we have that

$$|r_n(t)-r_n(s)|\le\varepsilon,\quad \forall n\in\mathbb{N},$$

whenever $|t-s|\le\delta_\varepsilon$. Since both $[0,T]$ and $[-5,5]$ equipped with the standard metric are compact and separable spaces, we can conclude by Part 1.

Part 3

(1) Let X be a nonempty set and let $d_X=d_{\text{disc}}$. Let (Y,d_Y) be an arbitrary metric space. Show that any function $f:X\to Y$ is uniformly continuous.

(2) Let X be a finite set i.e. $X=\{x_1,\dots,x_n\}$ for some $n\in\mathbb{N}$ and d_X any metric on X. Let (Y,d_Y) be a compact metric space. Show that any sequence of functions $(f_n)_{n\in\mathbb{N}}$, $f_n:X\to Y$ is equicontinuous. Can you say that any sequence of functions like this has a uniformly convergent subsequence? Justify your answer! Draw an analogy between these sequences of functions and bounded sequences in $(\mathbb{R}^n,d_{\ell^2})$.

Solution

(1) Take an arbitrary function $f:X\to Y$, take any $\varepsilon>0$ and set $\delta:=1/2$. By this choice one has that

$$d_Y(f(x),f(y))\le\varepsilon,$$

for all $x,y\in X$ such that $d_X(x,y)\le\delta$. Indeed, the only pairs that satisfy $d_X(x,y)\le\delta$ are such that $x=y$, for which $f(x)=f(y)$, hence $d_Y(f(x),f(y))=0$. This shows that any function is uniformly continuous.

(2) For any function $f:X\to Y$ and for any $\varepsilon>0$, set $\delta:=\min\{d_X(x,y):x,y\in X\}/2$, which is a well-defined positive number since X is a finite set. Now, similarly as in (1), one has that $d_X(x,y)\le\delta$ implies $x=y$, thus $d_Y(f(x),f(y))=0\le\varepsilon$. Since the choice of δ is independent of f and the points $x,y\in X$ (and actually of ε as well), we have that any sequence in this setting is equicontinuous. Since any finite metric space is compact and separable, we are in the framework of Part 1, so we can conclude.

This last scenario is analogous to the Bolzano-Weierstrass theorem in $(\mathbb{R}^n,d_{\ell^2})$. Indeed, any function that maps X to Y, it basically creates $n-$tuples in Y (because we have to describe its values at n different point only). So the whole space of function between X and Y behaves exactly as $Y\times Y\times\dots Y=:Y^n$. Since Y is compact, Y^n is also compact, so any sequence in Y^n has a convergent subsequence. This means that any function sequence between X and Y has a pointwise convergent subsequence. But all these functions are uniformly continuous and any sequence is equicontinuous, hence the convergence is uniform as well.

Exercise 4 (5+10=15 points)**.**

Let (X, d) be a metric space and $E \subseteq X$. We say that $x_0 \in E$ is an *isolated point* of E, if there exists $r > 0$ such that $B(x_0; r) \cap E = \{x_0\}$, i.e. the ball $B(x_0; r)$ does not contain any other points from E, but x_0. If E has only isolated points, we call it a *discrete set*.

Part 1

(1) Let us consider (X, d_{disc}). Show that any point of X is an isolated point, hence X is a discrete set (so the name of the metric is justified).

(2) Give an example (with justification) of a compact metric space and a discrete subset of it, which has infinitely many elements.

(3) Show that any discrete subset of the real line equipped with the standard metric has countably many elements! *Hint:* you may assume that there are countably many rationals and the rationals are dense in $\mathbb{R}$.

(4) Let (X, d) be an arbitrary metric space and let x_0 be an isolated point of X. Show that the set $\{x_0\}$ is both open and closed.

Solution.

(1) Clearly, for any $x \in X$, $B(x; 1/2) \cap X = \{x\}$, thus x is isolated.

(2) Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and d the standard metric on $\mathbb{R}$ This is clearly a compact space. Let $E = X \setminus \{0\}$. Then E is discrete and has infinitely many elements.

(3) Let E be a discrete set of $\mathbb{R}$. By definition for any $x \in E$, $\exists r > 0$ such that $B(x; r) \cap E = \{x\}$. We know also that $B(x; r)$ contains at least one rational, by density. By this fact, to any $x \in E$ one can associate a rational. Since the rationals are countable, E is countable.

(4) By definition of isolated point, there exists $r > 0$, such that $B(x_0; r) \cap X = \{x_0\}$. Thus, $\{x_0\}$ is open in X, since $B(x_0; r) \subseteq \{x_0\}$. Any singleton is a closed set. Therefore, $\{x_0\}$ is both open and closed.

Part 2

Let us consider a metric space (X, d) and a nonempty subset $E \subseteq X$. Let us consider a family $\mathcal{F}$ of subsets of X with the following properties

- if $A \in \mathcal{F}$, then $\mathrm{int}(A) \neq \emptyset$;
- for each $B \subset X$ nonempty and open there exists $A \in \mathcal{F}$ such that $A \subseteq B$.

Two players P_1 and P_2 play the following game: they alternatively choose sets $F_1 \supset F_2 \supset F_3 \supset \dots$ from the family $\mathcal{F}$. P_1 wins if and only if

$$E \cap \left(\bigcap_n F_n \right) \neq \emptyset,$$

otherwise P_2 wins. Notice that this game might not end in finitely many moves.

(1) Let $X = \{x_1, x_2, x_3\}$ be a set with 3 distinct elements, let $d = d_{\mathrm{disc}}$ and let $E = \{x_1\}$. Write down the family $\mathcal{F}$ associated to this configuration.

(2) Show that in the case of (1), P_1 has a winning strategy, i.e. there is a set of moves for P_1 such that no matter what P_2 does, P_1 wins using this set of moves.

(3) Show that for an arbitrary metric space (X, d) with an isolated point $x_0 \in X$, and for a subset E of X containing x_0, P_1 has a winning strategy. In how many moves does the game end in this case?

(4) ~~Show that for an arbitrary metric space (X, d) and $E \subset X$ nonempty and open, P_1 has a winning strategy~~. Give an example of a metric space and an open subset of it E s.t. P_2 has a winning strategy.

(5) Let (X, d) be an arbitrary metric space and let $U \subseteq X$ be a nonempty open set and let $x_0 \in U$. Show that $U \setminus \{x_0\}$ is an open set.

(6) Let (X, d) be an arbitrary metric space that has no isolated points. Let E be a nonempty countable subset of X. Show that in this case P_2 has a winning strategy. Describe this strategy. *Hint:* notice that the objective of P_1 is to choose sets which contain as many elements from E as possible, while the objective of P_2 if to choose sets which contain as few elements as possible from E.

(7) In the situation as in (6), what changes if E is a finite set? Can P_2 win in less moves?

Solution.

(1) A possible choice for the family $\mathcal{F}$ is to consider all possible subsets of X, i.e.

$$\mathcal{F} := \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\}.$$

Another choice that satisfies the properties is for instance $\mathcal{F} := \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_3\}\}$. Other correct choices are also possible

(2) P_1 choses $\{x_1\}$, then P_2 cannot choose anything, moreover $E \cap \{x_1\} = \{x_1\}$, so P_1 wins.

(3) Since x_0 is an isolated point of X, $\{x_0\}$ is open. By the second property, it will be an element of $\mathcal{F}$. The winning strategy for P_1, as in (2), is to choose $\{x_0\}$ at the first step. Then the game ends in one move.

(4) Let us consider $(\mathbb{Q}, d)$ as a metric space where d is a standard metric on $\mathbb{R}$ restricted to $\mathbb{Q}$. Let $E = \mathbb{Q}$, then P_2 has a winning strategy. Indeed, whatever P_1 choses, P_2 removes at least one element in the next step, so that the consider set is in $\mathcal{F}$. Since the rationals are countable, by this procedure eventually after infinitely many steps P_2 wins the game. Actually this strategy is the same as in (6).

(5) Notice that $X \setminus (U \setminus \{x_0\}) = (X \setminus U) \cup \{x_0\}$, moreover this last set is closed since it is the union of two closed sets. Thus the complement of $U \setminus \{x_0\}$ is closed, so this set it open.

(6) Let us denote $E = \{x_1, x_2, \dots\}$. Suppose that P_1 chooses the set F_1, which has U_1 as its nonempty interior. Then $U_1 \setminus \{x_1\}$ is an open set, and P_2 chooses a set from $\mathcal{F}$ that is contained in $U_1 \setminus \{x_1\}$ (this is possible by the second point). After, whatever P_1 does, P_2 behaves as described before, and after each move of P_2, at least one element of E is excluded from the chain of chosen sets. Thus, this is the winning strategy of P_2.

(7) Yes. P_2 can win in a finite number of steps.