Math 131B-1: Analysis

Instructor: Alpár R. Mészáros

Midterm exam, April 26, 2016

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the supervisors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs. You may lose points in the lack of justification of your answers.
- Theorems from the lectures and homework may be used in order to justify your solution. In this case state the theorem you are using.
- The problems are not necessarily ordered w.r.t. difficulty.
- This exam has 3 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **28 points**, which means that there are **8 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Total	

Exercise 1 (6 points)**.**

We define $\rho_1, \rho_2 : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ as

$$\rho_1(x, y) := |x - y|^2 \quad \text{and} \quad \rho_2(x, y) := \left|x^2 - y^2\right|.$$

(1) Show that ρ_1 cannot define a metric on $\mathbb{R}$, hence $(\mathbb{R}; \rho_1)$ cannot be a metric space.

(2) Show that ρ_2 defines a metric on $[0, +\infty)$, hence $([0, +\infty); \rho_2)$ is a metric space.

(3) Show that ρ_2 does not define a metric on $[-1, 1]$, hence $([-1, 1]; \rho_2)$ cannot be a metric space.

(4) Let $X \neq \emptyset$ and let $f : X \to \mathbb{R}$ be an injective (one-to-one) function. Show that $\rho_3 : X \times X \to [0, +\infty)$ defined as
$$\rho_3(x, y) := |f(x) - f(y)|$$
defines a metric on X, hence (X, ρ_3) is a metric space.

Solutions

(1) Here the triangle inequality will go wrong. Indeed, suppose that for all $x, y, z \in \mathbb{R}$ one has

$$|x - z|^2 \leq |x - y|^2 + |y - z|^2,$$

then developing the squares one obtains

$$0 \leq 2y^2 - (2x + 2z)y + 2xz.$$

Clearly, setting $y = 0$, $x = -1$ and $z = 1$ this inequality is violated, thus ρ_1 cannot define a metric on $\mathbb{R}$.

(2) Notice that the function $t \mapsto t^2$ is injective on $[0, +\infty)$). Thus, this problem is a consequence of (4) that we show below.

(3) Notice that $\rho_2(x, y) = 0$ implies that $|x| = |y|$ and on $[-1, 1]$ this property does not imply that $x = y$. Indeed, $\rho_2(1, -1) = 0$, thus ρ_2 cannot be a metric on $[-1, 1]$.

(4) Clearly, $\rho_3(y, x) = \rho_3(x, y) \geq 0$ and ρ_3 satisfies the triangle inequality as a consequence of the triangle inequality of the standard metric on $\mathbb{R}$. So one only needs to check that $\rho_3(x, y) = 0$ if and only if $x = y$.This is true, because f is injective and so $f(x) = f(y)$ if and only if $x = y$.

Exercise 2 (7 points)**.**

(1) Let X be a finite non-empty subset of $\mathbb{R}$, i.e. $X = \{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$ and $x_i \in \mathbb{R}$, for all $i \in \{1, \ldots, n\}$. Show that d_{ℓ^1} and d_{disc} are two equivalent metrics on X.

(2) Characterize the convergent sequences on the set $\mathbb{N}$ w.r.t. d_{ℓ^1} and d_{disc}.

(3) Show that d_{ℓ^1} and d_{disc} are not equivalent on the set $\mathbb{N}$, despite the results that you obtained in (2).

(4) Which are the compact sets in $(\mathbb{N}, d_{\ell^1})$ and in $(\mathbb{N}, d_{\text{disc}})$? Characterize all these sets; justify your answers!

(5) Show that both $(\mathbb{N}, d_{\ell^1})$ and $(\mathbb{N}, d_{\text{disc}})$ are complete metric spaces.

Solutions

(1) If X contains only one element, then the result is trivial. Otherwise, since X is a finite set, one can define $M := \max\{|x-y| : x, y \in X\}$ and $m := \min\{|x-y| : x \neq y \in X\}$ and these are finite positive numbers. Then

$$m d_{\text{disc}}(x, y) \leq d_{\ell^1}(x, y) \leq M d_{\text{disc}}(x, y), \ \forall x, y \in X, \tag{1}$$

which shows the equivalence of the two metrics on X.

(2) W.r.t. both metrics, clearly a sequence is convergent in $\mathbb{N}$ if and only if from a certain index on it is constant. This means in particular that they generate the same "notion of convergence".

(3) If one supposes that the two metrics are equivalent, there should exists two constants $m, M > 0$ such that (1) holds true for all $x, y \in \mathbb{N}$. However, there exists no such $M > 0$, because $d_{\ell^1}(x, y)$ can be arbitrary large if $x, y \in \mathbb{N}$. Thus the two metrics cannot be equivalent on $\mathbb{N}$.

(4) A set is compact, if every sequence in the set has a convergent subsequence. Clearly, sets with infinitely many elements will have sequences that do not converge neither w.r.t. d_{ℓ^1} nor w.r.t. d_{disc}. Hence the only candidates for compact sets are finite subsets of $\mathbb{N}$, which will indeed be compact, since any sequence defined on a finite sets has at least one element repeated infinitely many time. This will define a convergent subsequent.

(5) A sequence $(x_n)_{n \geq 0}$ w.r.t. d_{disc} is Cauchy if for any $\varepsilon > 0$ there exists N_ε such that

$$d_{\text{disc}}(x_n, x_m) < \varepsilon, \ \forall n, m \geq N_\varepsilon.$$

Take $\varepsilon = 1/2$ for instance, which implies that all such sequences have to be constant from a certain index onwards. (2) tells us precisely that these sequences are convergent, hence $(\mathbb{N}, d_{\text{disc}})$ is complete.

The very same reasoning shows also that $(\mathbb{N}, d_{\ell^1})$ is complete as well.

Exercise 3 (Dense and nowhere dense sets – 8+7 points).

Let (X, d) be an arbitrary metric space. We say that a set $E \subseteq X$ is *dense* in X, if for any $r > 0$ and any $x \in X$, $B(x; r) \cap E \neq \emptyset$. We say that $E \subseteq X$ is *nowhere dense* if $\mathrm{int}(\overline{E}) = \emptyset$.

Part 1

(1) Show that E is dense in X if and only if $\overline{E} = X$.

(2) Show that if $\partial E = X$, then E is dense. Give an example to show that the opposite implication is not true in general.

(3) Find all the dense sets in (X, d_{disc}). Justify your answer!

(4) Give two examples of dense sets in a metric space that are different from the whole space (the whole space is clearly always dense).

(5) Let $E \subset X$ be a dense set. Show that for any $x \in X$, there exists a sequence from E that converges to x.

(6) Can we say that dense sets are always: (a) open (b) closed (c) complete (d) compact? Justify your answers either with a proof or with a counterexample in each cases!

(7) Show that if E is dense in X and X is unbounded, so is E.

Solutions

(1) Suppose first that E is dense and show that its closure has to be the whole space. We know that

$$X = \mathrm{int}(E) \cup \partial E \cup \mathrm{ext}(E) = \overline{E} \cup \mathrm{ext}(E)$$

and suppose that there exists an element $x \in X \setminus \overline{E}$. Then by the previous decomposition $x \in \mathrm{ext}(E)$, which means in particular that there exists $r > 0$ such that $B(x; r) \cap E = \emptyset$, which contradicts to the fact that E is dense.

Conversely, suppose that $\overline{E} = X$ and show that E is dense in X. The assumption implies in particular that $\mathrm{ext}(E) = \emptyset$, so there exists no ball $B(x; r)$ such that $B(x; r) \cap E = \emptyset$, which means that E is dense.

(2) We know that $\overline{E} = \mathrm{int}(E) \cup \partial E = X$ by assumption, which by (1) implies that E is dense.

Let $X = [0, 1]$ with the standard metric on it and consider $E = (0, 1)$. Clearly, E is dense in X, because $\overline{E} = [0, 1] = X$, however $\partial E = \{0; 1\}$.

(3) The only dense set in (X, d_{disc}) is the set X itself. This is because a dense set has to intersect every ball in X, but balls with smaller radius than 1 are singletons.

(4) One example was given already in (2). Another example is $\mathbb{Q}$ in $\mathbb{R}$ w.r.t. the standard metric.

(5) Take $x \in X$ and for all $n \in N$ define $x_n \in E \cap B(x; 1/n)$, this element exists because E is dense and clearly this sequence converges to x.

(6) The answer for all four questions is negative and $\mathbb{Q}$ in $\mathbb{R}$ with the standard metric is a counterexample to all of these.

(7) Suppose that E is bounded, and such that $E \subseteq B(x_0; r)$ for some $x_0 \in X$ and $r > 0$. Now since X is unbounded, there are some elements in $X \setminus B(x_0; 2r)$. Clearly, such an element cannot be approximated with a sequence from E, which contradict to (5). Hence E has to be unbounded as well.

Part 2

(1) Give an example of a metric space and a subset of it, which is nowhere dense.

(2) By providing an example, show that a nowhere dense set is not compact in general.

(3) Give an example of a subset of a metric space which is neither dense nor nowhere dense.

(4) Are there nowhere dense sets in (X, d_{disc})? Give them all. *Hint:* a trivial example is also an example.

(5) Show that the boundary of every open set is nowhere dense. *Hint:* construct a proof by contradiction.

(6) Show that any finite union of nowhere dense sets is nowhere dense. Show that countable union of nowhere dense sets in general is not nowhere dense. *Hint:* provide an example for the second part.

(7) Show that the complement of a closed nowhere dense set is open and dense. *Hint:* construct a proof by contradiction.

Solutions

(1) Take $(\mathbb{R}, d)$ where d is the standard metric and set $E = \{1/n : n \in \mathbb{N}\}$. Clearly $\overline{E} = \{0\} \cup E$ and $\text{int}(\overline{E}) = \emptyset$, which implies that E is nowhere dense.

(2) The example provided in (1) gives such a set.

(3) Take for instance in $\mathbb{R}$ with the standard metric the set $E = [0, 1]$. This is not nowhere dense, since $\text{int}(E) = (0, 1)$ and it is not dense since it does not intersect for instance $B(10; 2)$.

(4) The only nowhere dense set is the empty set. All other sets are both open and closed, so the interior of their closure is the set itself which is not empty.

(5) Take (X, d) metric space and let $E \subseteq X$ be open. Then $E \cap \partial E = \emptyset$ which implies in particular that $\partial E \subseteq X \setminus E$. Suppose that there exists $x_0 \in \text{int}(\partial E) \subseteq \partial E \subseteq X \setminus E$, which implies that there exists $r > 0$ such that $B(x_0; R) \subseteq \partial E \subseteq X \setminus E$. But this last inclusion implies that x_0 is an exterior point, which is a contradiction to the fact that it is a boundary point. The result follows.

(6) Let $E_1, \ldots, E_n$ be nowhere dense sets in (X, d) for some $n \in \mathbb{N}$ finite. First, one has

$$\overline{\cup_{i=1}^n E_i} = \cup_{i=1}^n \overline{E_i}.$$

Second, let us show that E is nowhere dense if and only if $X \setminus \overline{E}$ is dense. For the first implication, suppose that $X \setminus \overline{E}$ is not dense, and so there exists $B(x; r) \subseteq X$ such that $B(x; r) \cap (X \setminus \overline{E}) = \emptyset$. This implies that $B(x; r) \subseteq \overline{E}$, which is clearly impossible by the fact that E is nowhere dense. For the converse implication, let $X \setminus \overline{E}$ be dense and we want to show that E is nowhere dense. Suppose the contrary, i.e. there exists $B(x; r) \subseteq \overline{E}$. But by the fact that $X \setminus \overline{E}$ is dense, one has that $B(x; r) \cap (X \setminus \overline{E}) = \emptyset$, which is impossible since $B(x; r)$ is entirely included in the complement of $X \setminus \overline{E}$.

Now, it is enough to show that $X \setminus (\cup_{i=1}^n \overline{E_i})$ is dense. We have by De Morgan's law that $X \setminus (\cup_{i=1}^n \overline{E_i}) = \cap_{i=1}^n (X \setminus \overline{E_i})$. Take a ball $B(x; r)$ and show that $\cap_{i=1}^n (X \setminus \overline{E_i}) \cap B(x; r) \neq \emptyset$. Since E_i is nowhere dense, and hence $(X \setminus \overline{E_i}) \cap B(x; r) \neq \emptyset$ for all $i \in \{1, \ldots, n\}$, and by the fact that this is the intersection of two open sets (hence open), one has that

$$(X \setminus \overline{E_j}) \cap (X \setminus \overline{E_i}) \cap B(x; r) \neq \emptyset, \quad \forall j \neq i.$$

This can be iterated, and thus one obtains that

$$\cap_{i=1}^n (X \setminus \overline{E_i}) \cap B(x; r) \neq \emptyset,$$

what we wanted.

Uncountable union of nowhere dense sets is in general not necessarily nowhere dense. As an example, consider $\mathbb{R}$ with the standard metric and write

$$\mathbb{Q} = \cup_{q \in \mathbb{Q}} \{q\},$$

since every singleton is nowhere dense, this provides the counterexample by the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

(7) The openness is trivial. Let us show that the density property. Let $F \subseteq X$ be a closed nowhere set in (X, d). Then since $\overline{F} = \mathrm{int}(F) \cup \partial F$ and $\mathrm{int}(\overline{F}) = \mathrm{int}(F) = \emptyset$, one has that $F = \partial F$. And so $X = \partial F \cup \mathrm{ext}(F) = F \cup (X \setminus F)$, hence $X \setminus F = \mathrm{ext}(F)$. Suppose now that $X \setminus F$ is not dense, which implies that there exists $B(x; r)$ in X such that $\mathrm{ext}(F) \cap B(x; r) = \emptyset$, which means that $B(x; r) \subseteq \partial F$. This implies in particular that x is an interior point of $F = \partial F$ and F is closed which is a contradiction to the fact that F is nowhere dense.